| **Advanced Distributed Algorithms** | **Spring 2019** |
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# Exercise 3 (June 16)

*Lecturer: Merav Parter*

**(Distributed) List Coloring via Symmteric LLL.** In the problem of *list coloring*, every vertex $v$ is associated with a list (or palette) of colors $Pal(v)$. It is required to compute a legal vertex coloring where the color of each vertex $v$ is taken from $Pal(v)$. (1a) Show that if for every $v$ it holds that: (i) $|Pal(v)| \geq \ell$ (ii) each color $c \in Pal(v)$ appears in at *most* $\ell/8$ of its neighbors, then there is a legal coloring (i.e., solution to the list coloring instance).
(1b) We consider a weighted variant of the list coloring problem. Given is a graph $G$ with maximum degree $\Delta$ where every vertex $v$ has a palette of colors $Pal(v)$. Each color $c \in Pal(v)$ has a weight $w_v(c)$ such that $\sum_{c \in Pal(v)} w_v(c) = 1$. Prove that if for every edge $uv$ we have $\sum_{c \in Pal(u) \cap Pal(v)} w_v(c) \cdot w_u(c) \leq 1/(8\Delta)$ then $G$ has a legal coloring.
(1c) Use the algorithms shown in class (as a black box) to devise distributed algorithms for computing the coloring in (1a) and (1b).

**Asymmetric LLL and Applications to $\beta$-balanced coloring.** In this exercise we consider a generalization of the symmetric LLL formula, known as the *asymmetric* LLL: Given is a collection of $m$ independent discrete random variables $\mathcal{X}=\{X_1, \ldots, X_m\}$, and a collection of $n$ bad events $\mathcal{A}=\{A_1, \ldots, A_n\}$. The variables of $A_i$ are denoted by $vbl(A_i) \subseteq \mathcal{X}$. Two events $A_i$ and $A_j$ are independent if $vbl(A_i) \cap vbl(A_j) = \emptyset$. Let $G$ be the dependency graph defined on these events. The neighbors of $A \in \mathcal{A}$ in the graph $G$ are denoted by $N(A)$, and $N^+(A) = N(A) \cup \{A\}$. The asymmetric LLL states that if there is a function $y : \mathcal{A} \to (0, 1)$ such that

$$\Pr[A] \leq y(A) \cdot \prod_{B \in N(A)} (1 - y(B))$$

then $\Pr[\bigcap_{A \in \mathcal{A}} \bar{A}] > 0$. That is, there is a satisfying assignment to the variables in $\mathcal{X}$ such that no bad event occurs. Note that the symmetric LLL can be derived simple corollary of the above: letting $p, d$ be such that $\Pr[A] \leq p$ and $|N(A)| \leq d$ for every $A \in \mathcal{A}$, if $ep(d + 1) < 1$ then $\Pr[\bigcap_{A \in \mathcal{A}} \bar{A}] > 0$.

(2) A $\beta$-balanced coloring is a legal coloring where *in addition*, each color appears at most $\beta$ times in each neighborhood. Use the asymmetric LLL to show the following. For any constant $\beta \geq 1$, if the graph $G$ has maximum degree $\Delta \geq \beta^\beta$, then it has $\beta$-balanced coloring using at most $16\Delta^{1+1/\beta}$ colors. Note: the case of $\beta = 1$ is easy, and can be shown w.o applying the LLL formula.