A number theoretic characterization of (FRS) morphisms: uniform estimates over finite rings of the form $\mathbb{Z}/p^k\mathbb{Z}$.

Abstract:

Let $f: X \to Y$ be a morphism between smooth algebraic varieties defined over the integers. We show its fibers satisfy an extension of the Lang-Weil bounds with respect to finite rings of the form $\mathbb{Z}/p^k\mathbb{Z}$ uniformly in $p$, $k$ and in the base point $y$ if and only if $f$ is flat and its fibers have rational singularities, a property abbreviated as (FRS). This characterization of (FRS) morphisms serves as a joint refinement of two results of Aizenbud and Avni; namely a similar characterization in the case of a single variety, and a characterization of (FRS) morphisms which is non-uniform in the prime $p$. Aizenbud and Avni's argument in the case of a variety breaks in the relative case due to bad behaviour of resolution of singularities in families with respect to taking points over $\mathbb{Z}$ and $\mathbb{Z}/p^k\mathbb{Z}$. To bypass this, we prove a key model theoretic statement on a certain satisfactory class of positive functions (formally non-negative motivic functions), which allows us to efficiently approximate suprema of such functions. Based on arXiv:2103.00282, joint with Raf Cluckers and Itay Glazer.