SUBEXPONENTIAL DECAY OF CORRELATIONS

OMRI SARIG

Dedicated to B. Weiss on the occasion of his 60th birthday

ABSTRACT. We describe a method for proving subexponential lower bounds for correlations functions, and apply it to study decay of correlations for maps with countable Markov partitions. One result is that LS Young's upper estimates [**Y2**] are optimal in many situations. Our method is based on a general result concerning the asymptotics of renewal sequences of bounded operators acting on Banach spaces, which we apply to the iterates of the transfer operator.

1. INTRODUCTION

A measure preserving transformation $(X, \mathcal{B}, m, T)$ is called strongly mixing if for every $f, g \in L^2$, $Cor(f, g\circ T^n) := \int f \cdot g \circ T^n - \int f \int g \xrightarrow[n\to\infty]{} 0$. The rate of convergence can be used to rigorously describe how fast the state of the system becomes uncorrelated with its future, and estimating it is therefore a basic problem in dynamical systems. In general, nothing can be said on the rate of convergence for arbitrary L^2-functions, so one usually restricts oneself to smaller Banach spaces of functions $\mathcal{L}$. We say that T has *exponential* decay of correlations in $\mathcal{L}$ if $\forall f, g \in \mathcal{L}$ there are $A > 0$ and $\theta \in (0, 1)$ such that $|Cor(f, g \circ T^n)| \le A\theta^n$ for all $n \ge 1$. We say that T has *subexponential* decay in $\mathcal{L}$ if there are $f, g \in \mathcal{L}$ for which there are no such A and θ.

This paper treats the problem of proving subexponential decay of correlations. This amounts to obtaining subexponential *lower* bounds for the correlation functions.

Subexponential *upper* bounds have been obtained by several authors, and there are now several general methods available for doing this: the coupling method of LS Young [**Y1**, **Y2**]; the Birkhoff projective metrics method of Ferrero & Schmitt [**FS**] and Liverani [**Li**] (for application in the subexponential case see Maume [**Ma**]); and Pollicott's conditional expectations method [**P**] (see also Pollicott & Yuri [**PY**]).

The key tool in these works is the transfer operator $\widehat{T} : L^1 \to L^1$ which is defined by $\forall f \in L^1, g \in L^\infty$, $\int g \cdot \widehat{T} f dm = \int g \circ T \cdot f dm$. By definition $|Cor(f, g \circ T^n)| = |\int g(\widehat{T}^n f - \int f) dm| \le \|g\|_\infty \|\widehat{T}^n f - \int f\|_1$, so if one can estimate $\|\widehat{T}^n f - \int f\|_{\mathcal{L}}$ for some norm $\|\cdot\|_{\mathcal{L}} \ge \|\cdot\|_1$, one immediately obtains an upper estimate for $Cor(g, f \circ T^n)$ (for $g \in L^\infty$, $f \in \mathcal{L}$). We refer the reader to [**B**] for a comprehensive and up to date discussion of this approach.

Lower estimates cannot be proved in this way, because even if $\|\widehat{T}^n f - \int f\|_1$ can be bounded from below, it is not clear how to use this to derive the existence of f and g. Indeed, we are not aware of any general methods for obtaining lower

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bounds, and the only examples known to us where subexponential lower bounds were obtained are certain maps of the interval similar to $x \mapsto x + x^s (mod\ 1)$, $s \in (1,2)$ (Hu [**H**] and Isola [**I1**]), certain piecewise linear maps of the interval (Mori [**Mo**] and Isola [**I2**]) and certain extensions of circle rotations (Courbage and Hadman [**CH**]).

2. Statement of Results

Recall that for $A \in \mathcal{B}$ with positive measure, the *induced transformation* on A is $(A, \mathcal{B} \cap A, m_A, T_A)$ given by $\mathcal{B} \cap A = \{B \cap A : B \in \mathcal{B}\}$, $m_A(E) = m(E|A)$ and $T_A(x) = T^{\varphi_A(x)}(x)$ where $\varphi_A(x) := 1_A(x) \inf\{n \geq 1 : T^n x \in A\}$. Our starting point is the following well-known observation (compare with [**PS**]). Here and throughout I is the identity operator and $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$:

Proposition 1 (The renewal equation)**.** *Let $(X, \mathcal{B}, m, T)$ be a conservative non-singular transformation, and assume that $A \in \mathcal{B}$ has finite positive measure. If T_n and R_n are given by $T_n f = 1_A \widehat{T}^n (f 1_A)$ and $R_n f = 1_A \widehat{T}^n (f 1_{[\varphi_A = n]})$, then $\forall z \in \mathbb{D}$,*

$$T(z) = \big(I - R(z)\big)^{-1} \text{ where } R = \sum_{n=1}^{\infty} z^n R_n \text{ and } T = I + \sum_{n=1}^{\infty} z^n T_n.$$

Furthermore, $R(1)$ is the transfer operator of T_A.

