EXPONENTIAL CHI SQUARED DISTRIBUTIONS IN INFINITE ERGODIC THEORY

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Abstract. We prove distributional limit theorems for random walk adic transformations obtaining ergodic distributional limits of exponential chi squared form.

§0 Introduction

As in [A1], for $(X,\mathcal{B},m)$ a σ-finite measure space, $F_n : X \to [0,\infty]$ measurable, and $Y \in [0,\infty]$ a random variable, we say that (F_n) *converges strongly in distribution to* Y, (written $F_n \xrightarrow[n\to\infty]{\mathfrak{d}} Y$,) if it converges in law with respect to all m-absolutely continuous probabilities; equivalently

$$g(F_n) \underset{n\to\infty}{\longrightarrow} \mathbb{E}(g(Y)) \text{ weak } * \text{ in } L^\infty(m)$$

for each bounded, continuous function $f : [0,\infty] \to \mathbb{R}$.

For discussion of strong distributional convergence, see [A1], [A2], [E] and [TZ].

Here, we study `distributional stability`. As in [A2], we'll call a conservative, ergodic measure preserving transformations $(X,\mathcal{B},m,T)$ *distributionally stable* if there are constants $a(n) > 0$ and a random variable Y on $(0,\infty)$ so that

$$\tfrac{1}{a(n)} S_n(f) \xrightarrow{\mathfrak{d}} Y m(f) \;\forall\; f \in L^1_+$$

where $S_n(f) := \sum_{k=0}^{n-1} f \circ T^k$ and $m(f) := \int_X f dm$.

By the ratio ergodic theorem, if the above convergence holds for some $f \in L^1_+$, then it holds $\forall\; f \in L^1_+$.

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If the *ergodic distributional limit* Y is integrable and normalized by $\mathbb{E}(Y) = 1$, the constants $a(n)$ are determined uniquely up to asymptotic equality and are also known as the *return sequence* of T. Both the (normalized) ergodic distributional limit and the return sequence are invariant under similarity (see [A1]).

By the Darling-Kac theorem ([DK]), pointwise dual ergodic transformations (e.g. Markov shifts) with regularly varying return sequences are distributionally stable with Mittag-Leffler ergodic distributional limits (see also [A1], [A2]).

Our present study concerns `random walk adic transformations`.

A *random walk adic transformation* is a conservative, ergodic measure preserving transformation associated to a Markov driven, aperiodic, random walk on a group of form $\mathbb{G} = \mathbb{Z}^k \times \mathbb{R}^{D-k}$. These were first considered in [HIK] (and are sometimes known as "generalized HIK transformations").

It is the (unique) $\mathbb{G}$-extension of the adic transformation on the underlying Markov shift which parametrizes the tail relation of the random walk. This definition is explained in §1.

The *degree* of a random walk adic transformation is the dimension of the associated group $\mathbb{G}$: $\dim(\mathtt{span}\,\mathbb{G})$. It appears as the number of degrees of freedom in the χ^2 distribution appearing in the limit.

We establish the following

Theorem

Suppose that $(X, \mathcal{B}, m, T)$ *is a random walk adic transformation with degree* $D \in \mathbb{N}$*, then*

$$\tfrac{1}{a_n(T)} S_n(f) \xrightarrow{\mathfrak{d}} (2^{\frac{D}{2}} e^{-\frac{1}{2}\chi_D^2})\mu(f) \;\; \forall \;\; f \in L^1_+$$

where $S_n(f) := \sum_{k=0}^{n-1} f \circ T^k,\ a_n(T) \propto \frac{n}{(\log n)^{D/2}}$*; and* $\chi_D^2 = \|\xi\|_2^2$ *for* ξ *a standard Gaussian random vector on* $\mathbb{R}^D$*.*

Here, for $a(n),\ b(n) > 0$, $a(n) \propto b(n)$ means $\exists\ \lim_{n\to\infty} \frac{a(n)}{b(n)} \in (0,\infty)$.

This ergodic distributional limit first appears in [LS] (see below).

Most of the paper is devoted to proving the theorem. In §1, we define adic transformations and random walk adic transformations as group extensions of adic transformations. In §2, we establish compact representation for (all) adic transformations and a uniform convergence theorem for stationary adic transformations which latter is needed in the proof of the theorem. We review the distributional limit theory of

Markov shifts in §3 and prove the theorem in §4, giving applications to exchangeability in §5 and horocycle flows in §6.

Related, earlier work can be found in [AW], [ANSS] and [LS] (see §6).

§1 Bratteli diagrams, adic and random walk adic transformations

Bratteli diagrams.
Fix $b_n \geq 2 \ \ (n \geq 1)$ and set $\mathcal{S}_k := \{0, 1, \dots, b_k - 1\}, \ \Omega := \prod_{k=1}^{\infty} \mathcal{S}_k$.
A *Bratteli diagram* is a subset $\Sigma \subset \Omega$ of form

$$\Sigma := \{\omega \in \Omega : \quad A_k(\omega_k, \omega_{k+1}) = 1 \quad \forall \ k \geq 1\}$$

where for $k \geq 1$, $A_k : \mathcal{S}_k \times \mathcal{S}_{k+1} \to \{0, 1\}$ is the k^{th} *transition matrix.*
We assume throughout that the transition matrices are *non-degenerate* in that they have no zero rows or columns.

Recall that Ω is compact when equipped with the standard metric $d(x, y) := \exp[-\min\{n : x_n \neq y_n\}]$ and Σ is a closed subset.
The Bratteli diagram Σ is called *stationary* if $\mathcal{S}_k = \mathcal{S}, \ A_k = A \ \forall \ k \geq 1$. In this case, Σ is a *topological Markov shift* (TMS) with *transtition matrix* A as in [LM].
The only result in this paper concerning non-stationary Bratteli diagrams is the compact representation lemma in §2.

Tail relation on a Bratteli diagram.
The *tail relation* on Σ is the equivalence relation

$$\mathfrak{T} = \mathfrak{T}(\Sigma) := \{(x, y) \in \Sigma \times \Sigma : \ \exists n \ \text{ such that } x_n^{\infty} = y_n^{\infty}\}$$

where $x_n^{\infty} := (x_n, x_{n+1}, \dots)$.
The equivalence classes of the tail relation are linearly ordered by the *reverse lexicographic order*, namely the partial order $\prec$ on Σ defined by

$$x \prec y \Leftrightarrow \exists n \text{ s.t. } x_{n+1}^{\infty} = y_{n+1}^{\infty} \text{ and } x_n < y_n.$$

If x is maximal, then $x_n = \max\{y \in \mathcal{S}_n : A_n(y, x_{n+1}) = 1\} \ \forall \ n \geq 1$; therefore the collection of non-maximal points is open and the collection $\Sigma_{\mathtt{max}}$ of maximal points is closed. A similar argument shows that the set $\Sigma_{\mathtt{min}}$ of minimal points is closed.
In case Σ is a TMS (stationary Bratteli diagram), more is true.

If x is maximal, then $x_n = \varphi_+(x_{n+1})$ where $\varphi_+(x) = \max\{y \in \mathcal{S} : A(y,x) = 1\}$ (well defined by non-degeneracy); and we claim that

(🚂) $$x \textit{ is periodic, with period } \le \#\mathcal{S}.$$

To see (🚂), note first that $\exists\ s \in \mathcal{S}$ so that $\#\{n \ge 1 :\ x_n = s\} = \infty$. The sequence $n \mapsto \varphi_+^n(s)$ is eventually periodic with a final period

$$(t, \varphi_+(t) \dots, \varphi_+^{\kappa-1}(t)) = (\varphi_+^J(s), \dots, \varphi_+^{J+\kappa-1}(s))$$

with $J \ge 1$ & $\kappa \le \#\mathcal{S}$.

Let $p := (\varphi_+^{\kappa-1}(t), \varphi_+^{\kappa-2}(t), \dots, t, \varphi_+^{\kappa-1}(t), \varphi_+^{\kappa-2}(t), \dots, t, \dots)$. We prove our claim by showing that $x = \sigma^\ell(p)$ for some $\ell \ge 1$. Let $n_k \uparrow \infty$ be so that $x_{n_k} = s\ \ \forall\ k \ge 1$. It follows that $x_{n_k-\nu} = \varphi_+^\nu(s)$ whence $\exists\ 1 \le \ell_k \le \kappa$ so that $x_1^{n_k-J} = \sigma^{\ell_k}(p)_1^{n_k-J}$. There is a subsequence $m_j = n_{k_j} \to \infty$ so that $\ell_{k_j} = \ell_{k_1} =: \ell\ \forall\ j \ge 1$ whence $x = \sigma^\ell(p)$ and (🚂) is established.

