

Fault-Tolerant Hotelling Games

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Abstract

The n -player Hotelling game calls for each player to choose a point on the line segment, so as to maximize the size of his Voronoi cell. This paper studies fault-tolerant versions of the Hotelling game. Two fault models are studied: line faults and player faults. The first model assumes that the environment is prone to failure: with some probability, a disconnection occurs at a random point on the line, splitting it into two separate segments and modifying each player's Voronoi cell accordingly. A complete characterization of the Nash equilibria of this variant is provided for every n . Additionally, a one to one correspondence is shown between equilibria of this variant and of the Hotelling game with no faults. The second fault model assumes the players are prone to failure: each player is removed from the game with i.i.d. probability, changing the payoffs of the remaining players accordingly. It is shown that for $n \geq 3$ this variant of the game has no Nash equilibria.

1 Introduction

Background. The Hotelling game, originated in Hotelling's seminal work in 1929 [8], modeled the competition of two servers on a linear market (e.g., two ice-cream vendors on a beach strip) as follows. Two servers choose a location on the line segment $[0, 1]$, and the payoff of each server is equal to the length of the segment of points closer to it than the other server (a.k.a. its Voronoi cell). Hotelling showed that if both servers locate themselves at the center of the line, then a Nash equilibrium would be reached, i.e., a situation in which neither server would rather relocate unilaterally. He next showed that three servers competing on the line reach no equilibrium state - in every configuration of servers there will be one server who can increase its profit by moving.

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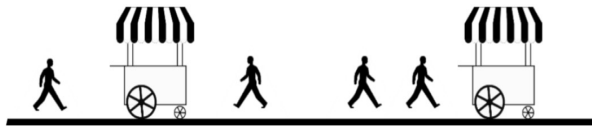


Figure 1: Ice cream vendors on a beach strip.

This initial idea sparked decades of research. Notably, in 1975, Eaton and Lipsey [4] completely characterized all Nash equilibria of the n server game, for every n . Over the years numerous variations were made to each component of the game, including the number of players, the pricing policy, the behavior of clients, and the geometry of the market. Eiselt, Laporte and Thisse [6] provide an annotated bibliography categorized by these features (for a more recent survey see Eiselt et al. [5]).

Motivation. In this paper, we consider the Hotelling game in a failure prone setting, and explore how the Hotelling game behaves differently in a world where faults may occur either in the environment, or in the servers themselves. In day to day life, uncertainty must be accounted for. For instance, one of the players may fail to open their store due to illness or vacation, or the road might be blocked due to infrastructure work or safety issues, denying clients access to their preferred vendor. While it is uncertain whether such an event occurs on a given day or not, it is certain to happen one day. It therefore stands to reason that failures would be accounted for in player strategies and payoffs. Indeed, fault-tolerant problems constitute a fertile area of research in Computer Science. Yet, to the best of our knowledge, this paper is the first to consider fault-tolerance aspects of the Hotelling game.

Contributions. We analyzed two types of failure models. In the first variant, we consider a failure prone *environment*, wherein it is possible that the line would be blocked at a random point, denying the passage of clients through it. We refer to this variant of the game with n players as the *Line Failure Hotelling game* and denote it by $H_{LF}(n)$. We characterize all existing equilibria of $H_{LF}(n)$ (Theorem 3.4). Moreover, we show that each Nash equilibrium of $H_{LF}(n)$ corresponds to a Nash equilibrium of the non-failure Hotelling game $H(n)$ (Theorem 3.2).

The second variant we consider assumes a reliable environment, but failure prone players. Each player (independently) has some probability of being removed from the game. This version of the game (with n players) is referred to as the *Player Failure Hotelling game*, denoted $H_{PF}(n)$. We show that if there are at least three players, then the game admits no Nash equilibrium (Theorem 4.1).

Related Work. Fault tolerant facility location problems have been studied extensively from an optimization perspective in the operations research and computer science communities [1, 2, 9, 10, 11, 12]. However, relatively little work has been done from a game theoretic approach. Two recent papers do consider fault tolerant location games. Wang and Ouyang studied a failure prone competitive location game in a two dimensional environment [13]. However, they studied a variant in which two players position several facilities each, and did not extend the model to a greater number of players. Zhang et al. considered a discrete competitive location model in which there are finitely many clients and finitely many potential facility locations [14]. Their model, while similar to our model in theme, bears little resemblance to Hotelling’s original game.

In 1987, De Palma et al. [3] introduced randomness into the Hotelling game, though in a manner different than we do. Rather than uncertainty in the reliability of the environment or the players, their paper studied uncertainty in client behaviors. That is, clients normally shop at the closest vendor, but with some probability, due to unquantifiable factors of personal taste, they skip a seller and travel further to the next one. This paper has a similar approach to our paper, but considers a different problem than ours.

Aside for location games, other types of fault tolerant games have recently been studied. For example, Gradwohl and Reingold [7] studied the immunity and tolerance of games with many players. Immunity means that faults have a small affect on the utility of non-faulty players. Tolerance means that optimal strategies remain optimal when faults are introduced to the game (even though the utilities may be different from the case without faults). The authors show that the *games themselves* are resilient to faults and quantify the strength of their resilience. We, on the other hand, consider a game that is very sensitive to faults and ask how would the players adapt their strategies in a given fault model.

Organization. The paper is organized as follows. Section 2 presents the Hotelling game formally along with known results. Sections 3 and 4 analyze the Hotelling game with line disconnections and server crashes, respectively. Section 5 concludes the paper and offers future directions of research.