The following theorem, which is analogous to a classic result in renewal theory due to Gelfond [**Ge**] (see also Rogozin [**R1**]), describes the asymptotics of renewal sequences of operators, and is the main result of this paper.

Theorem 1. *Let T_n be bounded linear operators on a Banach space $\mathcal{L}$ such that $T(z) = I + \sum_{n \geq 1} z^n T_n$ converges in $Hom(\mathcal{L}, \mathcal{L})$ for every $z \in \mathbb{D}$. Assume that:*

(1) **Renewal Equation:** *for every $z \in \mathbb{D}$, $T(z) = (I - R(z))^{-1}$ where $R(z) = \sum_{n \geq 1} z^n R_n$, $R_n \in Hom(\mathcal{L}, \mathcal{L})$ and $\sum \|R_n\| < \infty$.*
(2) **Spectral Gap:** *the spectrum of $R(1)$ consists of an isolated simple eigenvalue at 1 and a compact subset of $\mathbb{D}$.*
(3) **Aperiodicity:** *the spectral radius of $R(z)$ is strictly less than one for all $z \in \overline{\mathbb{D}} \setminus \{1\}$.*

Let P be the eigenprojection of $R(1)$ at 1. If $\sum_{k>n} \|R_k\| = O(1/n^{\beta})$ for some $\beta > 2$ and $PR'(1)P \neq 0$, then for all n

$$T_n = \frac{1}{\mu} P + \frac{1}{\mu^2} \sum_{k=n+1}^{\infty} P_n + E_n$$

where μ is given by $PR'(1)P = \mu P$, $P_n = \sum_{\ell > n} PR_\ell P$, and $E_n \in Hom(\mathcal{L}, \mathcal{L})$ satisfy $\|E_n\| = O(1/n^{\lfloor \beta \rfloor})$.

Our main application is for *Markov maps.* These are quintets $(X, \mathcal{B}, m, T, \alpha)$ where $(X, \mathcal{B}, m)$ is a measure space (which we always take to be a Lebesgue space), $T : X \to X$ is a measurable non-singular transformation (i.e. $m \sim m \circ T^{-1}$), and $\alpha \subset \mathcal{B}$ is a countable or finite partition such that (see [**ADU**])

(1) *Generator property:* $\mathcal{B}$ is the smallest σ-algebra which contains $\bigcup_{n \geq 0} T^{-n} \alpha$ which is complete with respect to m.
(2) *Markov property:* $\forall a, b \in \alpha$ if $m(Ta \cap b) > 0$, then $Ta \supseteq b \mod m$.
(3) *Local invertibility:* $\forall a \in \alpha$, $m(a) > 0$ and $T : a \to Ta$ is invertible with measurable inverse.

α is called the *Markov partition*. We often write (T,α) instead of $(X,\mathcal{B},m,T,\alpha)$.

For all $a_0,\ldots,a_{n-1}\in\alpha$ set $[a_0,\ldots,a_{n-1}]:=\bigcap_{i=0}^{n-1}T^{-i}a_i$. These are the *cylinders*. The (T,α)*-variations* of $\phi: X\to\mathbb{R}$ are

$$v_n(\phi):=\sup\{|\phi(x)-\phi(y)| : x,y\in[a_0,\ldots,a_{n-1}] \text{ where } a_i\in\alpha\}.$$

We say that ϕ has (T,α)*-summable variations*, if $\sum_{n\geq 2}v_n(\phi)<\infty$, and that it is (T,α)*-locally Hölder continuous* if $\exists A>0,\theta\in(0,1)$ such that $v_n(\phi)<A\theta^n$ for all $n\geq 1$. The reader should note the different ranges of n in these two conditions.

