Succinct Graph Structures and Applications **Spring 2022**

Exercise 2: June 12

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Trees with Small Average Stretch

We showed in class how to compute low-stretch tree embedding. A dual problem considers the construction of a *single* tree (either a subgraph of G or not) that has a small *average* stretch over all edges (u, v) in G. Formally, given an unweighted graph $G = (V, E)$ and a tree T with $V(G) \subseteq V(T)$, define the average stretch of T by:

$$1/|E(G)| \cdot \sum_{(u,v)\in E} \text{dist}_T(u, v) \ .$$

Exercise 1. (a) For a given even integer n, let W_n be the wheel graph consisting of n vertex ring C_n together with chords joining antipodal points on the ring. Find a tree $T \subseteq W_n$ with average stretch at most 8/3. (b) Show that the 2-dimensional $\sqrt{n} \cdot \sqrt{n}$ grid has a spanning tree with average stretch $O(\log n)$.

Exercise 2. We showed a randomized construction of a tree T such that $V(G) \subseteq V(T)$ and $\text{dist}_G(u, v) \leq \mathbb{E}(\text{dist}_T(u, v)) \leq \alpha \cdot \text{dist}_G(u, v)$ for every $u, v \in V(G)$. Adapt this construction to provide a tree T (where $V(G) \subseteq V(T)$) with *average stretch* at most α.

Cut Sparsification

Exercise 3. You are given a graph G that has a good *edge-expansion* such that for every $S \subset V$, $|S| \leq n/2$, it holds that:

$$|E(S, V \setminus S)|/|S| \geq \alpha, \quad \text{where} \quad \alpha = \Omega(\log n) \ .$$

Show that if we sample each edge $e \in G$ with probability $p = \Omega(\log n/(\alpha \cdot \epsilon^2))$ then all cuts are preserved within $(1 \pm \epsilon)$ of their expectation with high probability (at least $1 - 1/n^5$). That is, show that w.h.p. for every $S \subseteq V$, $|S| \leq n/2$, the number of sampled edges in the cut $(S, V \setminus S)$ is $(1 \pm \epsilon) \cdot p \cdot |E(S, V \setminus S)|$. Instructions: you should *not* use the cut counting argument that we saw in class, i.e., do not use the fact that there are at most $n^{O(\alpha)}$ cuts of size $\alpha \cdot c$ where c in the min-cut in G.