Succinct Graph Structures and Their Applications **Spring 2018**

Exercise 1 (March 28)

Lecturer: Merav Parter

Girth and Short Cycles

Recall that the girth $g(G)$ of a graph G is the length of the shortest cycle in G. Erdős girth conjecture states that for every $k \geq 1$ and sufficiently large n, there exist n-vertex graphs with $\Omega(n^{1+1/k})$ edges and girth at least $2k+1$. A weaker lower bound can be shown via the probabilistic approach. Specifically, we will prove that there exists an n-vertex graph G^* with $\Omega(n^{1+1/(2k-1)})$ edges and girth $g(G)$ at least $2k+1$.

Exercise 1. The existence of G^* can be shown in two steps. (I) Consider a $G(n,p)$ graph[1] with $p = \Theta(1/n^{1-1/(2k-1)})$ and bound the expected (total) number of cycles of length $t \leq 2k$ in this graph. (II) Prove the existence of n-vertex graph G' with $\Theta(n^{1+1/(2k-1)})$ edges and small number of cycles and turned it into the desired graph G^* while keeping the same order of the number of edges as in G'.

Exercise 2. The number of $2k$-cycles in a graph grows with number of edges. In this exercise, we will understand this function for the case $k = 2$.
(2a) Show that every graph with no 4-cycles has $O(n^{3/2})$ edges. Hint: A cherry in a graph is an ordered set $\langle u, \{v, w\}\rangle$ where v, w are neighbors of u. Bound the number of distinct cherries in the graph from below and above and use it to bound the number of edges in 4-cycle free graph.
(2b*) Prove that any n-vertex graph G with average degree $\Delta = \Omega(\sqrt{n})$ has $\Omega(\Delta^4)$ 4-cycles. Hint: Consider a randomly chosen pair u, v in G and show that there are Δ^2/n 2-paths between u and v, use it to bound the number of 4-cycles with u, v on the opposite corners of these cycles.
Remark: The two claims above imply that the constant factor hidden in the $O(n^{3/2})$ edges is important. A graph with less than $c_1 \cdot n^{3/2}$ edges has *no* 4-cycle and every graph with at least $c_2 \cdot n^{3/2}$) edges has $\Omega(n^2)$ 4-cycles for $c_2 > c_1$.

Multiplicative Spanners

Exercise 3. Show that the greedy spanner algorithm we saw in class has a runtime of $O(m \cdot n^{1+1/k})$. For simplicity, you may assume G being unweighted.

Exercise 4. Show that every n-vertex unweighted graph with minimum degree $\Theta(\sqrt{n}\log n)$ has a 5-spanner H with $O(n)$ edges. Hint: use the randomized clustering approach as shown in class for 3-spanners.

[1] In $G(n,p)$ graph, each of the $\binom{n}{2}$ edges exists with probability p.