2 Model and Preliminaries

The Game. The Hotelling game on the line segment $[a, b]$, denoted as $H(n, [a, b])$, involves n servers, s_i for $i = 1, \dots, n$, who place themselves at different points x_i on the segment. Each point on the line also represents a client, who will be served by the nearest server. The *market* of each server is the line segment containing the clients that

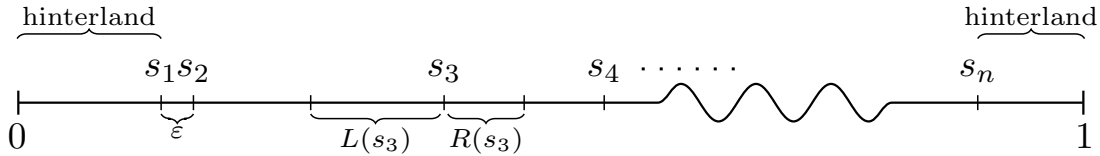


Figure 2: A possible configuration of the game. Servers s_1 and s_n are the peripheral servers, s_1 and s_2 are paired, and the half-markets of s_3 are marked in the figure.

will be served by it (this segment is also known as the server's *Voronoi cell*). The payoff of each server s_i , denoted $p(s_i)$, is the length of his market. Servers strive to maximize their payoff.

We assume clients are uniformly distributed over the line. We also assume that no more than one server can occupy a given location; the minimal distance between two servers is some arbitrarily small $\epsilon > 0$.¹ When two servers are separated by a distance of ϵ they are said to be *paired*. We say two servers s_i and s_j are *paired at location* x if $x_i = x - \epsilon/2$ and $x_j = x + \epsilon/2$; with a slight abuse of notation, we hereafter denote this as $x_i = x_j = x$. Conversely, a server is *isolated* if it is not paired to another server.

Each server s_i divides its market into two sides, referred to as *half-markets*. We denote by $L(s_i)$ and $R(s_i)$ the lengths of s_i 's left and right half-markets respectively. Therefore, the payoff of s_i is

$$p(s_i) = L(s_i) + R(s_i).$$

Two servers are said to be *neighbors* if no server is located between them. A server that has neighbors on both sides is called an *interior server*. A server that has one neighbor is called a *peripheral server*. That is, the two peripheral servers are the server closest to 0 and the server closest to 1. (See Fig. 2.)

By definition, the market of an interior server extends half the distance to its two neighbors. The length of an interior server's market is thus half the distance between its neighbors, wherever the server is located between those neighbors. The line segment between the boundary and the corresponding peripheral server is called a *hinterland*.

¹Setting a minimal distance between servers is common practice in the Hotelling game literature. This prevents servers from infinitely moving closer and closer to each other to slightly improve their payoff. However, the servers choose their location simultaneously and thus two servers might inadvertently choose the same location. We assume that in this case players make small corrections until they meet the constraint. Alternatively, we could say that two servers can be located at the same point and split the payoff in half. However, this leads to unnatural equilibria and thus makes the analysis more complicated. For example, in the two-server game, locating both servers at the same point, anywhere on the line, yields an equilibrium.

The market of a peripheral server includes its hinterland in one direction, and extends half way to its neighbor in the other direction.

The game is in a *Nash equilibrium* if no server can increase its payoff by moving to a location other than its present location.

As an example, consider the basic Hotelling game which is played on the line $[0, 1]$, and involves four servers, s_i for $i = 1, 2, 3, 4$, who need to place themselves at different points on the line, x_i for $i = 1, 2, 3, 4$. Consider a solution in which servers s_1 and s_2 place themselves at the point $1/4$, with s_1 to the left of s_2 , and servers s_3 and s_4 place themselves at the point $3/4$, with s_3 to the left of s_4 . (See Figure 3.) It is easy to verify

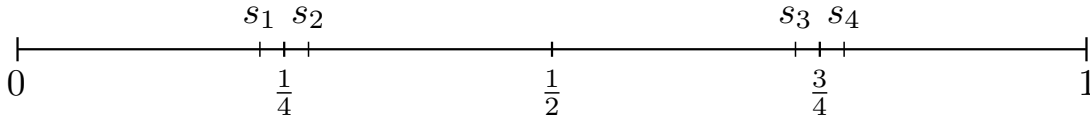


Figure 3: Equilibrium with 4 servers.

that this solution of the game, whose payoff is $1/4$ per server, is a Nash equilibrium (i.e., none of the servers is motivated to change its location).

Known Results. Eaton and Lipsey (1975) [4] proved that the following are necessary and sufficient conditions for an equilibrium:

(EL1) Each peripheral server is paired.

(EL2) No server's whole market is smaller than any other server's half-market.

The proof these conditions are sufficient is a bit involved, but it is easy to see why they are necessary. If (EL1) does not hold, a peripheral server would increase its profit by locating closer to its neighbor. (EL2) follows from the fact that any server can obtain a market equal to any other server's half-market by pairing with it.

By applying these equilibrium conditions, Eaton and Lipsey determined the equilibria of the game depending on the number of servers. We present their results in what follows.

One Server. The payoff of a single server s_1 is $p(s_1) = 1$, wherever it is located.

Two Servers. As shown by Hotelling, there is a unique equilibrium, where the two servers are paired at $x_1 = x_2 = 1/2$, with payoffs $p(s_1) = p(s_2) = 1/2$.

Three Servers. No equilibrium exists. It follows from the fact that (EL1) and (EL2) contradict one another in this case.

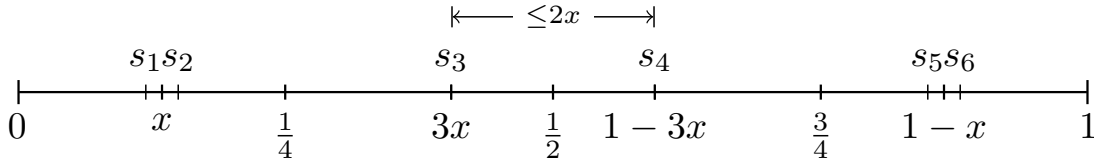


Figure 4: Equilibrium with 6 servers.

Four Servers. There is a unique equilibrium, with equal payoffs of $1/4$, where two servers are paired at $x_1 = x_2 = 1/4$, and the other two at $x_3 = x_4 = 3/4$.