Let $m\circ T$ be the σ-finite measure $(m\circ T)=\sum_{a\in\alpha}m\big(T(A\cap a)\big)$. For Markov maps, the transfer operator of m is always of the form $\widehat{T}f=\sum_{Ty=x}g_m(y)f(y)$ where $g_m=\frac{dm}{dm\circ T}$ ([**W**], [**Ke**], see also [**S1**] lemma 10). When m is invariant, g_m is called the *g-function* of m. It follows from results in [**ADU**] that if T is conservative and $\log g_m$ has (T,α)-summable variations, then T is exact and so strongly mixing.

Note that if T is conservative and $a\in\alpha$, then the induced map $T_a: a\to a$ is a Markov map with the partition $\alpha_a:=\{[a,\xi_1,\ldots,\xi_{i-1},a]:\xi_i\in\alpha,\xi_i\neq a\}\setminus\{\varnothing\}$. Let $[\cdot]_a$ denote the cylinders with respect to (T_a,α_a) and define for all different $x,y\in\bigcup\alpha_a$, $s(x,y):=\sup\{n\geq 0: x,y\in[b_0,\ldots,b_{n-1}]_a,\ b_0,\ldots,b_{n-1}\in\alpha_a\}$ and $D_af=\sup|f(x)-f(y)|/\theta^{s(x,y)}$ where the supremum ranges over all $x,y\in[a]$ different such that $T^ix,T^jy\in[a]$ for infinitely many i and j. Define

$$\|f\|_{\mathcal{L}}:=\|f\|_\infty+D_af.$$

Recall that m_a is the measure $m_a(E)=m(E|a)$ and let g_{m_a} denote the g-function of m_a with respect to T_a. Finally, we say that Markov map is *irreducible* if $\forall a,b\in\alpha$, $m(a\cap T^{-n}b)>0$ for some n.

Theorem 2. *Let $(X,\mathcal{B},m,T,\alpha)$ be an irreducible probability preserving Markov map, and assume that $\log g_{m_a}$ has a (T_a,α_a)-locally Hölder continuous version for some $a\in\alpha$. If g.c.d.$\{\varphi_a(x)-\varphi_a(y): x,y\in\bigcup\alpha_a\}=1$ and $m[\varphi_a>n]=O(1/n^\beta)$ where $\beta>2$, then $\exists\theta\in(0,1), C>0$ s.t. $\forall f,g$ integrable supported inside $[a]$,*

$$\left|Cor(f,g\circ T^n)-\left(\sum_{k=n+1}^{\infty}m[\varphi_a>k]\right)\int f\int g\right|\leq\frac{C}{n^{\lfloor\beta\rfloor}}\|g\|_\infty\|f\|_{\mathcal{L}}\tag{1}$$

Note that when $m[\varphi_a>n]\asymp\frac{1}{n^\beta}$, $\int f,\int g\neq 0$ and $\|g\|_\infty,\|f\|_{\mathcal{L}}<\infty$ this estimate implies $Cor(f,g\circ T^n)\sim\left(\sum_{k=n+1}^\infty m[\varphi_a>k]\right)\int f\int g$ and we have obtained our goal of obtaining a lower bound for the correlation functions.

2.1. **Example 1: the Liverani-Saussol-Vaienti map [LSV].** This is

$$T:[0,1]\to[0,1],\quad T(x)=\begin{cases}x(1+2^\alpha x^\alpha) & 0\leq x<\frac12\\ 2x-1 & \frac12\leq x\leq 1\end{cases}\qquad(0<\alpha<1).$$

It is known that T admits an integrable invariant density which is Lipschitz away from zero (see e.g. [**LSV**]). Let $h(x)$ be this density and assume it is normalised to have integral one. The following result sharpens estimates obtained in [**LSV**], [**PY**], [**H**], [**I1**] and [**Y2**] (we refer the reader to [**B**], §3.5 for complete references and historic discussion).

Corollary 1. *If $\alpha\in(0,\frac12)$, f is Lipschitz, g is bounded measurable, $\int f,\int g>0$ and $[f\neq 0],[g\neq 0]\subseteq[\frac12,1]$, then $Cor(f,g\circ T^n)\sim\frac14 h(\frac12)\alpha^{-\frac{1}{\alpha}}(\frac{1}{\alpha}-1)^{-1}n^{1-\frac{1}{\alpha}}\int f\int g$ with respect to the invariant probability measure.*