Thus the set of maximal points (and the set of minimal points) is finite.

Adic transformations.

The *adic transformation* (generated by $\prec$) on the Bratteli diagram Σ is

$$\tau : \Sigma \smallsetminus \Sigma_{\max} \to \Sigma \smallsetminus \Sigma_{\min} \texttt{ defined by } \tau(x) := \min\{y : y \succ x\}.$$

It is called *stationary* if the underlying Bratteli diagram is stationary.

As shown in [V], every ergodic, probability preserving transformation is isomorphic to some adic transformation.

Stationary adic transformations are

- isomorphic to odometers or primitive substitutions, have zero entropy but can be weakly mixing (see [L1]); and
- are always *uniquely ergodic*; moreover the unique τ–invariant probability measure ν_0 is globally supported, non-atomic, Markov and equivalent to the Parry measure μ (of maximal entropy) for the associated TMS (see [BM]).

The (Σ, f)-random walk.

Let Σ be a topologically mixing TMS on the (ordered) finite state space $\mathcal{S}$, let $\sigma : \Sigma \to \Sigma$ be the shift and let $\tau : \Sigma' \to \Sigma'$ be the corresponding stationary adic transformation where

$$\Sigma' := \Sigma \smallsetminus \bigcup_{n\ge0} \sigma^{-n}(\Sigma_{\texttt{max}} \cup \Sigma_{\texttt{min}}) = \bigcap_{n\in\mathbb{Z}} \tau^n(\Sigma \smallsetminus (\Sigma_{\texttt{max}} \cup \Sigma_{\texttt{min}})).$$

The tail relation of Σ is the tail relation of σ:

$$\mathfrak{T}(\Sigma) = \mathfrak{T}(\sigma) := \bigcup_{n\ge0} \{(x,y) \in \Sigma \times \Sigma :\ \sigma^n(x) = \sigma^n(y)\};$$

and this is parametrized by the adic transformation:

$$\mathfrak{T}(\sigma) \cap (\Sigma' \times \Sigma') = \{(x, \tau^n(x)) :\ x \in \Sigma',\ n \in \mathbb{Z}\}.$$

A function $f : \Sigma \to \mathbb{R}^d$ is *Hölder continuous* if $\exists\ \theta \in (0,1),\ M > 1$ so that

$$(\text{✔}) \qquad \|f(x) - f(y)\| \le M\theta^n \quad \forall\ x, y \in \Sigma,\ x_n^\infty = y_n^\infty.$$

Specifically for we call $f : \Sigma \to \mathbb{R}^d$ *θ-Hölder continuous* ($\theta \in (0,1)$) if (✔) is satisfied for some $M > 1$.

For $f : \Sigma \to \mathbb{R}^d$ Hölder continuous let

$$\mathbb{H} := \overline{\langle \{f_n(x) :\ n \ge 1,\ x \in \Sigma,\ \sigma^n x = x\} \rangle}$$

where $\langle \cdot \rangle$ means "the group generated by" and

$$f_n(x) = f_n^{(\sigma)}(x) := \sum_{k=0}^{n-1} f(\sigma^k x).$$

Let

$$\mathbb{G} := \overline{\langle \{f_n(x) - f_n(y) :\ n \ge 1,\ x,\ y \in \Sigma,\ \sigma^n x = x,\ \sigma^n y = y\} \rangle},$$

then $\mathbb{G}$, $\mathbb{H}$ are both closed subgroups of $\mathbb{R}^d$ and $\mathbb{G} \le \mathbb{H}$.

It follows from Livšic's cohomology theorem [L2], (see e.g. [ANS], [SA], [PS]) that

$$f = g - g \circ \sigma + h + a \text{ where}$$

- $g : \Sigma \to \mathbb{R}^d$ is Hölder continuous;
- $a \in \mathbb{H}$ is such that $\langle \mathbb{G} + a \rangle = \mathbb{H}$;
- $h : \Sigma \to \mathbb{G}$ is Hölder continuous and σ-*aperiodic* in the sense that if $\gamma \in \widehat{\mathbb{G}},\ \lambda \in \mathbb{S}^1,\ g : \Sigma \to \mathbb{S}^1$ Hölder continuous and $\gamma \circ f = \lambda \frac{g \circ \sigma}{g}$, then $\gamma \equiv 1$ $\lambda = 1$ and g is constant.

It follows that $\dim(\mathbb{G}) = \dim(\mathbb{H}) =: D$ where for $A \subset \mathbb{R}^d$, $\dim(A)$ denotes the dimension of the closed linear subspace spanned by A.

Any closed subgroup $\mathbb{G} \le \mathbb{R}^D$ with $\dim(\mathbb{G}) = D$ is conjugate by linear map to a group of form $\mathbb{Z}^k \times \mathbb{R}^{D-k}$ where $0 \le k \le D := \dim(\mathbb{G})$ and $\mathbb{Z}^0,\ \mathbb{R}^0 := \{0\}$.

Now suppose that $f : \Sigma \to \mathbb{G} = \mathbb{Z}^k \times \mathbb{R}^{D-k}$ is Hölder continuous and σ-aperiodic and consider the (Σ, f)-*random walk* $(\Sigma \times \mathbb{G}, \mathcal{B}(\Sigma \times \mathbb{G}), \widetilde{m}, \sigma_f)$ where $\sigma_f : \Sigma \times \mathbb{G} \to \Sigma \times \mathbb{G}$ is defined by

$$\sigma_f(x, y) := (\sigma(x), y + f(x)) \ \&\ \ d\widetilde{m}(x, y) := d\mu(x) dy$$

where μ is the σ-invariant Parry measure (with maximal entropy) and dy is Haar measure on $\mathbb{G}$.

We note for future reference that $\sigma_f^n(x,y) = (\sigma^n(x), y + f_n(x))$.

As shown in [G], by the aperiodicity of f, $(\Sigma \times \mathbb{G}, \sigma_f, \widetilde{m})$ is exact.

Random walk adic transformation over (Σ, f, τ).

The *random walk adic transformation over* (Σ, f, τ) is that skew product over τ which parametrizes the tail $\mathfrak{T}(\sigma_f)$ of the (Σ, f)-random walk. To see existence and uniqueness of such:

$$\begin{aligned}\mathfrak{T}(\sigma_f) &:= \bigcup_{n\geq 0}\{((x,y),(x',y')) \in (\Sigma \times \mathbb{G})^2 : \ \sigma_f^n(x',y') = \sigma_f^n(x,y)\} \\ &= \bigcup_{n\geq 0}\{((x,y),(x',y')) \in (\Sigma \times \mathbb{G})^2 : \ \sigma^n(x') = \sigma^n(x) \ \& \ y' + f_n(x') = y + f_n(x)\} \\ &= \{((x,y),(x',y')) \in (\Sigma \times \mathbb{G})^2 : \ (x,x') \in \mathfrak{T}(\sigma) \ \& \ y' = y + \psi(x,x')\}\end{aligned}$$

where

$$\psi(x,x') := \sum_{k=0}^{\infty}(f(\sigma^k x') - f(\sigma^k x)).$$

Thus

$$\begin{aligned}\mathfrak{T}(\sigma_f) \cap (\Sigma' \times \mathbb{G})^2 &= \{((x,y),(x',y')) \in (\Sigma' \times \mathbb{G})^2 : \ \sigma_f^n(x',y') = \sigma_f^n(x,y)\} \\ &= \{((x,y),T^n(x,y)) : \ (x,y) \in (\Sigma' \times \mathbb{G})^2, \ n \in \mathbb{Z}\}\end{aligned}$$

holds for $T : \Sigma' \times \mathbb{G} \to \Sigma' \times \mathbb{G}$ of form

$$T(x,y) = \tau_\phi(x,y) = (\tau(x), y + \phi(x))$$

if and only if

$$\phi(x) := \psi(x,\tau(x)) = \sum_{k=0}^{\infty}(f(\sigma^k \tau x) - f(\sigma^k x)).$$

We consider T with the invariant measure

$$dm(x,y) := d\nu(x)dy$$

where $\nu \in \mathcal{P}(\Sigma)$ is the τ–invariant Markov measure (equivalent to the Parry measure μ) and dy is Haar measure on $\mathbb{G}$.

As mentioned above, it was shown in [G] that $(\Sigma \times \mathbb{G}, \sigma_f, \widetilde{m})$ is exact, whence $(\Sigma \times \mathbb{G}, T, m)$ is ergodic and also conservative (being an invertible, ergodic, measure preserving transformation of a non-atomic space).