Five Servers. There is a unique equilibrium, where two servers are paired at $x_1 = x_2 = 1/6$, two others are paired at $x_4 = x_5 = 5/6$, and an isolated server is located at $x_3 = 1/2$. Note that here, the payoffs are not uniform: s_3 has payoff $p(s_3) = 1/3$, while all other servers have payoff $1/6$.

More than five Servers. There exist infinitely many equilibria, characterized as follows: peripheral servers are paired with their neighbors and have identical hinterlands. Each peripheral pair is separated from the closest server by a distance twice as long as the hinterland. The interior servers are paired or isolated.

As an example, consider the game $H(6, [0, 1])$. Let the length of the hinterland be x , and without loss of generality let the servers be ordered such that $x_1 < x_2 < \dots < x_6$. From the above characterization of equilibria, it follows that in every equilibrium s_1 and s_2 are paired at $x_1 = x_2 = x$, s_3 is located at $x_3 = 3x$, s_4 is located at $x_4 = 1 - 3x$ and s_5 and s_6 are paired at $x_5 = x_6 = 1 - x$. The distance between s_3 and s_4 is at least ε and at most $2x$ (by condition (EL2)).

That is, the above described configuration is an equilibrium for every x such that

$$\varepsilon \leq (1 - 3x) - 3x \leq 2x ,$$

or rather $1/8 \leq x < 1/6$. (See Figure 4.)

3 Hotelling Game on the Line with Link Failures

Let us now consider the game with environmental failures. For concreteness, let us assume that the only possible failure is a disconnection of the line at point f , chosen uniformly from $[0, 1]$, which severs the line into two separate markets, and forces some of the clients (specifically, those disconnected from their originally chosen server) to change their server selection. For simplicity, assume at most one disconnection may occur, with constant probability $0 < r < 1$, at a location $f \in [0, 1]$ chosen uniformly at random.

We call this game the *Line Failure Hotelling game* and denote it as $H_{\text{LF}}(n, r, [0, 1])$. The new payoff function is denoted as p_{LF} and becomes the expected profit under these assumptions (i.e., the payoff of player s_i is $1 - r$ times its payoff in the fault-free case plus r times its expected payoff in case a disconnection occurred at point $0 < f < 1$).

Let us begin by considering this game with only *one server*. Unlike the basic Hotelling game, in which the location of a single server is inconsequential, in this setting the optimal location of a single server is at the center of the line, $x_1 = 1/2$, as we show next. Let $x_1 \in [0, 1]$ be the location of the server s_1 . If no failure occurs, the payoff is 1. If the line is disconnected at $0 < f < x_1$, then the payoff is $1 - f$. If the line is disconnected at $x_1 < f < 1$, then the payoff is f . It follows that the payoff of s_1 is

$$p_{\text{LF}}(s_1) = \mathbb{E}[p(s_1)] = (1 - r) \cdot 1 + r \cdot \left[\int_0^{x_1} (1 - f)df + \int_{x_1}^1 fdf \right] = 1 - \frac{r}{2} + rx_1 - rx_1^2.$$

That is, $p_{\text{LF}}(s_1)$ is a function of x_1 that attains its maximum at $x_1 = 1/2$. Hence the only equilibrium is when the server is at the center of the market.

We next consider the *two server* variant. Without loss of generality let $x_1 < x_2$, i.e., s_1 is located to the left of s_2 . Observe that the payoff of s_1 is $p_{\text{LF}}(s_1) = L_{\text{LF}}(s_1) + R_{\text{LF}}(s_1)$, where $L_{\text{LF}}(s_1)$ is the length of its hinterland, and $R_{\text{LF}}(s_1)$ is the length of its half-market on the right. (See Fig. 5.)

- **Calculating $L_{\text{LF}}(s_1)$:** if the line is disconnected at location $0 < f < x_1$, then the length of s_1 's hinterland is $L(s_1) = x_1 - f$. Otherwise, $L(s_1) = x_1$. The expected length is therefore

$$L_{\text{LF}}(s_1) = \mathbb{E}[L(s_1)] = (1 - r) \cdot x_1 + r \cdot \left[\int_0^{x_1} (x_1 - f)df + \int_{x_1}^1 x_1df \right] = x_1 - \frac{r \cdot x_1^2}{2}.$$

- **Calculating $R_{\text{LF}}(s_1)$:** if no failure occurs, or if the line is disconnected at location $0 < f < x_1$ or $x_2 < f < 1$, then the length of the half-market is $R(s_1) = (x_2 - x_1)/2$. If, on the other hand, an edge is disconnected at $x_1 < f < x_2$, then $R(s_1) = f - x_1$. Therefore,

$$R_{\text{LF}}(s_1) = \mathbb{E}[R(s_1)] = (1 - r) \cdot \frac{x_2 - x_1}{2} + r \cdot \frac{\int_{x_1}^{x_2} (f - x_1)df}{x_2 - x_1} = \frac{x_2 - x_1}{2}.$$

It follows that the payoff of s_1 is

$$p_{\text{LF}}(s_1) = L_{\text{LF}}(s_1) + R_{\text{LF}}(s_1) = x_1 - \frac{r \cdot x_1^2}{2} + \frac{x_2 - x_1}{2} = \frac{x_2 + x_1}{2} - \frac{r \cdot x_1^2}{2}.$$

This function attains its maximum at $x_1 = 1/(2r)$. That is, as long as s_1 remains on the left of s_2 , s_1 would move to $1/(2r)$. If $1/2 \leq x_2 < 1/(2r)$, then s_1 's best response would be pairing with s_2 .

