Exercise 1: May 01

Lecturer: Merav Parter

Girth and Short Cycles

Recall that the girth $g(G)$ of a graph G is the length of the shortest cycle in G. Erdős girth conjecture states that for every $k \geq 1$ and sufficiently large n, there exist n-vertex graphs with $\Omega(n^{1+1/k})$ edges and girth at least $2k+1$. A weaker lower bound can be shown via the probabilistic approach. Specifically, we will prove that there exists an n-vertex graph G^* with $\Omega(n^{1+1/(2k-1)})$ edges and girth $g(G)$ at least $2k+1$.

Exercise 1. The existence of G^* can be shown in two steps. (I) Consider a $G(n,p)$ graph[1] with $p = \Theta(1/n^{1-1/(2k-1)})$ and bound the expected (total) number of cycles of length $t \leq 2k$ in this graph. (II) Prove the existence of n-vertex graph G' with $\Theta(n^{1+1/(2k-1)})$ edges and a small number of cycles and turned it into the desired graph G^* while keeping the same order of the number of edges as in G'.

Distance Oracles

This section is devoted for the Thorup and Zwick distance oracles that we have covered in the last class.

Exercise 2 [DO w. No Swap]. Assume that the query algorithm does not swap the roles of u and v. Specifically, consider the following simplified query algorithm. Let i^* be min value such that $p_{i^*}(v) \in B_{i^*}(u)$. Return $\delta(u,v) = \text{dist}_G(v, p_{i^*}(v)) + \text{dist}_G(p_{i^*}(v), u)$. Note that the value i^* can be detected in $O(k)$ time. Show that $\delta(u,v) \leq (4k-3)\text{dist}_G(u,v)$.

Exercise 3 [Sourcewise DOs] The distance oracle presented in class supports distance queries u, v for any $u, v \in V \times V$. Our goal is to construct a *smaller* oracle of size $o(n^{1+1/k})$ that would handle only a subset of query pairs. In particular, you are now given a subset of vertices, called hereafter, *sources*, $S \subseteq V$ and it is required to design a *source-wise* approximate distance oracle scheme. In such a scheme, the query algorithm receives only queries of the form $u, v \in S \times V$ (i.e., your oracle should only answer distance queries between one of the sources and some other vertex in the graph). Show that in such a case, the preprocessing algorithm can be adapted to yield a data structure of size $\widetilde{O}(k|S|^{1/k} \cdot n)$. Your answer should include a modified preprocessing algorithm, query algorithm and a correctness analysis.

[1] In $G(n,p)$ graph, each of the $\binom{n}{2}$ edges exists with probability p.