The *degree* of the random walk adic transformation $(\Sigma \times \mathbb{G}, T, m)$ is

$$\mathtt{deg}\,(T) := \dim\,(\mathrm{span}\,\mathbb{G}).$$

In this paper we ignore the other invariant measures for T (which are considered in [ANSS]).

§2 Uniform Convergence

Uniform Convergence Lemma

Let Σ be a mixing TMS and let $\tau : \Sigma' \to \Sigma'$ be the associated stationary adic transformation with τ–invariant Borel probability measure $\nu_0 \in \mathcal{P}(\Sigma)$, then

$$\frac{1}{n}\sum_{k=0}^{n-1} F \circ \tau^k \xrightarrow[n\to\infty]{} \int_{\Sigma'} F d\nu_0 \quad \textit{uniformly on } \Sigma' \quad \forall \quad F \in C(\Sigma).$$

Although τ is a uniquely ergodic homeomorphism on Σ', this space is not compact.

The main part of the proof is to provide a suitable continuous transformation of a related compact space. This latter construction is made for any adic transformation.

Compact Representation Lemma

Let Σ be a Bratteli diagram and let τ be the associated adic transformation. There are:

- *a compact metric space $(\widehat{\Sigma}, \widehat{d})$;*
- *a continuous injection $\pi : \Sigma \setminus \Sigma_{\max} \to \widehat{\Sigma}$ and*
- *continuous surjections $\varpi : \widehat{\Sigma} \to \Sigma$, $\widehat{\tau} : \widehat{\Sigma} \to \widehat{\Sigma}$ so that*

$$\varpi \circ \pi = \mathrm{Id}|_{\Sigma\setminus\Sigma_{\max}}, \;\; \pi \circ \tau = \widehat{\tau} \circ \pi \;\&\; \varpi \circ \widehat{\tau} = \tau \circ \varpi.$$

Proof. For $\omega \in \Sigma_{\max}$, set

$$A(\omega) := \{\alpha \in \Sigma : \; \exists \text{ sequence } x^{(n)} \in \Sigma\setminus\Sigma_{\max} \text{ s.t. } x^{(n)} \to \omega \text{ and } \tau(x^{(n)}) \to \alpha\},$$

then $A(\omega) \subset \Sigma_{\min}$.

Let

$$\Sigma_0 := \Sigma \setminus \Sigma_{\max} \;\;\&\;\; \Sigma_1 := \bigcup_{\omega\in\Sigma_{\max}} \{\omega\} \times A(\omega).$$

Define the metric space $(\widehat{\Sigma}, \widehat{d})$ by

$$\widehat{\Sigma} := \Sigma_0 \uplus \Sigma_1; \quad \widehat{d}|_{\Sigma_0\times\Sigma_0} \equiv d \;\&\; \text{for } (\omega,\alpha) \in \Sigma_1 :$$

$$\widehat{d}((\omega,\alpha), z) = \begin{cases} d(\omega, z) + d(\alpha, \tau z) & z \in \Sigma_0; \\ d(\omega,\omega') + d(\alpha,\alpha') & z = (\omega',\alpha') \in \Sigma_1. \end{cases}$$

To see that this is a compact metric space, let $(z_n)_{n\geq 1}$ be a sequence in $\widehat{\Sigma}$, then:

- If $\exists\ n_k \to \infty,\ z_{n_k} = (\omega_k, \alpha_k) \in \Sigma_1$, then (possibly passing to a subsequence) we may assume that $(\omega_k, \alpha_k) \to (\omega, \alpha) \in \Sigma_{\max} \times \Sigma_{\min}$. To see that $(\omega, \alpha) \in \Sigma_1$, for each $k \ge 1$, choose $x_k \in \Sigma_0$ so that $d(x_k, \omega_k) + d(\tau(x_k), \alpha_k) \to 0$. It follows that $(\omega, \alpha) \in \Sigma_1$ and $(\omega_k, \alpha_k) \to (\omega, \alpha)$ in $\widehat{\Sigma}$.
- Otherwise $\exists\ n_k \to \infty,\ z_{n_k} \in \Sigma_0,\ z_{n_k} \to r \in \Sigma$ and
  - if $r \in \Sigma_0$, then $z_{n_k} \to r$ in $\widehat{\Sigma}$;
  - if $r \in \Sigma_1$, then $\exists\ m_\ell = n_{k_\ell} \to \infty$ so that $\tau(z_{n_k}) \to s \in \Sigma$; whence $(r, s) \in \Sigma_1$ and $z_{m_\ell} \to (r, s)$ in $\widehat{\Sigma}$.

Let $\pi : \Sigma_0 \to \widehat{\Sigma}$ be the identity map. The following map is a continuous left inverse: $\varpi : \widehat{\Sigma} \to \Sigma$ defined by the identity map on Σ_0 and by $(\omega, \alpha) \mapsto \omega$ on Σ_1.

Now define $\widehat{\tau} : \widehat{\Sigma} \to \widehat{\Sigma}$ by $\widehat{\tau}(x) := \tau(x)$ for $x \in \Sigma_0$ and $\widehat{\tau}(\omega, \alpha) = \alpha$ for $(\omega, \alpha) \in \Sigma_1$, then $\widehat{\tau}$ is continuous and $\pi \circ \tau = \widehat{\tau} \circ \pi$.

To see that $\widehat{\tau}$ is onto, it suffices to show that $\widehat{\tau}(\widehat{\Sigma} \supset \Sigma_{\min}$. To this end, fix $\alpha \in \Sigma_{\min}$, then $\exists\ x_n \in \Sigma \setminus \Sigma_{\min},\ x_n \to \alpha$. Without loss of generality, $x_n \notin \Sigma_{\max}$ and so $x_n = \tau(y_n)$ for some $y_n \in \Sigma$ where $y_n \to \omega \in \Sigma$. It follows that $\omega \in \Sigma_{\max}$ (else $\alpha = \tau(\omega) \notin \Sigma_{\min}$) whence $(\omega, \alpha) \in \widehat{\Sigma}$ and ϖ is onto. □

Proof of the Uniform Convergence Lemma

Since $\nu_0 \in \mathcal{P}(\Sigma')$, it lifts to a $\widehat{\tau}$–invariant measure $\nu_1 \in \mathcal{P}(\widehat{\Sigma})$: $\nu_1 \circ \pi^{-1} = \nu_0$. We claim that $\widehat{\tau}$ is uniquely ergodic on $\widehat{\Sigma}$ with invariant measure ν_1. Else $\exists\ \nu_1 \ne \nu_2 \in \mathcal{P}(\widehat{\Sigma})$ with $\nu_2 \circ \widehat{\tau}^{-1} = \nu_2$. This entails $\nu_2 \circ \varpi^{-1} = \nu_0$ whence $\nu_2 = \nu_1$.

It follows that

$$\Delta_n(F) := \sup_{\widehat{\Sigma}} \left| \frac{1}{n} \sum_{k=0}^{n-1} F \circ \widehat{\tau}^k - \int_{\widehat{\Sigma}} F d\nu_1 \right| \xrightarrow[n\to\infty]{} 0\ \forall\ F \in C(\widehat{\Sigma}).$$

If $f \in C(\Sigma)$, then $f \circ \varpi \in C(\widehat{\Sigma})$ and for every $\omega \in \Sigma'$ and $k \ge 1$, we have $\tau^k \omega \in \Sigma'$ whence $f(\tau^k \omega) = f \circ \varpi(\widehat{\tau}^k \pi \omega)$. Thus

$$\sup_{\omega \in \Sigma'} \left| \tfrac{1}{n} \sum_{k=0}^{n-1} f(\tau^k \omega) - \int_{\widehat{\Sigma}} f d\nu_1 \right| \le \Delta_n(f \circ \phi) \xrightarrow[n\to\infty]{} 0. \quad V$$

§3 Limit theory for the shift

Let $(\Sigma, \mathcal{A}, \mu, \sigma)$ be a mixing TMS with μ a σ-invariant, Markov measure let $\widehat{\sigma} = \widehat{\sigma}_\mu$ be its transfer operator, let $\mathbb{G} \subset \mathbb{R}^d$ be a closed subgroup of dimension D; and let $f = (f^{(1)}, \ldots, f^{(D)}) : \Sigma \to \mathbb{G}$ be Hölder continuous and σ-aperiodic. Let $\overline{f} := f - \mathbb{E}(f)$. Note that $\overline{f}$ may have values outside $\mathbb{G}$.

We'll need the following results for the sequel. The results are not new although we did not find references for their statements. The proofs are standard and will only be sketched.