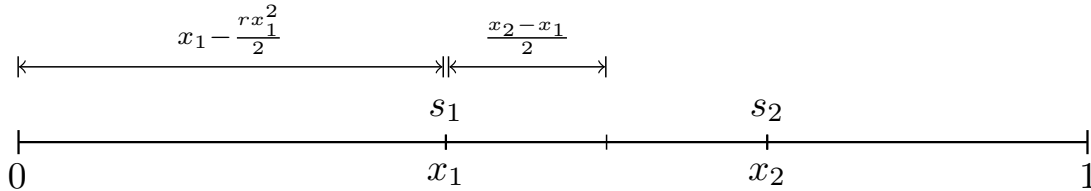


Figure 5: Calculating the expected payoff in the two-player game with link failures.

Note that $1/(2r) > 1/2$ for every $0 < r < 1$ and thus if $x_1 = 1/(2r)$ and $x_2 > 1/(2r)$, then s_2 would prefer to move to the left of s_1 . It follows that $x_2 < 1/(2r)$. By symmetry, it also holds that $x_1 > 1 - 1/(2r)$. Hence, condition (EL1) holds, and as in the basic Hotelling game, the only equilibrium is when both servers are paired at the center of the line, i.e., $x_1 = x_2 = 1/2$. (See Fig. 6.)

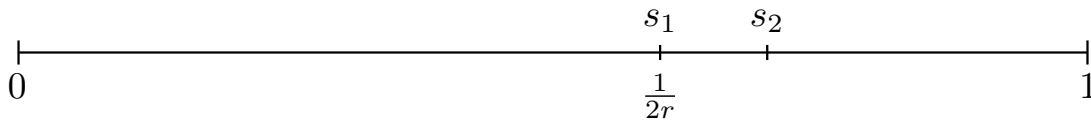


Figure 6: s_1 is located such that its payoff is maximal, and moving closer to s_2 would reduce s_1 's payoff. However, s_2 would maximize its payoff by moving to the left of s_1 .

As a result of the above calculations, for the general game, with $n \geq 3$ servers, we have the following.

Observation 3.1 *In the Hotelling game $H_{LF}(n, r, [0, 1])$, on the line with edge disconnections, the following holds:*

1. *A peripheral server located at distance x from the boundary has a hinterland with an expected length of*

$$x - \frac{r \cdot x^2}{2} .$$

2. *Two neighboring servers at distance x gain a half-market of expected length $x/2$ each in the direction of the other (as in the basic Hotelling game).*
3. *Each peripheral server would increase his hinterland up to $1/(2r)$. That is, condition (EL1) holds unless the neighbor of a peripheral server is at a distance of more than $1/(2r)$ from the boundary.*

In light of the observation above, if the game is played with *three servers*, then no equilibrium exists. To see why, observe that if the interior server s_2 is located between $1 - 1/(2r)$ and $1/(2r)$, then the peripheral servers s_1 and s_3 would pair with s_2 on both sides leaving it with 0 payoff, and thus s_2 would move. If, on the other hand, s_2 is located at $x_2 > 1/(2r)$, then s_1 would locate at $1/(2r)$ and s_3 would pair with s_2 . But then s_3 could improve its payoff and would thus move. Due to symmetry, a similar argument holds if we suppose s_2 is located at $x_2 < 1 - 1/(2r)$.

Next, consider the *four server* game $H_{\text{LF}}(4, r, [0, 1])$. Let us consider the strategy profile where the location $0 < x < 1$ satisfies that when s_1 and s_2 are paired at x and s_3 and s_4 are paired at $1 - x$ all four servers receive the same expected payoff. Since s_1 and s_2 are symmetric to s_4 and s_3 respectively it suffices to compare the expected payoffs of s_1 and s_2 .

The following table compares the payoffs $p(s_1)$ and $p(s_2)$ according to possible values of f , as well as with the failure-free case.

f	$\mathbb{P}[f]$	s_1	s_2
no failure	$1 - r$	x	$1/2 - x$
$[0, x]$	rx	$x/2$	$1/2 - x$
$[x, 1 - x]$	$r(1 - 2x)$	x	$1/2 - x$
$[1 - x, 1]$	rx	x	$1/2 - x$

We now solve $\mathbb{E}[p(s_1)] = \mathbb{E}[p(s_2)]$ for x , namely,

$$(1 - r) \cdot x + rx \cdot \frac{x}{2} + rx \cdot x + r(1 - 2x) \cdot x = \frac{1}{2} - x .$$

This yields

$$rx^2 - 4x + 1 = 0,$$

and recalling that $0 < x < 1$ we obtain

$$x = \frac{2 - \sqrt{4 - r}}{r}.$$

By the definition of x , this means that for every r , when $x_1 = x_2 = (2 - \sqrt{4 - r})/r$ and $x_3 = x_4 = 1 - (2 - \sqrt{4 - r})/r$ (See Fig. 7), each server s_i receives the same expected payoff: $\mathbb{E}[p(s_i)] = 1/2 - x$. We claim that this is a Nash equilibrium, since no server may increase his payoff by moving to another location. This follows from Observation 3.1. As a motivating exercise, in Appendix A we explicitly show that no improving move exists.

Moreover, this Nash equilibrium is unique due to the following considerations. First, by Observation 3.1, the peripheral servers must be paired with their neighbors. Second,

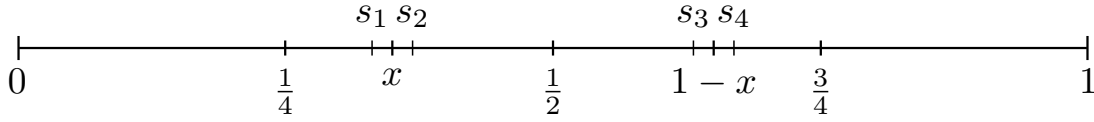


Figure 7: Equilibrium with 4 servers ($x = \frac{2-\sqrt{4-r}}{r}$).

the peripheral servers must have hinterlands of the same length, otherwise one would take the hinterland of the other. Third, every two paired servers must have the same expected payoff, otherwise one would take the half-market of the other. It follows that the configuration above is the only Nash equilibrium.

