Succinct Graph Structures and Applications **Spring 2020**

Exercise 3: June 14

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Low-Diameter Decomposition

A *low-diameter* decomposition of a graph $G = (V, E)$ and a parameter D is a randomized partitioning of the vertices V into $V_1, \ldots, V_t$ such that:

(P1) the *weak-diameter*[1] of each $G[V_i]$ is at most D, and

(P2) for every $u, v \in V$, $Pr(u \in V_i \text{ and } v \in V_{j \neq i}) = \alpha \cdot \text{dist}_G(u, v)$ for $\alpha = O(\log n/D)$.

Exercise 1. In this exercise, we consider a candidate algorithm for computing a low-diameter decomposition. For a vertex v and integer r, let $B(u, r) = \{v \in V \mid \text{dist}_G(u, v) \leq r\}$ be the r-radius ball of u in G. Prove or disprove: the sets $G[V_1], \ldots, G[V_n]$ satisfy the properties (P1) and (P2) w.h.p. In case you think

Algorithm $\mathsf{Decomp}(G, D)$

1. Pick a radius $\delta \in [D/8, D/4]$ at random.
2. Consider the vertices in an *arbitrary* order π.
3. The i^{th} set V_i is the set of all vertices in $B(\pi(i), \delta) \setminus \bigcup_{j<i} B(\pi(j), \delta)$

Figure 3.1: A Low-Diameter Decomposition Algorithm?

the algorithm is incorrect, suggest how to fix it (along with a proof that your fix works).

Trees with Small Average Stretch

We presented in class, the construction of a distribution over trees such that the expected stretch of each pair u, v (when sampling a tree from the distribution) is bounded by α. A dual problem considers the construction of a *single* tree T (not necessarily a subgraph) that has a small *average* stretch over all edges (u, v) in G. Formally, given an unweighted graph $G = (V, E)$ and a tree T with $V(G) \subseteq V(T)$, define the average stretch of T by: $1/|E(G)| \cdot \sum_{(u,v)\in E} \text{dist}_T(u, v)$.

Exercise 2. (a) Given a even integer n, let W_n be the wheel graph consisting of n vertex ring C_n together with chords joining antipodal points on the ring. Find a tree $T \subseteq W_n$ with average stretch at most 8/3. (b) Show that the 2-dimensional $\sqrt{n} \cdot \sqrt{n}$ grid has a spanning tree with average stretch $O(\log n)$.

Cut Sparsification

Exercise 3. You are given a graph G that has a good *edge-expansion* such that for every $S \subset V$, $|S| \leq n/2$, it holds that:

$$|E(S, V \setminus S)|/|S| \geq \alpha, \quad \text{where} \quad \alpha = \Omega(\log n) .$$

[1] The weak diameter of a subgraph $G' \subseteq G$ with respect to G is $\max_{u,v\in G'} \text{dist}(u, v, G)$.

Show that if we sample each edge $e \in G$ with probability $p = \Omega(\log n/(\alpha \cdot \epsilon^2))$ then all cuts are preserved within $(1 \pm \epsilon)$ of their expectation with high probability (at least $1 - 1/n^5$). That is, show that w.h.p. for every $S \subseteq V$, $|S| \leq n/2$, the number of sampled edges in the cut $(S, V \setminus S)$ is $(1 \pm \epsilon) \cdot p \cdot |E(S, V \setminus S)|$. Instructions: you should *not* use the cut counting argument that we saw in class, i.e., do not use the fact that there are at most $n^{O(\alpha)}$ cuts of size $\alpha \cdot c$ where c in the min-cut in G.