3.1 Asymptotic variance theorem

$\exists$ a $D \times D$, symmetric, positive definite matrix $\Gamma = \Gamma_f$ so that

$$\texttt{Rank}\,\Gamma_f = D \ \& \ \frac{1}{n}\mathbb{E}(\overline{f}_n^{(i)}\overline{f}_n^{(j)}) \underset{n\to\infty}{\longrightarrow} \Gamma_{i,j} \quad \forall\ 1 \le i,j \le D$$

where $\overline{f}_n^{(i)} := \sum_{k=0}^{n-1} \overline{f}^{(i)} \circ \sigma^k$.

Since Γ is symmetric and positive definite, it can be put in the form

$$\Gamma = (U^t M)(U^t M)^t$$

with U unitary and $M > 0$ diagonal.

3.2 Central limit theorem

$$\text{(CLT)} \qquad \widehat{\sigma}^n 1_{[\frac{\overline{f}_n}{\sqrt{n}} \in I]} \underset{n\to\infty}{\longrightarrow} \frac{1}{\sqrt{(2\pi)^D \det\Gamma}} \int_I \exp[-\tfrac{1}{2}u^t\Gamma^{-1}u]du$$

whenever $I \subset \mathbb{R}^D$ is Riemann integrable (i.e. with Riemann integrable indicator function).

3.3 Local limit theorem (mixed lattice-nonlattice case)

Let $0 \in I \subset \mathbb{G}$ be Riemann integrable; then

(LLT)

$$(\sqrt{n})^D \widehat{\sigma}^n 1_{[f_n \in n\mathbb{E}(f) + \sqrt{n}x_n + I]} \underset{n\to\infty,\ x_n\to u}{\longrightarrow} \frac{m_{\mathbb{G}}(I)}{\sqrt{(2\pi)^D \det\Gamma}} \cdot \exp[-\tfrac{1}{2}u^t\Gamma^{-1}u].$$

Proof sketches

Suppose that f is θ-Hölder continuous and let $\rho = \rho_\theta$ be the metric on Σ defined by

$$\rho(x,y) := \theta^{\inf\{n\ge1:\ x_n \ne y_n\}}.$$

Let $\mathbb{L} = \mathbb{L}_\theta$ be the Banach space of ρ-Lipschitz continuous (equivalently, θ-Hölder continuous functions) on Σ equipped with the norm

$$\|F\|_{\mathbb{L}} := \sup_{x\in\Sigma}|F(x)| + \sup_{x,\ y\in\Sigma} \frac{|F(x) - F(x)|}{\rho(x,y)}.$$

As shown in [D-F], [GH], for some $N \geq 1$, $\widehat{\sigma}^N : \mathbb{L} \to \mathbb{L}$ satisfies the *Doeblin-Fortet inequality*, namely $\exists\ r \in (0,1)$ & $H > 0$ so that

$$\text{(DF)} \qquad \|\widehat{\sigma}^N F\|_{\mathbb{L}} \leq r\|F\|_{\mathbb{L}} + H \sup_{x\in\Sigma}|F(x)| \quad \forall\ F \in \mathbb{L},$$

whence ([D-F], [IT-M]) $\widehat{\sigma} : \mathbb{L} \to \mathbb{L}$ is quasi-compact in that $\exists\ \theta \in (0,1)$ & $M > 1$ so that

$$\text{(QC)} \qquad \|\widehat{\sigma}^n F - \int_\Sigma F d\mu\|_{\mathbb{L}} \leq M\theta^n\|F\|_{\mathbb{L}} \quad \forall\ F \in \mathbb{L}.$$

Proof of the asymptotic variance theorem

The absolute convergence of the series $\sum_{n\geq 1}|\int_\Sigma f^{(i)} \cdot f^{(j)} \circ \sigma^n d\mu|$ for $1 \leq i, j \leq D$ follows from (QC) and the convergence follows from this. The non-singularity of the limit matrix follows from the aperiodicity of each $t \cdot f : \Sigma :\to \mathbb{R} \quad (t \in \mathbb{R}^D)$ via Leonov's theorem (as in [RE]). ☑

Proof of the central and local limit theorems

Both of these results are established using Nagaev's perturbation method as in [HH], [PP], [RE] (also known as characteristic function operators [AD]).

For $t \in \mathbb{C}^D$ define $P_t : L^1(\Sigma) \to L^1(\Sigma)$ by $P_t(F) := \widehat{\sigma}(e^{it\cdot f}F)$, then $P_t(F) : \mathbb{L} \to \mathbb{L}$ and the map $t \mapsto P_t$ is analytic $\mathbb{C}^D \to \text{hom}(\mathbb{L}, \mathbb{L})$ with

$$\left(\frac{\partial^r}{\partial t_{k_1} \cdots \partial t_{k_r}} P_t\right)(F) := i^r \widehat{\sigma}\left(\prod_{j=1}^r f^{(k_j)} e^{it\cdot f} F\right);$$

where $\text{hom}(\mathbb{L}, \mathbb{L})$ is equipped with the uniform topology. We have (see [N], [PP] &/or [RE])

- $\|P_t\| \leq 1 \quad \forall\ t \in \widehat{\mathbb{G}} = \mathbb{T}^k \times \mathbb{R}^{D-k}$ with equality iff $t = 0$;
- P_t satisfies (DF) for $|t|$ small; whence

Nagaev's Theorem [N] *There are constants* $\epsilon > 0$, $K > 0$ *and* $\theta \in (0,1)$*; and analytic functions* $\lambda : (-\epsilon, \epsilon) \to B_{\mathbb{C}}(0,1)$, $N : (-\epsilon, \epsilon) \to \text{hom}(L, L)$ *such that*

$$\|P_t^n h - \lambda(t)^n N(t)h\|_L \leq K\theta^n\|h\|_L \qquad \forall\ |t| < \epsilon,\ n \geq 1,\ h \in L$$

where $\forall |t| < \epsilon$*,* $N(t)$ *is a projection onto a one-dimensional subspace,* $\lambda(0) = 1$ *&* $N(0)F := \int_\Sigma F d\mu$*.*

The expansion of λ is obtained by considering $t \cdot f : \Sigma \to \mathbb{R} \quad (t \in \mathbb{R}^D)$ as in [GH]. It is given by

$$\lambda(t) = 1 + it \cdot \mathbb{E}(f) - \frac{<\Gamma t, t>}{2} + o(|t|^2) \quad \text{as } t \to 0.$$

The central limit theorem follows from this in the standard manner (see [GH], [RE]); and the local limit theorem follows with a proof as in [S] (see [AD]).

§4 Proof of the theorem

For $n \geq 1$, set $\ell_n := [\log_\lambda n]$ where $\lambda = e^{h_{\text{top}}(\Sigma)}$. Let $0 \in I \subset \mathbb{G}$ be Riemann integrable with $0 < |I| < \infty$ of form $I = \{0^{(k)}\} \times J$ where $0^{(k)} \in \mathbb{Z}^k$, $0_j^{(k)} = 0$ $(1 \leq j \leq k)$ and $0 \in J \subset \mathbb{R}^{D-k}$ is Riemann integrable with $0 < |J| < \infty$. Here, $|\cdot|$ denotes Haar measure on $\mathbb{G}$.

To achieve our goal, we'll establish:

(⚽) $\forall\ R > 0,\ \forall\ x \in \Sigma'$,

$$\frac{\ell_n^{D/2}}{n} \cdot S_n(1_{\Sigma' \times I})(x,0) 1_{B(R)}\left(\frac{\overline{f}_{\ell_n}(x)}{\sqrt{\ell_n}}\right) \approx \frac{|I|}{\sqrt{(2\pi)^D \det \Gamma}} \cdot \exp\left[-\frac{\|M^{-1}U\overline{f}_{\ell_n}(x)\|^2}{2\ell_n}\right] 1_{B(R)}\left(\frac{\overline{f}_{\ell_n}(x)}{\sqrt{\ell_n}}\right)$$

where $\overline{f} := f - \mathbb{E}(f)$ and $B(R) := \{z \in \mathbb{R}^D :\ \|z\| < R\}$ and $a_n \approx b_n$ means $a_n - b_n \underset{n \to \infty}{\longrightarrow} 0$.

We'll show first that (⚽) holds, and then we'll prove that (⚽) $\Longrightarrow$ the theorem.

Overview of the proof of (⚽).

The proof uses a process of `block splitting` where in order to estimate

$$S_n(1_{\Sigma' \times I})(x,0) = \sum_{j=0}^{n-1} 1_{\Sigma' \times I}(\tau^k x, \phi_k(x))$$

we split the τ-orbit block $\{\tau^k x :\ 0 \leq k \leq n-1\}$ into simpler blocks on which it is easy to apply the results of §3.