Applying the considerations above to the game with *five servers* we find that the only equilibrium is when s_1 and s_2 are paired at $x_1 = x_2 = x$, s_3 is isolated at $x_3 = 1/2$, and s_4 and s_5 are paired at $x_4 = x_5 = 1 - x$, such that the distance between s_3 and each peripheral pair is exactly twice $x - r \cdot x^2/2$. (See Fig. 8.)

Hence x satisfies $1/2 - x = 2 \cdot (x - rx^2/2)$, which yields $rx^2 - 3x + 1 = 0$, and recalling that $0 < x < 1$, we obtain Hence x satisfies

$$\frac{1}{2} - x = 2 \cdot \left(x - \frac{r \cdot x^2}{2} \right) ,$$

which yields

$$rx^2 - 3x + 1 = 0 ,$$

and recalling that $0 < x < 1$, we obtain

$$x = \frac{3 - \sqrt{9 - 4r}}{2r} .$$

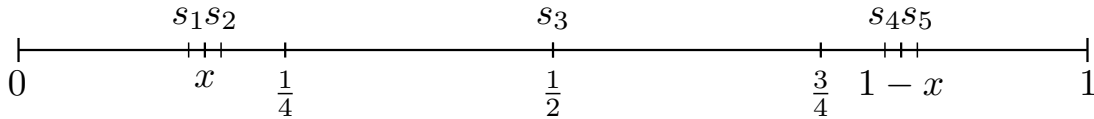


Figure 8: Equilibrium with 5 servers ($x = \frac{3-\sqrt{9-4r}}{2r}$).

We next consider the general n -server game $H_{\text{LF}}(n, r, [0, 1])$. By Obs. 3.1, for any given configuration of servers, adding link failures to the game only affects the expected payoff gained from the hinterlands. Namely, a hinterland of length x shrinks by $rx^2/2$

in expectation, while every other half-market retains its length in expectation. This leads to the following theorem. (Hereafter, proofs are deferred to the full paper.) Let $a = rx_1^2/2$ and $b = 1 - rx_n^2/2$, hence

$$[a, b] = \left[\frac{r \cdot x_1^2}{2}, 1 - \frac{r \cdot (1 - x_n)^2}{2} \right] .$$

Theorem 3.2 *Let $0 \leq x_1 \leq x_2 \leq \dots \leq x_n \leq 1$. The configuration (x_1, x_2, \dots, x_n) is a Nash equilibrium of the game $H_{\text{LF}}(n, r, [0, 1])$ if and only if it is a Nash equilibrium of the game $H(n, [a, b])$.*

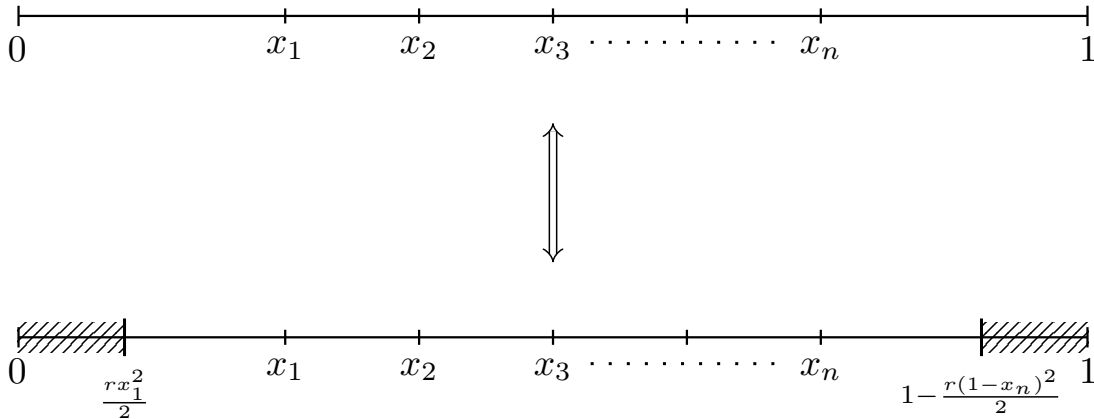


Figure 9: Illustration of Theorem 3.2. The top line refers to the configuration in the game $H_{\text{LF}}(n, r, [0, 1])$ and the bottom line refers to the configuration in the game $H(n, [\frac{rx_1^2}{2}, 1 - \frac{r \cdot (1-x_n)^2}{2}])$.

Proof. Suppose that the servers s_1, s_2, \dots, s_n are located at x_1, x_2, \dots, x_n respectively. Note first that both peripheral servers s_1 and s_n must be paired with their respective neighbors. In the forward direction of the claim this holds by Observation 3.1; in the inverse direction this follows from condition (EL1).

We start by showing that the payoff of each server given the configuration (x_1, x_2, \dots, x_n) is the same in both games $H_{\text{LF}}(n, r, [0, 1])$ and $H(n, [a, b])$, i.e., for each s_i , $p_{\text{LF}}(s_i) = p(s_i)$. Consider an interior server s_i , for $2 \leq i \leq n - 1$. In the game $H_{\text{LF}}(n, r, [0, 1])$, its payoff is

$$p_{\text{LF}}(s_i) = \frac{x_{i+1} - x_{i-1}}{2} ,$$

by Observation 3.1. In the game $H(n, [a, b])$, its payoff is also

$$p(s_i) = \frac{x_{i+1} - x_{i-1}}{2} .$$

Consider the payoff of the peripheral server s_1 . The servers s_1 and s_2 are paired, and thus, by Observation 3.1, the payoff of s_1 in the game $H_{\text{LF}}(n, r, [0, 1])$ is

$$p_{\text{LF}}(s_1) = x_1 - \frac{r \cdot x_1^2}{2}.$$

In the game $H(n, [a, b])$, the left border of the line is at $a = \frac{r \cdot x_1^2}{2}$ and thus the payoff of s_1 is also

$$p(s_1) = x_1 - \frac{r \cdot x_1^2}{2}.$$

Similarly, the server s_n gets the same payoff in both games. This proves that, in the given configuration (x_1, x_2, \dots, x_n) , each server gets the same payoff in both games $H_{\text{LF}}(n, r, [0, 1])$ and $H(n, [a, b])$.

