Lecture 9: June 06

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Additive Spanners

A β-additive spanner H of an unweighted[1] graph $G=(V,E)$ is a subgraph of G satisfying $\texttt{dist}(u,v,H) \leq \texttt{dist}(u,v,G)+\beta$ for every vertex pair $u,v \in V$. In contrast to multiplicative spanners where for every integer $k \geq 1$, there is a $(2k-1)$-spanner with $O(n^{1+1/k})$ edges, there are only 3 additive spanners for stretch values of $2,4$ and 6 of size $O(n^{3/2}), \widetilde{O}(n^{7/5})$ and $O(n^{4/3})$ respectively [ACIM99, Che13, BKMP10]. Understanding the general size-stretch tradeoff for additive spanners was one of the biggest open problems in the area for the last twenty years. Recently, Abboud and Bodwin have made a quite shocking breakthrough which essentially implies that there are no new additive spanners to be revealed, unless settling for a polynomially large stretch.

Theorem 9.1 *[AB17] There is no $+n^{o(1)}$ additive spanners with $O(n^{4/3-\epsilon})$ edges, for any fixed ϵ.*

2-Additive Spanners. We start by showing a construction of 2-additive spanners with $\widetilde{O}(n^{3/2})$ edges due to Aingworth et al. [ACIM99]. For simplicity, we present a randomized construction, however, it is easy to derandomize it using standard hitting-set tools. In this context, we say that a vertex v is *high-degree* if $\deg(v,G) \geq \sqrt{n}$. Let V_h be the subset of high vertices in G. First, we add to the spanner H all the edges incident to the low-degree vertices. Next, we sample a subset S of $O(\sqrt{n}\log n)$ vertices, by adding each vertex $v \in V$ into S with probability $\Theta(\log n/n)$. We then add to spanner a BFS tree rooted at each $s \in S$. Formally, $H = \bigcup_{s \in S} BFS(s) \cup \bigcup_{v \notin V_h} E(v,G)$. It is easy to see that H has $O(n^{3/2} \cdot \log n)$ edges. We next show that H is a 2-additive spanner.

Lemma 9.2 *W.h.p., for every $u,v \in V$, $\texttt{dist}(u,v,H) \leq \texttt{dist}(u,v,G)+2$.*

Proof: Unlike multiplicative spanners, here it is not sufficient to make the stretch argument only for neighboring pairs $(u,v) \in E$, but rather we need to show it for every pair. Fix $u,v \in V$ and let P be some u-v shortest path in G. If $P \subseteq H$, we are done as $\texttt{dist}(u,v,H) = |P|$. Otherwise, P must include at least one high-degree vertex w (actually, it must have at least two!). By Chernoff bound (a hitting-set argument), w.h.p., w has a neighbor $s \in N(w) \cap S$. Let T be the BFS tree T rooted at s that was added into H and let $P_1 = \pi(u,s,T)$, $P_2 = \pi(s,v,T)$ be the $s-u$ and $s-v$ paths in this tree T. Consider the u-v path $P_3 = P_1 \circ P_2$ obtained by concatenating P_1 and P_2. Since $T \subseteq H$, we have that $P_3 \subseteq H$ and thus by the triangle inequality:

$$\texttt{dist}(u,v,H) \leq |P_3| = |P_1|+|P_2| \leq \texttt{dist}(u,w,G)+1+\texttt{dist}(w,v,G)+1 \leq \texttt{dist}(u,v,G)+2.$$

The lemma follows. ■

4-Additive Spanners. We next show a construction of 4-additive spanners with $\widetilde{O}(n^{7/5})$ due to Chechik [Che13]. We now call vertex v high-degree if $\deg(v,G) \geq n^{2/5}$ and low-degree otherwise. Let V_h be the subset of high-degree vertices. For every u,v, let $\pi(u,v)$ be the unique shortest-path between u and v in G (we break shortest-path ties in a consistent manner).

(1) Add to H all edges incident to low-degree vertices.