This is done as follows.

For $x \in \Sigma'$, $N \geq 1$

$$\sigma^{-N}\{\sigma^N x\} = \{\tau^k x_{\mathtt{min}} : 0 \leq k \leq \#\sigma^{-N}\{\sigma^N x\} - 1\}$$

where $x_{\mathtt{min}} := \min \sigma^{-N}\{\sigma^N x\}$ with respect to the reverse lexicographic order and

$$\sum_{j=0}^{\#\sigma^{-N}\{\sigma^N x\}-1} 1_{\Sigma' \times I} \circ T^k(x_{\mathtt{min}}, 0) = \#\{y \in \sigma^{-N}\{\sigma^N x\} :\ f_N(y) \in f_N(x_{\mathtt{min}}) + I\}.$$

Quantities appearing, such as

$$\#\{y\in\sigma^{-N}\{x\}:\quad f_N(y)\in f_N(z)+I\}$$

where $I\subset\mathbb{G}$ is Riemann integrable, are estimated using (LLT) as in lemma 4.1 (below).

The arbitrary blocks are estimated from the simple ones of suitably smaller size. This is calibrated using the Perron-Frobenius theorem:

$$\#\sigma^{-N}\{\sigma^N x\}=\sum_{0\le s\le d-1}A^N_{s,x_{N+1}}\ \sim\ c(x_{N+1})\lambda^N$$

where A is transition matrix of Σ, $\lambda=e^{h_{\text{top}}(\Sigma)}$ is its leading eigenvalue and $c:\mathcal{S}\to(0,\infty)$.

Fix $M\ge 1$ large. For each $n\ge 1$ large, let $N=N_n$ be such that $M\lambda^N=\lambda^{\pm 1}n$, then

$$\{\tau^j x:\ 0\le j<n\}=\bigcup_{k=0}^{M-1}{}^{\!\!\cdot}\ \sigma^{-N}\{\tau^k\sigma^N x\}$$

up to relatively small edge effects (estimated in the proof below using lemma 4.2) and

$$S_n(1_{\Sigma'\times I})(x,0)=$$
$$\sum_{k=0}^{M-1}\#\{y\in\sigma^{-N}\{\tau^k\sigma^N x\}:\ f_N(y)=f_N(\min\sigma^{-N}\{\tau^k\sigma^N x\})\}$$

up to relatively small errors (estimated in lemma 4.3 below).

Proof of (⚽) on page 11**.**

For $x\in\Sigma,\ t^{(n)}\in\mathbb{G},\ \ \sup_n\frac{\|t^{(n)}\|}{\sqrt{n}}<\infty$, set

$$N_n(x):=\#\{z\in\sigma^{-n}\{\sigma^n x\}:\ \ \overline{f}_n(z)\in t^{(n)}+I\}.$$

Lemma 4.1

$$N_n(x)\quad\sim\quad\frac{\lambda^n h(\sigma^n x)|I|}{\sqrt{(2\pi n)^D\det\Gamma}}\exp[-\frac{\|M^{-1}Ut^{(n)}\|^2}{2n}]$$

uniformly on Σ where $h=\frac{d\mu}{dm}$; m and μ being the τ- and σ-invariant probabilities (respectively).

Proof

Let $\widehat{\sigma}_m,\ \widehat{\sigma}_\mu$ be the transfer operators of σ with respect to m & μ respectively, then $\widehat{\sigma}_m f=h\widehat{\sigma}_\mu(\frac{f}{h})$ and

$$
\begin{aligned}
N_n(x) &= \lambda^n \widehat{\sigma}_m^n 1_{[\overline{f}_n \in t^{(n)}+I]}(\sigma^n x) \\
&= \lambda^n h(\sigma^n x) \widehat{\sigma}_\mu^n \Big(\frac{1_{[\overline{f}_n \in t^{(n)}+I]}}{h} \Big)(\sigma^n x) \\
&\sim \frac{\lambda^n h(\sigma^n x)|I|}{\sqrt{(2\pi n)^D \det \Gamma}} \exp[-\frac{1}{2n} \| M^{-1} U t^{(n)} \|^2]
\end{aligned}
$$

uniformly on Σ by (LLT). ☑

Block splitting.

For $n \geq 1$, let $\Sigma_{n,s} := \{(x_1, \ldots, x_n) : \ x \in \Sigma, \ x_{n+1} = s\}, \ J_n(s) := \#\Sigma_{n,s}$, then

$$J_n(s) = \sum_{u \in \mathcal{S}} A_{u,s}^n \underset{n\to\infty}{\sim} c(s)\lambda^n$$

uniformly in $s \in \mathcal{S}$ where $\lambda = e^{h_{\text{top.}}(\Sigma,\sigma)}$ and $c : \mathcal{S} \to \mathbb{R}_+$.

It will be convenient also to set $\widehat{J}_n(z) := \# \sigma^{-n}\{z\}$. Here

$$\widehat{J}_n(x) = \#\Sigma_{n,x_1} = J_n(x_1)$$

and

$$\widehat{J}_n(x) \sim \mathfrak{c}(x)\lambda^n$$

uniformly in $x \in \Sigma$ where $\mathfrak{c} : \Sigma \to \mathbb{R}_+, \ \mathfrak{c}(x) := c(x_1)$.

For $n \geq 1$ fixed, we call a point $x \in \Sigma$

- *n-minimal* if $x = \min \sigma^{-n}\{\sigma^n x\} = \min\{y \in \Sigma : \ y_{n+1} = x_{n+1}\}$ &
- *n-maximal* if $x = \max \sigma^{-n}\{\sigma^n x\} = \max\{y \in \Sigma : \ y_{n+1} = x_{n+1}\}$;

Now define

$$K_n : \Sigma \to \mathbb{N} \ \& \ \tau_n : \Sigma \to \Sigma \ \text{ by}$$

$$K_n(x) := \min\{k \geq 1 : \ \tau^k x \ \text{ is } n\text{-maximal}\} \ \& \ \tau_n(x) := \tau^{K_n(x)+1},$$

then:

- $\sigma^n \tau_n(x) = \tau(\sigma^n x)$;
- $\tau_n(x)$ is n-minimal and
- $K_n(x) \leq \widehat{J}_n(\sigma^n x) = \#\sigma^{-n}\{\sigma^n x\}$ with equality if x is n-minimal.

It follows that for $j \geq 1$,

$$\sigma^n \tau_n^j(x) = \tau^j(\sigma^n x)$$

and

$$K_n(\tau_n^j(x)) = \widehat{J}_n(\tau^j \sigma^n x).$$

For $n \geq 1$ fixed and $r \geq 1$, set

$$K_n^{(r)}(x) := \sum_{j=0}^{r-1} K_n(\tau_n^j(x)) = K_n(x) + \sum_{j=1}^{r-1} \widehat{J}_n(\tau^j(\sigma^n x)).$$

Lemma 4.2 $\exists\ \eta_n,\ \theta_r \downarrow\ 0$ *so that*

$$K_n^{(r)}(x) = e^{\pm(\eta_n+\theta_r)} r\lambda^n E(\mathfrak{c}) \quad \forall\ n, r \geq 1,\ x \in \Sigma'.$$

Proof By the uniform convergence lemma $\exists\ \theta_r \downarrow 0$ such that

$$\sum_{j=1}^{r-1} \mathfrak{c}(\tau^j(\sigma^n x)) = e^{\pm\theta_r} r E(\mathfrak{c}) \ \forall\ x \in \Sigma',\ n,\ r \geq 1.$$

Suppose that $J_n(s) = e^{\pm\eta_n}\lambda^n \mathfrak{c}(s)$ where $\eta_n \downarrow 0$, then for $x \in \Sigma'$,

$$\begin{aligned} K_n^{(r)}(x) &= K_n(x) + \sum_{j=1}^{r-1} \widehat{J}_n(\tau^j(\sigma^n x)) \\ &= K_n(x) + e^{\pm\eta_n}\lambda^n \sum_{j=1}^{r-1} \mathfrak{c}(\tau^j(\sigma^n x)) \\ &= K_n(x) + e^{\pm\eta_n} e^{\pm\theta_r} r\lambda^n E(\mathfrak{c}). \end{aligned}$$

Since $K_n(x) \leq \widehat{J}_n(\sigma^n x) = O(\lambda^n)$, the lemma follows. ☑

Lemma 4.3 *For $r \in \mathbb{N}$ fixed, $x \in \Sigma'$ and $R > 0$, as $n \to \infty$:*

(1) $$S_{K_n^{(r)}(x)}(1_{\Sigma\times I})(x,0) 1_{B(R)}(\tfrac{\overline{f}_n(x)}{\sqrt{n}}) \ \gtrsim \ \frac{h_{r-1}(\sigma^n x)|I|\lambda^n}{n^{D/2}} \exp[-\frac{\|M^{-1}U\overline{f}_n(x)\|^2}{2n}] 1_{B(R)}(\tfrac{\overline{f}_n(x)}{\sqrt{n}})$$

and

(2) $$S_{K_n^{(r)}(x)}(1_{\Sigma\times I})(x,0) 1_{B(R)}(\tfrac{\overline{f}_n(x)}{\sqrt{n}}) \ \lesssim \ \frac{h_{r+1}(\sigma^n x)|I|\lambda^n}{n^{D/2}} \exp[-\frac{\|M^{-1}U\overline{f}_n(x)\|^2}{2n}] 1_{B(R)}(\tfrac{\overline{f}_n(x)}{\sqrt{n}})$$

where $h_r(z) := \sum_{j=0}^{r-1} h(\tau^j z)$.