We next show that for each move available to the server s_i , for $1 \leq i \leq n$, in one of the two games $H_{\text{LF}}(n, r, [0, 1])$ and $H(n, [a, b])$, there exists a corresponding move in the other game for which s_i obtains a greater or equal payoff.

Consider first a move within the interval defined by s_i 's present neighbors. If s_i is an interior server then in both games $H_{\text{LF}}(n, r, [0, 1])$ and $H(n, [a, b])$ such a move does not affect s_i 's payoff, as we have previously shown. If s_i is a peripheral server, since it must be paired with its neighbor, such a move reduces s_i 's payoff.

Second, consider a move to an interior interval between two new neighbors s_j and s_{j+1} . In both games, the payoff is equal to half the length of the interval, and thus the move yields the same payoff in both games.

Finally, consider a move to either one of the hinterlands. In both games, the best a server s_i could do is pair with the peripheral server, in which case s_i gets the payoff of this peripheral server. But, as we have shown above, the payoff of each peripheral server is the same in both games $H_{\text{LF}}(n, r, [0, 1])$ and $H(n, [a, b])$, and thus s_i obtains the same payoff by making this move in both game. This proves that for each move s_i may make in one of the games $H_{\text{LF}}(n, r, [0, 1])$ and $H(n, [a, b])$, there exists a move in the other game for which s_i gets greater or equal payoff.

Assume towards contradiction that (x_1, x_2, \dots, x_n) is a Nash equilibrium in the game $H_{\text{LF}}(n, r, [0, 1])$ and but not in the game $H(n, [a, b])$. It follows that there exists a server s_i that can improve his payoff by moving in the game $H(n, [a, b])$. But, as we have shown above, s_i has the same payoff before moving in both games, and by moving s_i can increase its payoff at least as much in the game $H_{\text{LF}}(n, r, [0, 1])$ as in the game $H(n, [a, b])$. Therefore, s_i would also move in the game $H_{\text{LF}}(n, r, [0, 1])$, contradicting the assumption. The inverse direction is similar. This concludes the proof that (x_1, x_2, \dots, x_n) is a Nash equilibrium in the game $H_{\text{LF}}(n, r, [0, 1])$ if and only if it is a Nash equilibrium in the game $H(n, [a, b])$. \square

Theorem 3.2, in conjunction with conditions (EL1) and (EL2) for an equilibrium of the basic Hotelling game, yield the following conditions for an equilibrium of the Hotelling game with line disconnections.

Corollary 3.3 *The following conditions are sufficient and necessary for an equilibrium of the Hotelling game on the line with line disconnections.*

- (1) *The peripheral servers are paired, and are located at an identical distance from the boundary, x .*
- (2) *No interior server's whole market is smaller than $x - r \cdot x^2/2$.*
- (3) *No interior server's half market is larger than $x - r \cdot x^2/2$.*

Proof. Suppose (x_1, x_2, \dots, x_n) , for $x_1 \leq x_2 \leq \dots \leq x_n$, is an equilibrium of $H_{\text{LF}}(n, r, [0, 1])$. Without loss of generality, let the server s_i be located at x_i , for $1 \leq i \leq n$. By Theorem 3.2, it follows that (x_1, x_2, \dots, x_n) is an equilibrium of $H(n, [a, b])$, where $a = rx_1^2/2$ and $b = 1 - rx_n^2/2$ as above. Therefore, conditions (EL1) and (EL2) must hold for this configuration in $H(n, [a, b])$.

By condition (EL1), the servers s_1 and s_2 are paired and the servers s_{n-1} and s_n are paired. By condition (EL2), the hinterlands are equal in length, so

$$x_1 - \frac{r \cdot x_1^2}{2} = (1 - x_n) - \frac{r \cdot (1 - x_n)^2}{2},$$

which yields

$$x_1 = (1 - x_n).$$

This proves condition (1) holds.

By condition (EL2), no interior server's whole market is smaller than any other server's half market, including the hinterland. This proves condition (2) holds.

By condition (EL2), no interior server's half-market is larger than any other server's half market, including the hinterland. This proves condition (3) holds.

Conversely, suppose the configuration (x_1, x_2, \dots, x_n) satisfies conditions (1),(2) and (3), for $x_1 \leq x_2 \leq \dots \leq x_n$, and without loss of generality, the server s_i is located at x_i , for $1 \leq i \leq n$. We show that (x_1, x_2, \dots, x_n) is an equilibrium in the game $H(n, [a, b])$.

Given condition (1), condition (EL1) follows immediately in the game $H(n, [a, b])$. Furthermore, it holds that

$$x_1 = (1 - x_n) = x.$$

By condition (2), the whole market of each interior server s_i , for $2 \leq i \leq n$ is at least $x - r \cdot x^2/2$, and by condition (3) the half market of each interior server s_j is at most

$x - r \cdot x^2/2$, for $2 \leq j \leq n$. By transitivity, it follows that the whole market of s_i is at least the size of the half-market of s_j . This proves that condition (EL2) holds.

It follows that (x_1, x_2, \dots, x_n) is an equilibrium of the game $H(n, [a, b])$, and thus, by Theorem 3.2, (x_1, x_2, \dots, x_n) is an equilibrium of the game $H_{\text{LF}}(n, r, [0, 1])$ as well. This concludes the proof of the Corollary. \square

To summarize, this section established the following theorem.

