Succinct Graph Structures and Applications **Spring 2020**

Exercise 4: July 2

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Gomory-Hu Trees

We saw last week the Gomory-Hu tree algorithm which encodes all the s-t cuts in G in a single tree structure, using only $n-1$ min-cut computations. It is known that one can extend the Gomory-Hu construction to encode the "minimum-cut" equivalent[1] of any symmetric and sub-modular function $f: 2^V \rightarrow \mathbb{R}$.

Exercise 1. Let $G=(V,\mathcal{E})$ be a hypergraph where each edge $W \in \mathcal{E}$ is simply a subset of vertices. (In standard graphs, $|W|=2$, but in hypergraph, the cardinality of an hyperedge can be arbitrarily large). For a subset of vertices S, the value of the cut $\delta_G(S)$ is the number of all hyperedges that contain both vertices in S and vertices not in S. That is, $\delta_G(S) = |\{W \in \mathcal{E} \ \mid W \setminus S \neq \emptyset \ \text{ and } \ W \cap S \neq \emptyset\}|$.

Consider the function $f: 2^V \rightarrow \mathbb{N}$ where $f(S) = \delta_G(S)$ for every $S \subseteq V$. Show that f is a symmetric and sub-modular function.

Exercise 2. Using the Gomory-Hu trees one can show several facts on minimum cuts that are otherwise hard to prove directly. Here is one such a example. Let G be a graph with minimum-degree k. Show that there is some pair $s,t \in V(G)$ with $\alpha_G(s,t) \geq k$, where $\alpha_G(s,t)$ is the minimum s-t cut in G.

FT-BFS Structures

Exercise 3. Given a graph $G=(V,E)$ and a subset of sources S, a subgraph $H \subseteq G$ is a *Multi-Source FT-BFS* w.r.t S if

$$\text{dist}_{H\setminus\{e\}}(s,t) = \text{dist}_{G\setminus\{e\}}(s,t), \forall s,t \in V, e \in E\ .$$

We saw in class today a simple algorithm for the case where $|S|=1$ which is based on the notion of near and far edge faults. Show that a slight modification of this algorithm provides Multi-Source FT-BFS w.r.t S sources with $\widetilde{O}(\sqrt{|S|n} \cdot n)$ edges. **Bonus** (5 points): Show that this edge bound is tight by modifying the lower-bound graph example as well.

[1]For a given function $f: 2^V \rightarrow \mathbb{R}$, the "minimum-cut" equivalent is defined by $\alpha_{f,G}(u,v) = \min_{W \in V, |W \cap \{u,v\}|=1} f(W)$.