Here, for $A_n,\ B_n > 0,\ A_n \gtrsim B_n$ means $\underline{\lim}_{n\to\infty} \frac{A_n}{B_n} \geq 1$ and $A_n \lesssim B_n$ means $\overline{\lim}_{n\to\infty} \frac{A_n}{B_n} \leq 1$

Proof

Writing $K_n^{(0)} \equiv 0$, we have

$$\begin{aligned} & S_{K_n^{(r)}(x)}(1_{\Sigma\times I})(x,0) \\ & = \sum_{j=0}^{r-1}\Big(S_{K_n^{(j+1)}(x)}(1_{\Sigma\times I})(x,0) - S_{K_n^{(j)}(x)}(1_{\Sigma\times I})(x,0)\Big) \\ & = S_{K_n(x)}(1_{\Sigma\times I})(x,0)+ \\ & \quad + \sum_{j=1}^{r-1}\Big(S_{K_n^{(j+1)}(x)}(1_{\Sigma\times I})(x,0) - S_{K_n^{(j)}(x)}(1_{\Sigma\times I})(x,0)\Big). \end{aligned}$$

For fixed $j \geq 1$,

$$\begin{aligned} & S_{K_n^{(j+1)}(x)}(1_{\Sigma\times I})(x,0) - S_{K_n^{(j)}(x)}(1_{\Sigma\times I})(x,0) \\ & = \sum_{k=K_n^{(j)}(x)}^{K_n^{(j+1)}(x)-1} 1_{\Sigma\times I}(\tau^k x, \phi_k(x)) \\ & = \sum_{\ell=0}^{K_n(\tau_n^j(x))-1} 1_{\Sigma\times I}(\tau^\ell(\tau^{K_n^{(j)}(x)}x), \phi_{K_n^{(j)}(x)+\ell}(x)) \\ & = \sum_{\ell=0}^{\widehat{J}_n(\tau^j\sigma^n x)-1} 1_{\Sigma\times I}(\tau^\ell(\tau_n^j(x)), \phi_{K_n^{(j)}(x)+\ell}(x)) \end{aligned}$$

Now,

$$\{\tau^\ell(\tau_n^j(x) :\ 0 \leq \ell \leq \widehat{J}_n(\tau^j\sigma^n x) - 1\} = \sigma^{-n}\{\tau^j\sigma^n x\},$$

$$\phi_{K_n^{(j)}(x)+\ell}(x) = \psi(z,x) = \sum_{k=0}^{\infty}(f(\sigma^k x) - f(\sigma^k z)) \quad (0 \leq \ell \leq \widehat{J}_n(\tau^j\sigma^n x) - 1);$$

and so

$$\begin{aligned} & S_{K_n^{(j+1)}(x)}(1_{\Sigma\times I})(x,0) - S_{K_n^{(j)}(x)}(1_{\Sigma\times I})(x,0) \\ & \qquad = \#\{z \in \sigma^{-n}\{\sigma^n\tau_n^j(x)\} :\ \psi(z,x) \in I\}. \end{aligned}$$

For $z = \tau^\ell(\tau_n^j(x) \in \sigma^{-n}\{\tau^j\sigma^n x\}$ we have

$$\begin{aligned} \phi_{K_n^{(j)}(x)+\ell}(x)) & = \psi(z,x) \\ & = \sum_{k=0}^{\infty}(f(\sigma^k x) - f(\sigma^k z)) \\ & = f_t(x) - f_t(z) \end{aligned}$$

where $t = t(x,z) := \min\{N \geq 1 :\ \sigma^N x = \sigma^N z\}$.

Here, $\sigma^n z = \tau^j\sigma^n x$ and

$$t(x,z) \leq n + t(\tau^j\sigma^n x, \sigma^n x).$$

Thus

$$\psi(z,x) = f_n(z) - f_n(x) + \kappa_{n,j}(x)$$

where

$$|\kappa_{n,j}(z)| \le 2\sup|f|(N-n) \le 2\sup|f|t(\tau^j\sigma^n x, \sigma^n x).$$

We claim that for a.e. $x \in \Sigma$,

(Å) $$\max_{1\le j\le r} t(\tau^j\sigma^n x, \sigma^n x) = O(\log n) \quad \text{as} \quad n \to \infty.$$

Proof of (Å)

For $M > 0$ set $A_n(M) := \{x \in \Sigma : \ t(\tau\sigma^n x, \sigma^n x) > M\log n\}$, then

$$\begin{aligned} m(\{x \in \Sigma : \max_{1\le j\le r} t(\tau^j\sigma^n x, \sigma^n x) > M\log n\}) &\le m(\bigcup_{0\le j\le r-1} \tau^{-j}A_n(M)) \\ &\le rm(A_n(M)). \end{aligned}$$

Now $t(\tau\sigma^n x, \sigma^n x) > M\log n$ iff $\exists\ z \in \Sigma_{\max}$ so that

$$x_{n+j} = z_j \ \forall\ 1 \le j \le M\log n.$$

Thus

$$\begin{aligned} m(A_n(M)) &\le \sup\ h\mu(A_n(M)) \\ &= \sup\ h \sum_{z\in\Sigma_{\max}} \mu([z_1, \dots, z_{\lfloor M\log n\rfloor}]) \\ &= O(\lambda^{-M\log n}) \end{aligned}$$

and $\sum_{n\ge1} m(A_n(M)) < \infty$ for $M > \frac{1}{\log\lambda}$. The claim (Å) now follows from the Borel-Cantelli lemma. ☑

In view of (Å), we have by lemma 4.1 that for a.e. x:

$$\begin{aligned} &1_{B(R)}(\tfrac{\overline{f}_n(x)}{\sqrt{n}})(S_{K_n^{(j+1)}(x)}(1_{\Sigma\times I})(x,0) - S_{K_n^{(j)}(x)}(1_{\Sigma\times I})(x,0)) \\ &= 1_{B(R)}(\tfrac{\overline{f}_n(x)}{\sqrt{n}})\#\{z \in \sigma^{-n}\{\sigma^n\tau_n^j(x)\} : \ f_n(z) \in f_n(x) - \kappa_{n,j}(x) + I\} \\ &\sim |I|1_{B(R)}(\tfrac{\overline{f}_n(x)}{\sqrt{n}})\tfrac{\lambda^n h(\sigma^n x)}{n^{D/2}}\exp[-\tfrac{\|M^{-1}U(\overline{f}_n(x)-\kappa_{n,j}(x))\|^2}{2n}] \\ &\sim |I|1_{B(R)}(\tfrac{\overline{f}_n(x)}{\sqrt{n}})\tfrac{\lambda^n h(\sigma^n x)}{n^{D/2}}\exp[-\tfrac{\|M^{-1}U\overline{f}_n(x)\|^2}{2n}]. \end{aligned}$$

The lemma follows from this. ☑

Completion of the proof of (⚽) on page 11

Given $0 < \epsilon < \frac{1}{3}$,

- use the uniform convergence lemma to fix r_ϵ such that $\forall\ y \in \Sigma',\ r \geq r_\epsilon,\ \ e^{\theta_r} < 1 + \epsilon$ where θ_r is as in lemma 4.2, and

$$\mathfrak{c}_r(y),\ \mathfrak{c}_{r+2}(y) = (1 \pm \epsilon) r \mathbb{E}_m(\mathfrak{c})\ \&\ h_r(y),\ h_{r+2}(y) = (1 \pm \epsilon) r \mathbb{E}_m(h).$$