Theorem 3.4 *In the n -server Hotelling game $H_{\text{LF}}(n, r, [0, 1])$ with a single line disconnection occurring with probability $0 < r < 1$ at a location chosen uniformly at random:*

- (i) *When there is a single server, a Nash equilibrium exists for which the server is located at the center of the line, i.e., $x = 1/2$.*
- (ii) *When two servers compete, there exists an equilibrium for which they are paired at the center of the line, i.e., $x_1 = x_2 = 1/2$.*
- (iii) *If there are three servers, then no equilibrium exists.*
- (iv) *With four servers, there exists an equilibrium for which the servers are organized in two pairs, at*

$$x_1 = x_2 = \frac{2 - \sqrt{4 - r}}{r}, \quad x_3 = x_4 = 1 - \frac{2 - \sqrt{4 - r}}{r}.$$

- (v) *If $n = 5$, then there exists an equilibrium for which four of the servers are paired at*

$$x_1 = x_2 = \frac{3 - \sqrt{9 - 4r}}{2r}, \quad x_4 = x_5 = 1 - \frac{3 - \sqrt{9 - 4r}}{2r},$$

and an isolated server is located at $x_3 = 1/2$.

- (vi) *When more than five servers occupy the line, there exist infinitely many equilibria, characterized as follows: peripheral servers are paired with their neighbor and have identical hinterlands of length x . Each peripheral pair is separated from the closest server by a distance twice as long as $x - r \cdot x^2/2$. The interior servers are paired or isolated such that no server's whole market is smaller than $x - r \cdot x^2/2$, and no server's half-market is larger than $x - r \cdot x^2/2$.*

4 Hotelling Game on the Line with Server Crashes

Let us now consider a different failure setting, where the environment is resilient, but the servers might crash. For concreteness, let us assume that each server might fail with

probability $0 < r < 1$ independently of the others. Once a server has failed, the clients who chose this server originally must change their server selection. We call this game the *Player Failure Hotelling game* and denote it as $H_{\text{PF}}(n, r, [0, 1])$. The new payoff function is again the expected profit under these assumptions and is denoted as p_{PF} .

If a *single server* s_i occupies the line, then it is clear that the server's expected payoff is $p_{\text{PF}}(s_1) = 1 - r$ wherever its location is. If there are *two servers*, then it is easy to see that server crashes have no impact on the game. When one server fails, the other inherits the entire line; when both fail, they both get nothing. Hence, the game is equivalent to the basic Hotelling game and the only Nash equilibrium is when the servers are paired at the center.

Let us now analyze the case where there are *three servers* on the line. In every equilibrium, the peripheral servers are paired with the interior server on both sides, and the interior server is located at $1/2$. This is because when pairing with the interior server, each peripheral server is closest to its neighbor regardless of which servers crash. Moreover, if they are not located at the center, then one peripheral server could improve its payoff by taking the hinterland of the other. Note that, in this state, the expected payoff of the peripheral server s_1 is

$$p_{\text{PF}}(s_1) = \mathbb{E}[p(s_1)] = (1 - r) \left(\frac{1}{2} + \frac{1}{2} \cdot r^2 \right) ,$$

since it always gets its hinterland (provided it did not fail), and it gets the remainder of the line only if both other servers have failed. By a similar case analysis, the expected payoff of the interior server s_2 is

$$p_{\text{PF}}(s_2) = \mathbb{E}[p(s_2)] = (1 - r) \left(0 + \frac{1}{2} \cdot r(1 - r) \cdot 2 + 1 \cdot r^2 \right) ,$$

since (provided it did not fail) it gets nothing if both other servers do not fail, it gets half the line if one of the other servers has failed, and it gets the entire line if both other servers have failed. Note that s_2 can move to take s_1 's hinterland, hence this state would be a Nash equilibrium only if

$$p_{\text{PF}}(s_1) = p_{\text{PF}}(s_2) ,$$

i.e.,

$$(1 - r) \left(\frac{1}{2} + \frac{1}{2} \cdot r^2 \right) = (1 - r) \left(0 + \frac{1}{2} \cdot r(1 - r) \cdot 2 + 1 \cdot r^2 \right) ,$$

which yields

$$r^2 - 2r + 1 = 0,$$

whose only solution is $r = 1$, i.e., all servers crash in every game, which is obviously an equilibrium, but not an interesting one. This proves that no equilibrium exists when there are three servers in the game.

We expand this logic to make a general claim about the game.

Theorem 4.1 *For every $n \geq 3$, the Player Failure Hotelling game $H_{\text{PF}}(n, r, [0, 1])$ has no Nash equilibrium in pure strategies.*

Proof. We have shown the claim holds for $n = 3$ so suppose $n \geq 4$. Suppose there exists a Nash equilibrium in pure strategies for $H_{\text{PF}}(n, r, [0, 1])$, and let the three leftmost servers be s_1 , s_2 and s_3 , ordered from left to right (see Figure 10). Let s_1 's hinterland be of length x_1 , let s_2 be paired with s_1 (otherwise s_1 would move closer to s_2) and let s_3 be located at x_3 .

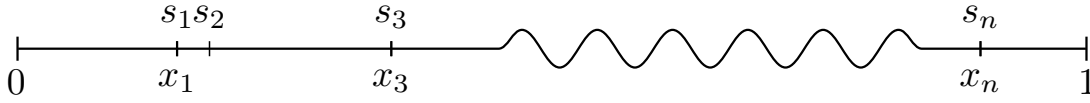


Figure 10: The three leftmost servers in an n -server game.

Let us consider a move of s_2 within the interval between its present neighbors s_1 and s_2 . In all fault scenarios where s_2 remains an interior server, such a move has no effect. In case s_2 either crashes or remains as the only non-faulty server, its location does not matter. But in cases where s_2 is a peripheral server, moving and increasing its hinterland would increase its payoff. However, s_2 is much more likely to become the left peripheral server than the right peripheral server, since the former requires only s_1 to crash while the latter requires $n - 2$ servers to crash. For this reason, s_2 would profit by moving closer to s_3 . A detailed formal analysis follows.