Proof. Set $x_{-1} := 1$, $x_0 = \frac{1}{2}$ and $x_n = x_{n+1}(1 + 2^\alpha x_{n+1}^\alpha)$. A standard argument shows that $x_n \sim \frac{1}{2}\big(\alpha n\big)^{-1/\alpha}$ and that $\alpha := \{(x_n, x_{n-1}] : n \geq 0\}$ is a Markov partition. The induced transformation on $(\frac{1}{2}, 1]$ admits the Markov partition $\alpha_{(\frac{1}{2},1]} = \{(y_n, y_{n-1}] : n \geq 0\}$ where $y_n = \frac{1}{2}(1 + x_n)$, Since the induced transformation is uniformly expanding with derivative 2 or more, the diameters $(T_{(\frac{1}{2},1]}, \alpha_{(\frac{1}{2},1]})$-cylinders of length n are $O(\frac{1}{2^n})$. Therefore, every function which is Lipschitz on $(\frac{1}{2}, 1]$ is $(T_{(\frac{1}{2},1]}, \alpha_{(\frac{1}{2},1]})$-locally Hölder continuous.

By definition, $g_{m_a} = \frac{h}{h\circ T^\varphi}\frac{1}{(T^\varphi)'}$ where $\varphi : (\frac{1}{2}, 1] \to \mathbb{N}$ is the first return function with respect to $(\frac{1}{2}, 1]$. By [**LSV**] $\log h$ is Lipschitz on $[\frac{1}{2}, 1]$ and so $\log h - \log h \circ T^\varphi$ is $(T_{(\frac{1}{2},1]}, \alpha_{(\frac{1}{2},1]})$-locally Hölder continuous. So is $\log \frac{1}{(T^\varphi)'}$, by an argument similar to [**S2**] page 564. Therefore, $\log g_{m_a}$ is $(T_{(\frac{1}{2},1]}, \alpha_{(\frac{1}{2},1]})$-locally Hölder continuous.

One checks that $[\varphi = n] = (y_{n-1}, y_{n-2}]$, and so, since h is continuous,

$$m[\varphi > n] = \int_{\frac{1}{2}}^{\frac{1}{2}(x_{n-1}+1)} h(t)dt \sim \frac{1}{2}h(\tfrac{1}{2})x_{n-1} \sim \frac{h(\frac{1}{2})}{4(\alpha n)^{\frac{1}{\alpha}}}$$

and so $\sum_{k>n} m[\varphi > k] \sim \frac{1}{4}h(\frac{1}{2})\alpha^{-\frac{1}{\alpha}}(\frac{1}{\alpha} - 1)^{-1}n^{1-\frac{1}{\alpha}}$. The result follows since $g.c.d\{\varphi(x) - \varphi(y)\} = g.c.d.(\mathbb{Z}) = 1$. $\square$

2.2. **Example 2: LS Young towers.** A *LS Young tower* is a non-singular conservative transformation $(\Delta, \mathcal{B}, m, F)$ with a generating measurable partition $\{\Delta_{\ell,i} : i \in \mathbb{N}, \ell = 0, \ldots, R_i - 1\}$ with the following properties:

(T1) The measure of $\Delta_{\ell,i}$ is positive and finite for every i and ℓ, and $m(\Delta_0) < \infty$ where $\Delta_0 = \biguplus_{i\geq 1} \Delta_{0,i}$.
(T2) $g.c.d.\{R_i : i = 1, 2, 3, \ldots\} = 1$.
(T3) If $\ell + 1 < R_i$, then $F : \Delta_{\ell,i} \to \Delta_{\ell+1,i}$ is a measurable bijection, and $m|_{\Delta_{\ell+1,i}} \circ F|_{\Delta_{\ell,i}} = m|_{\Delta_{\ell,i}}$.
(T4) If $\ell + 1 = R_i$, then $F : \Delta_{\ell,i} \to \Delta_0$ is a measurable bijection.
(T5) Let $R : \Delta_0 \to \mathbb{N}$ be the function $R|_{\Delta_{0,i}} \equiv R_i$ and set $\phi := \log \frac{dm|_{\Delta_0}}{dm|_{\Delta_0}\circ F^R}$. ϕ has version for which $\exists C > 0, \theta \in (0, 1)$ such that $\forall i$ and $\forall x, y \in \Delta_{0,i}$,

$$\left|\sum_{k=0}^{R(x)-1} \phi(F^k x) - \sum_{k=0}^{R(y)-1} \phi(F^k y)\right| < C\theta^{s(F^R x, F^R y)}$$

where $s(x, y) = \min\{n \geq 0 : (F^R)^n x, (F^R)^n y \text{ lie in distinct } \Delta_{0,i}\}$.