(2) Sample a subset S of $\Theta(n^{2/5}\log n)$ vertices, by adding each v into S independently with probability $p_S = \Theta(\log n/n^{3/5})$. Add to H, a BFS tree of each $s \in S$.

[1] Unlike multiplicative spanners, additive spanners are interesting only for the unweighted case.

(3) Sample a subset T of $\Theta(n^{3/5}\log n)$ vertices, by adding each v into T independently with probability $p_T = \Theta(\log n/n^{2/5})$. For each $v \in V_h$, let $s(v) \in N(v) \cap T$ and for every $t \in T$, let $B(t) = \{v \in V_h \mid s(v) = t\}$. Add the edges $\{(v, s(v)) \mid v \in V_h\}$ to H.

(4) For each $t, t' \in T$, define $\Phi_{t,t'}$ by:

$$\Phi_{t,t'} = \{\pi(u,v) \mid u \in B(t), v \in B(t'), \pi(u,v) \text{ has at most } n^{1/5} \text{ high-deg vertices}\}.$$

(5) In each set $\Phi_{t,t'}$ for $t, t' \in T$, pick the shortest path and add it to H.

We next analyze the construction and start with size analysis.

Observation 9.3 (Size) *H has $\widetilde{O}(n^{7/5})$ edges.*

Proof: Steps (1,2,3) clearly add $\widetilde{O}(n^{7/5})$ edges. In Step (5), for each pair t, t', we add $O(n^{1/5})$ edges to H. Since $|T| = O(\log n \cdot n^{3/5})$, overall this adds $O(|T|^2 \cdot n^{1/5}) = \widetilde{O}(n^{7/5})$ edges. ■

Claim 9.4 (Stretch) $\texttt{dist}(u, v, H) \leq \texttt{dist}(u, v, G) + 4$.

Case 1: $\pi(u,v)$ has more than $n^{1/5}$ high-deg vertices. We claim that the number of vertices that have a neighbor on the path $\pi(u,v)$ is $\Omega(n^{3/5})$. Formally, letting $N(\pi(u,v)) = \bigcup_{x\in\pi(u,v)} N(x)$, we will show that $|N(\pi(u,v))| = \Omega(n^{3/5})$. The claim follows by noting that each vertex can have at most 3 neighbors on a given shortest-path, and thus the neighborhood sets of $N(x), N(x')$ for $x, x' \in \pi(u,v)$ are effectively almost vertex-disjoint (i.e., each vertex is counted at most 3 times in these $N(x)$ subsets). By the hitting set argument, we get that there exists $s \in N(\pi(u,v)) \cap S$. Since we added a BFS tree rooted at s to H, we get a stretch of +2 (same argument as in the 2-additive case).

Case 2: $\pi(u,v)$ has at most $n^{1/5}$ high-deg vertices. Let x the closest high-degree vertex to u on $\pi(u,v)$ and let y be the closest high-degree vertex to v on this path. Since $\pi(u,v)$ has at least one edge not in H (otherwise, we are done), we get that $x \neq y$. Let $t = s(x)$ and $t' = s(y)$. By definition, $\pi(x,y) \in \Phi_{t,t'}$. If $\pi(x,y)$ is added in step (5), then $\pi(u,v) \subset H$, and we are done. Otherwise, in Step (5), the algorithm picks the shortest path $\pi(x', y') \in \Phi_{t,t'}$ and adds it to H. We now consider the following $u - v$ path P in H where

$$P = \pi(u,x) \circ (x,t) \circ (t,x') \circ \pi(x',y') \circ (y',t') \circ (t',y) \circ \pi(y,t).$$

Let $\ell = |\pi(x',y')|$, we have:

$$\texttt{dist}(u,v,H) \leq |P| = \texttt{dist}(u,x,G) + 2 + \ell + 2 + \texttt{dist}(y,v,G) \leq \texttt{dist}(u,v,G) + 4 \ ,$$

where the last inequality follows by the fact that $|\pi(x,y)| \geq |\pi(x',y')| = \ell$, by the selection of $\pi(x',y')$ in Step (5). The stretch argument follows.

References

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