- fix $J > e^{\theta_r}\ \forall\ r \geq 1$ and for $n \geq 1$ let

$$L_n = L_{n,\epsilon} := \lfloor \log_\lambda \tfrac{n}{2JE(c) r_\epsilon} \rfloor$$

and let $r_n = r_{n,\epsilon}$ be such that

$$K_{L_n}^{(r_n)}(\tau\sigma^n x) \leq n < K_{L_n}^{(r_n+1)}(\tau\sigma^n x) < K_{L_n}^{(r_n+2)}(\sigma^n x).$$

It follows that

$$S_{K_{L_n}^{(r_n)}(\tau\sigma^n x)}(1_{\Sigma\times I})(x,0) \leq S_n(1_{\Sigma\times I})(x,0) \leq S_{K_{L_n}^{(r_n+2)}(\sigma^n x)}(1_{\Sigma\times I})(x,0).$$

By lemma 4.2,

$$n \leq K_{L_n}^{(r_n+2)}(\sigma^n x) \leq e^{(\eta_{L_n}+\theta_{r_n})} r_n \lambda^{L_n} E(c) \lesssim e^{\theta_{r_n}} r_n \tfrac{n}{2JE(c) r_\epsilon} E(c) \leq \tfrac{n r_n}{2 r_\epsilon}$$

whence for large n, $r_n > r_\epsilon$ and

$$\begin{aligned}
S_n(1_{\Sigma\times I})(x,0) 1_{B(R)}\left(\tfrac{\overline{f}_{L_n}(x)}{\sqrt{L_n}}\right) &\geq S_{K_{L_n}^{(r_n)}(\tau\sigma^n x)}(1_{\Sigma\times I})(x,0) 1_{B(R)}\left(\tfrac{\overline{f}_{L_n}(x)}{\sqrt{L_n}}\right)\\
&\gtrsim \ \tfrac{h_{r_n}(\tau\sigma^{L_n}x)\lambda^{L_n}}{L_n{}^{D/2}} \exp\left[-\tfrac{\|M^{-1}U\overline{f}_{L_n}(x)\|^2}{2L_n}\right] 1_{B(R)}\left(\tfrac{\overline{f}_{L_n}(x)}{\sqrt{L_n}}\right)\\
&\geq (1-\epsilon)|I| \tfrac{\lambda^{L_n} r_n}{L_n{}^{D/2}} \exp\left[-\tfrac{\|M^{-1}U\overline{f}_{L_n}(x)\|^2}{2L_n}\right] 1_{B(R)}\left(\tfrac{\overline{f}_{L_n}(x)}{\sqrt{L_n}}\right)
\end{aligned}$$

and similarly

$$\begin{aligned}
S_n(1_{\Sigma\times I})(x,0) 1_{B(R)}\left(\tfrac{\overline{f}_{L_n}(x)}{\sqrt{L_n}}\right) &\leq S_{K_{L_n}^{(r_n+2)}(\sigma^n x)}(1_{\Sigma\times I})(x,0) 1_{B(R)}\left(\tfrac{\overline{f}_{L_n}(x)}{\sqrt{L_n}}\right)\\
&\lesssim \ \tfrac{h_{r_n+2}(\sigma^{L_n}x)\lambda^{L_n}}{L_n{}^{D/2}} \exp\left[-\tfrac{\|M^{-1}U\overline{f}_{L_n}(x)\|^2}{2L_n}\right] 1_{B(R)}\left(\tfrac{\overline{f}_{L_n}(x)}{\sqrt{L_n}}\right)\\
&\leq (1+\epsilon)|I| \tfrac{\lambda^{L_n} r_n}{L_n{}^{D/2}} \exp\left[-\tfrac{\|M^{-1}U\overline{f}_{L_n}(x)\|^2}{2L_n}\right] 1_{B(R)}\left(\tfrac{\overline{f}_{L_n}(x)}{\sqrt{L_n}}\right).
\end{aligned}$$

Now,

- $L_n \sim \ell_n = \log_\lambda n\ \forall\ \epsilon > 0$ and since $r_n > r_\epsilon$,

$$n \geq K_{L_n}^{(r_n)}(\tau\sigma^n x) \geq e^{-(\eta_n+\theta_{r_n})} r_n \lambda^{L_n} E(c) \gtrsim \tfrac{1}{1+\epsilon} r_n \lambda^{L_n} E(c).$$

$$n < K_{L_n}^{(r_n+1)}(\tau\sigma^n x) \lesssim (1+\epsilon) r_n \lambda^{L_n} E(c) < \tfrac{1}{1-\epsilon} r_n \lambda^{L_n} E(c)$$

whence

$$\tfrac{\lambda^{L_n} r_n}{L_n{}^{D/2}} = (1 \pm \epsilon)^2 \tfrac{n}{\ell_n^{D/2} E(c)}.$$

This proves (⚽) (on page 11). ☑

Proof that (⚽) $\implies$ the theorem. Let Γ be as in §3 and write $\Gamma = VV^t$ where $V := U^t M$ with U unitary and $M > 0$ diagonal.

Let $\xi = (\xi_1, \dots, \xi_d)$ where $\xi_1, \dots, \xi_d$ are independent, identically distributed, Gaussian random variables with $E(\xi_j) = 0$, $E(\xi_j^2) = 1$, then $Z := U^t M\xi = V\xi$ is Gaussian with correlation matrix

$$E(Z_i Z_j) = E(\sum_{s,t} V_{i,s}\xi_s V_{j,t}\xi_t) = \sum_s V_{i,s}V_{j,s} = \Gamma_{i,j}.$$

By (CLT)

$$\frac{\overline{f}_n}{\sqrt{n}} \xrightarrow{\mathfrak{d}} U^t M\xi =: Z.$$

Now suppose that (⚽) (as on page 11) holds. We'll show that for some $a(n) \propto \frac{n}{\ell_n^{D/2}}$ and for $g \in C([0,\infty])$, $f \in L^1(m)_+$,

$$g(\tfrac{1}{a(n)} \cdot S_n(f)) \underset{n\to\infty}{\longrightarrow} \mathbb{E}(g(e^{-\frac{1}{2}\chi_{D^2}} \cdot m(f))) \text{ weak } * \text{ in } L^\infty(m).$$

By the asymptotic variance theorem, $\mathbb{E}(\|\overline{f}_n\|_2) = O(\sqrt{n})$ and $\forall\ \epsilon > 0\ \exists\ R$ so that $m_{\Sigma'}(C_n(R)) > 1 - \epsilon\ \forall\ n \ge 1$ where $C_n(R) := [\frac{\overline{f}_{\ell_n}(x)}{\sqrt{\ell_n}} \in B(R)]$.

Thus for $n \in \mathbb{N}$ & $R > 0$ both large enough and $x \in C_n(R)$ we have

$$g(\tfrac{\ell_n^{D/2}}{n} S_n(1_{\Sigma'\times I})(x,0)) = g(\tfrac{|I|}{\sqrt{(2\pi)^D \det\Gamma}} \cdot \exp[-\tfrac{\|M^{-1}U\overline{f}_n(x)\|^2}{2n}]) \pm \epsilon.$$

Next, by (CLT),

$$\begin{aligned}\int_{\Sigma'\times I} g(\tfrac{|I|}{\sqrt{(2\pi)^D \det\Gamma}} \cdot \exp[-\tfrac{\|M^{-1}U\overline{f}_n\|^2}{2n}])dm &\underset{n\to\infty}{\longrightarrow} \mathbb{E}(g(\tfrac{|I|}{\sqrt{(2\pi)^D \det\Gamma}} \cdot \exp[-\tfrac{\|M^{-1}UZ\|^2}{2}]))\\ &= \mathbb{E}(g(\tfrac{|I|}{\sqrt{(2\pi)^D \det\Gamma}} \cdot \exp[-\tfrac{\chi_D^2}{2}])).\end{aligned}$$

Thus, $\exists\ a(n) \propto \frac{n}{\ell_n^{D/2}}$,

$$\int_{\Sigma'\times I} g(\tfrac{1}{a(n)} \cdot S_n(1_{\Sigma'\times I}))dm \underset{n\to\infty}{\longrightarrow} \mathbb{E}(g(m_{\mathbb{G}}(I) \cdot 2^{D/2}e^{-\frac{1}{2}\chi_D^2})).$$

Using Corollary 3.6.2 of [A1], we obtain that $\forall\ F \in L^1(m)_+$, $g \in C([0,\infty])$,

$$g(\tfrac{1}{a(n)} \cdot S_n(F)) \underset{n\to\infty}{\longrightarrow} \mathbb{E}(g(m(F) \cdot 2^{D/2}e^{-\frac{1}{2}\chi_D^2})) \text{ weak } * \text{ in } L^\infty(m)$$

where $m(F) := \int_{\Sigma\times\mathbb{G}} F dm$. ☑

§5 Application to exchangeability

Let $\mathcal{S} = \{0, 1, \ldots, d-1\}$ and let $\Sigma \subset \mathcal{S}^{\mathbb{N}}$ be a mixing TMS. Define $F^\natural : \Sigma \to \mathbb{Z}^{d-1}$ by $F^\natural(x)_k := \delta_{x_1,k}$.

