A central question in the field of optimal transport studies optimization problems involving two measures on a common metric space, a source and a target. The goal is to find a mapping from the source to the target, in a way that minimizes distances. A remarkable fact discovered by Caffarelli is that, in some specific cases of interest, the optimal transport maps on a Euclidean metric space are Lipschitz. Lipschitz regularity is a desirable property because it allows for the transfer of analytic properties between measures. This perspective has proven to be widely influential, with applications extending beyond the field of optimal transport. In this talk, we will further explore the Lipschitz properties of transport maps. Our main observation is that, when one seeks Lipschitz mappings, the optimality conditions mentioned above do not play a major role. Instead of minimizing distances, we will consider a general construction of transport maps based on interpolation of measures, and introduce a set of techniques to analyze the Lipschitz constant of this construction. In particular, we will go beyond the Euclidean setting and consider Riemannian manifolds as well as infinite-dimensional spaces. Some applications, such as functional inequalities and normal approximations will also be discussed.