Exercise 1: Multiplicative Spanners. Show that every $n$-vertex graph $G$ with minimum degree $\sqrt{n}$ admits a 5-spanner $H \subseteq G$ with $O(n \log^2 n)$ edges. In addition, show that such a spanner can be computed in $O(1)$ rounds in the LOCAL model. The algorithm can be randomized, and it is required to output a 5-spanner with the desired number of edges, with high probability.

Exercise 2: Additive Spanners. Consider a $D$-diameter $n$-vertex graph $G = (V, E)$, and let $S \subseteq V$ be a random sample of $O(\sqrt{n \log n})$ vertices. In this exercise, we compute a 2-additive spanner $H$ for $G$ obtained by taking the union of two subsets of edges: $T_S$ and $H_{low}$. The set $T_S$ is a union of $|S|$ BFS trees rooted at each source $s \in S$. The set $H_{low}$ consists of all edges incident to low-degree vertices, namely, vertices with degree at most $\sqrt{n}$. Formally,

$$T_S = \bigcup_{s \in S} \text{BFS}(s) \quad \text{and} \quad H_{low} = \{(u, v) \in E(G) \mid \text{deg}(u) \leq \sqrt{n}\} .$$

(i) Show that $H = T_S \cup H_{low}$ is a 2-additive spanner. I.e., $H$ should satisfy that

$$\text{dist}_H(u, v) \leq \text{dist}_G(u, v) + 2, \forall u, v \in V .$$

Hint: fix a pair $u, v$ and consider its shortest path $P$ in $G$. Zoom into one specific edge on $P \setminus H_{low}$.

(ii) Provide a randomized CONGEST algorithm for computing $H$ in $\tilde{O}(D + \sqrt{n})$ rounds (w.h.p.). You may use theorems shown in class in a black-box manner, but still required to formally state them.

Exercise 3: Low Congestion Shortcuts. Given is an $n$-vertex graph $G = (V, E)$ and a collection of vertex-disjoint subsets $S_1, \ldots, S_N$ where $G[S_i]$ (i.e., the induced subgraph) is connected for every $S_i$. Recall that $(\alpha, \beta)$ shortcuts for these sets is collection of subgraphs $H_1, \ldots, H_N$ that satisfies the following: (1) the diameter of each subgraph $G[S_i] \cup H_i$ is at most $\alpha$ and (2) each edge $e \in G$ appears on at most $\beta$ subgraphs. Show that every graph with minimum degree $k$ has $(\alpha, \beta)$ shortcuts with $\alpha = O(n/k)$ and $\beta = 2$. 