As shown in [ADSZ], $F^\natural : \Sigma \to \mathbb{Z}^{d-1}$ is σ-aperiodic iff Σ is *almost onto* in the sense that

$\forall\ b, c \in \mathcal{S},\ \exists\ n \geq 1,\ b = a_0, a_1, \ldots, a_n = c \in \mathcal{S}$ such that

$$\sigma[a_k] \cap \sigma[a_{k+1}] \neq \varnothing \quad (0 \leq k \leq n-1).$$

Define $\varphi : \Sigma \to \mathbb{N}$ and $R : \Sigma \to \Sigma$ by

$$\varphi(x) := \min\{n \geq 1 :\ \tau^n(x)_i = x_{\sigma(i)} \ \text{ some finite permutation } \sigma \text{ of } \mathbb{N}\}$$

and

$$R(x) := \tau^{\varphi(x)}(x).$$

Corollary 5.1

Suppose that Σ is almost onto, then $(\Sigma, \mathcal{B}(\Sigma), R, m)$ is an ergodic, probability preserving transformation and $\exists\ b(n) \propto \frac{n}{(\log n)^{(d-1)/2}}$ such that

$$\frac{1}{b(n)} \sum_{k=0}^{n-1} \varphi \circ R^k \xrightarrow{\ \mathfrak{d}\ } e^{\frac{1}{2}\chi^2_{d-1}}.$$

Proof.

The random walk adic

$$(\Sigma \times \mathbb{Z}^{d-1}, \mathcal{B}(\Sigma \times \mathbb{Z}^{d-1}), m \times m_{\mathbb{Z}^{d-1}}, T)$$

over $(\Sigma, F^\natural, \tau)$ is conservative and ergodic. Calculation shows that $T_{\Sigma \times \{0\}}(x, 0) = (Rx, 0)$ whence $(\Sigma, \mathcal{B}(\Sigma), R, m)$ is an ergodic, probability preserving transformation.

By the theorem,

$$\frac{1}{a(n)} S_n(f) \xrightarrow{\ \mathfrak{d}\ } e^{-\frac{1}{2}\chi^2_{d-1}} \mu(f)\ \ \forall\ f \in L^1_+$$

where $a(n) \propto \frac{n}{(\log n)^{(d-1)/2}}$ (we absorbed the factor in $a(n)$).

In particular,

$$\frac{1}{a(n)} S_n(1_{\Sigma \times \{0\}}) \xrightarrow{\ \mathfrak{d}\ } e^{-\frac{1}{2}\chi^2_{d-1}}$$

whence by inversion (proposition 1 in [A2]),

$$\frac{1}{b(n)} \sum_{k=0}^{n-1} \varphi \circ R^k \xrightarrow{\ \mathfrak{d}\ } e^{\frac{1}{2}\chi^2_{d-1}}$$

where $b(n) = a^{-1}(n) \propto \frac{n}{(\log n)^{(d-1)/2}}$. ☑

§6 Chi squared laws for horocycle flows

Let M_0 be a compact, connected, orientable, smooth, Riemannian surface with negative sectional curvature, and let T^1M_0 denote the set of unit tangent vectors to M_0. The *geodesic flow* on T^1M_0 is the flow which moves a vector $\underline{v} \in T^1M$ at unit speed along its geodesic.

Margulis [Mrg] and Marcus [Mrc] constructed a continuous flow $h^t : T^1M \to T^1M$ such that

(a) The h–orbit of $\vec{v} \in T^1M_0$ equals

$$W^{ss}(\vec{v}) := \{\vec{u} \mid \operatorname{dist}(g^s(\vec{v}), g^s(\vec{u})) \xrightarrow[s\to\infty]{} 0\}$$

(b) $\exists\mu$ s.t. $g^{-s} \circ h^t \circ g^s = h^{\mu^s t}$

In the special case when M_0 is a hyperbolic surface, h is the *stable horocycle flow.* Properties (a) and (b) should be compared to the relation between the odometer and the left shift.

A $\mathbb{Z}^D$*–cover of* M_0 is a surface M together with a continuous map $p : M \to M_0$ such that p is a local isometry at every point, the group of *deck transformations*

$$G := \{A : M \to M | D \text{ an isometry s.t. } p \circ A = p\}$$

is isomorphic to $\mathbb{Z}^D$, and for every $x \in M_0$, $p^{-1}(x)$ is a G–orbit of some point in M.

The flows $g, h : T^1M_0 \to T^1M_0$ lift to flows $g, h : T^1M \to T^1M$ which commute with the elements of G, and which satisfy (a),(b). Now (a) and (b) could be compared to the relation between the HIK transformation and a $\mathbb{Z}^D$–skew-product over the left shift map [Po].

The locally finite ergodic invariant measures for h are described in [BL] and [S]. There are infinitely many, but only one up to normalization, is non-squashable [LS]. This measure, which we call m_0, is rationally ergodic, and it is invariant under the action of the geodesic flow and the deck transformations.

We choose a normalization for m_0 as follows. Let $\widetilde{M_0}$ be a connected pre-compact subset of M s.t. $p : \widetilde{M_0} \to M_0$ is one-to-one and onto, then we normalize m_0 so that $m_0[T^1\widetilde{M_0}] = 1$.

The following can be extracted from [LS]:

Theorem 6.1

There exists $a(T) \propto T/(\ln T)^{D/2}$ such that for every $f \in L^1(m_0)$ with positive integral,

$$\frac{1}{a(T)}\int_0^T f[h^s(\omega)]ds \xrightarrow[T\to\infty]{\mathfrak{d}} (2^{\frac{D}{2}}e^{-\frac{1}{2}\chi_D^2})m_0(f).$$

Proof sketch

Enumerate $G = \{A_{\underline{\xi}} : \underline{\xi} \in \mathbb{Z}^D\}$ such that $A_{\underline{\xi}_1} \circ A_{\underline{\xi}_2} = A_{\underline{\xi}_1+\underline{\xi}_2}$, then $M = \biguplus_{\underline{\xi}\in\mathbb{Z}^D} A_{\underline{\xi}}[\widetilde{M_0}]$. The $\mathbb{Z}^D$*–coordinate* of $\vec{v} \in T^1M$ is the unique $\underline{\xi}(\vec{v}) \in \mathbb{Z}^D$ such that $\vec{v} \in T^1[A_{\underline{\xi}}(\widetilde{M_0})]$.

It is known that $\frac{1}{\sqrt{T}}\underline{\xi} \circ g^T \xrightarrow[T\to\infty]{\mathfrak{d}} \mathcal{N}$, where $\mathcal{N}$ is a D–dimensional Gaussian random variable with positive definite covariance matrix $\mathrm{Cov}(\mathcal{N})$ (Ratner [R], Katsuda & Sunada [KS]).

Let $\|\cdot\|_H$ denote the norm on $\mathbb{R}^D$ given by $\|\underline{v}\|_H := \sqrt{\underline{v}^t\mathrm{Cov}(\mathcal{N})^{-1}\underline{v}}$. The following is proved in [LS] (Theorem 5): Suppose $f \in L^1(m_0)$, then for every $\epsilon > 0$, for m_0–a.e. $\vec{v} \in T^1M$, for all T large enough

$$2^{\frac{D}{2}-\epsilon}e^{-\frac{1}{2}(1+\epsilon)\left\|\frac{\underline{\xi}(g^{\log_\mu T}\vec{v})}{\sqrt{\log_\mu T}}\right\|_H^2} \le \frac{1}{a(T)}\int_0^T f[h^s(\vec{v})]ds \le 2^{\frac{D}{2}+\epsilon}e^{-\frac{1}{2}(1-\epsilon)\left\|\frac{\underline{\xi}(g^{\log_\mu T}\vec{v})}{\sqrt{\log_\mu T}}\right\|_H^2}$$

where $a(T) = \mathrm{const}\, T/(\ln T)^{D/2}$ (the value of the constant is known, see [LS]).

This is the version of (⚽) (see page 11) needed to deduce the theorem as above.

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