Suppose s_2 pairs with s_3 and consider the following cases:

1. s_2 becomes the left peripheral server and has a non-faulty neighbor s_i . In this case pairing with s_3 increases s_2 's payoff from $x_1 + (x_i - x_1)/2$ to $x_1 + (x_3 - x_1) + (x_i - x_3)/2$. This occurs with probability $r \cdot (1 - r)^2$.
2. s_2 becomes the right peripheral server and s_1 is non-faulty. In this case pairing with s_3 decreases s_2 's payoff from $1 - x_1$ to $1 - x_1 - (x_3 - x_1)/2$. This occurs with probability $r^{n-2} \cdot (1 - r)^2$.
3. *Otherwise*, pairing with s_3 does not change s_2 's payoff, because s_2 either crashed or remained the only non-faulty server or remained an interior server.

In conclusion, when s_2 pairs with s_3 , in case 1 his payoff is increased by $(x_3 - x_1)/2$ with probability $r \cdot (1 - r)^2$, and in case 2 his payoff is decreased by $(x_3 - x_1)/2$ with

probability $r^{n-2} \cdot (1-r)^2$. In cases 3, s_2 's payoff is unchanged. In total, s_2 's payoff increases as a result of this relocation by

$$r \cdot (1-r)^2 \cdot \frac{x_3 - x_1}{2} - r^{n-2} \cdot (1-r)^2 \cdot \frac{x_3 - x_1}{2} = (r - r^{n-2}) \cdot (1-r)^2 \cdot \frac{x_3 - x_1}{2},$$

which is strictly positive for $0 < r < 1$ and $n \geq 4$ as long as $x_3 > x_1$.

Therefore, s_2 must be paired with s_3 . However, the peripheral server s_1 must be paired with s_2 as well. But then s_2 would be paired on both sides and as we have seen in the three server game this cannot be an equilibrium. Namely, s_2 would pair with s_1 on the left or with s_3 on the right. We thus reach a contradiction, proving that no equilibrium exists if there are more than three servers in the game. \square

5 Conclusion

In this paper we considered two fault tolerant variants of the Hotelling game: link failures and player faults. On the one hand, we have shown that the game is resilient to link failures in some sense - each equilibrium is related to an equilibrium of the no-faults Hotelling game by rescaling the interval along with player positions. On the other hand, we have shown that the game is vulnerable to player failures. No equilibrium exists because players tend to converge towards the center. Table 1 compares the numbers of Nash equilibria in each model

Table 1: The number of Nash equilibria that exist in each variant of the game depending on the number of players.

n	Server Crashes	Line Disconnect	No Faults
1	∞	1	∞
2	1	1	1
3	0	0	0
4	0	1	1
5	0	1	1
≥ 6	0	∞	∞

There are many possible future directions for this research. A large number of variants of the Hotelling game have been studied and each would be interesting to consider in a faulty setting, such as: the Hotelling game on graphs, on the plane or over \mathbb{R}^n , the Hotelling game with sequential entry, and so on. Another interesting direction would be to try other fault models. Some examples are models where the number of faulty players is bounded, where faulty players remain in the game but act unexpectedly (or ‘‘Byzantinely’’), or where faults are injected adversarially rather than at random.

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Appendix

A Nash Equilibrium of the 4-Server Game with Link Failures

Consider the solution presented in Figure 7, and suppose that one of the servers, denoted s^* , moves to location y between 0 and x . The identity of s^* does not change his payoff since one of the other players will be located at x .

The following table compares the payoff of s^* to the payoff of s_1 in the original state (wlog, because all servers have the same expected payoff).

f	$\mathbb{P}[f]$	s_1	s^*
—	$1 - r$	x	$(x + y)/2$
$[0, y]$	ry	$x - y/2$	$x/2$
$[y, x]$	$r(x - y)$	$(x - y)/2$	$(x + y)/2$
$[x, 1]$	$r(1 - x)$	x	$(x + y)/2$

We claim that $\mathbb{E}[p(s_1)] \geq \mathbb{E}[p(s^*)]$, and thus obtain the following inequality.

$$(1 - r) \cdot x + ry \cdot \left(x - \frac{y}{2}\right) + r(x - y) \cdot \frac{x - y}{2} + r(1 - x) \cdot x \geq (1 - r) \cdot \frac{x + y}{2} + ry \cdot \frac{x}{2} + r(x - y) \cdot \frac{x + y}{2} + r(1 - x) \cdot \frac{x + y}{2}.$$

Shifting all terms to the left hand side and simplifying yields

$$(1 - r) \cdot \left(\frac{x - y}{2}\right) + r \cdot \frac{x - y}{2} \cdot (1 - y - x) \geq 0.$$

Recall that $0 < r < 1$ and $0 \leq y < x < 1/2$, therefore all terms on the left hand side are non-negative and the inequality holds. Hence, no server would gain from moving between 0 and x .

It is left to show that no server would gain by moving between x and $1/2$, since the right half of the segment is symmetrical.

Suppose now server s^* relocates at y between x and $1/2$, we then obtain the following table.

f	$\mathbb{P}[f]$	s_2	s^*
—	$1 - r$	$1/2 - x$	$1/2 - x$
$[0, x]$	rx	$1/2 - x$	$1/2 - x$
$[x, y]$	$r(y - x)$	$(y - x)/2$	$1/2 - x$
$[y, 1 - x]$	$r(1 - x - y)$	$(1 + y - 3x)/2$	$1/2 - x$
$[1 - x, 1]$	rx	$1/2 - x$	$1/2 - x$

We conclude that any server moving between x and $1/2$ would have an expected payoff of $1/2 - x$, which does not improve his payoff (rather, his payoff remains unchanged).

In summary, when the servers s_1, s_2, s_3 and s_4 are located at $x_1 = x_2 = (2 - \sqrt{4 - r})/r$ and $x_3 = x_4 = 1 - (2 - \sqrt{4 - r})/r$ respectively (see Figure 7), the system is at a Nash equilibrium. No server may improve his expected payoff by changing his location (assuming the other servers stay in the same place).