These systems where introduced by Young in for the purpose of estimating the rate of decay of correlations in the presence of singularities or regions of non-hyperbolicity. We refer the reader to [**Y1**],[**Y2**] and [**BSL**] for numerous concrete examples whose decay of correlations can be treated by studying a suitable LS Young tower.

Henceforth we assume for simplicity that $\int Rdm < \infty$ and that m is an F-invariant probability measure. This can be done, because if $\int Rdm < \infty$, then m has an integrable invariant density h such that $c_0^{-1} \leq h \leq c_0$ on Δ_0 ([**Y2**], theorem 1; see also [**ADU**], theorem 3.1).

Set $C_\theta(\Delta) := \{f : \Delta \to \mathbb{R} : \exists C \forall x, y \in \Delta, |f(x) - f(y)| \leq C\theta^{s(x,y)}\}$ and

$$(2) \quad C_\theta^+(\Delta) := \Big\{ f \in C_\theta(\Delta) : \exists C^+ \text{ s.t. on each } \Delta_{\ell,i}, \text{ either } f \equiv 0 \text{ or } f > 0, \\ \text{and } \left| \frac{f(x)}{f(y)} - 1 \right| \leq C^+ \theta^{s(x,y)} \forall x, y \in \Delta_{\ell,i} \Big\}.$$

The following result is an immediate corollary of theorem 2:

Corollary 2. *Let $(\Delta, \mathcal{B}, m, F)$ be a LS Young tower with g.c.d.$\{R_i - R_j\} = 1$ and $m[R > n] = O(1/n^\beta)$ where $\beta > 2$. If $f \in C_\theta^+(\Delta), f \in L^\infty$ are supported inside $\Delta_{0,i}$ and $\int f, \int g > 0$, then $Cor(f, g \circ T^n) = \sum_{k>n} m[\varphi_{\Delta_{0,i}} > k] \int f \int g + O\left(\frac{1}{n^{\lfloor \beta \rfloor}}\right)$.*

This improves the estimate $Cor(f, g \circ T^n) = O(\sum_{k>n} m[R > k])$ which was obtained by Young in [**Y2**], although she did not have to assume that $\beta > 2$ and that $g.c.d\{R_i - R_j\} = 1$. Theorem 2 shows that in many situations Young's estimate is sharp.

3. Proof of proposition 1

Proof. First note that the norms of $R_n, T_n : L^1 \to L^1$ are all at most one, so $T(z)$ and $R(z)$ make sense for $z \in \mathbb{D}$. Next note that if we use the convention $T_0 = I$, then it is enough to check that for every $n \geq 1$

$$T_n = \sum_{k=1}^{n} R_k T_{n-k} = \sum_{k=0}^{n-1} T_k R_{n-k}$$

because these relations imply that $T(z)(I - R(z)) = (I - R(z))T(z) = I$.

For every $g \in L^\infty(A, \mathcal{B} \cap A, m_A)$ and $f \in L^1(A, \mathcal{B} \cap A, m_A)$

$$\begin{aligned}
\int g \sum_{k=1}^{n} R_k T_{n-k} f dm_A &= \sum_{k=1}^{n} \frac{1}{m(A)} \int 1_A g \widehat{T}^k \left(1_{[\varphi_A = k]} \widehat{T}^{n-k}(1_A f) \right) dm \\
&= \sum_{k=1}^{n} \frac{1}{m(A)} \int 1_A \circ T^k g \circ T^k 1_{[\varphi_A = k]} \widehat{T}^{n-k}(1_A f) dm \\
&= \sum_{k=1}^{n} \frac{1}{m(A)} \int_{A_{nk}} g \circ T^n \cdot f dm
\end{aligned}$$

where $A_{nk} = A \cap T^{-n}A \cap T^{-(n-k)}[\varphi_A = k]$.

Now, $A_{nk} = \{x \in A \cap T^{-n}A : \max\{0 \leq \ell \leq n-1 : T^\ell x \in A\} = n-k\}$. For every n, $A_{n1}, \ldots, A_{nn}$ are pairwise disjoint, and their union is $A \cap T^{-n}A$. Consequently, since f and g are supported inside A,

$$\int g \sum_{k=0}^{n-1} R_k T_{n-k} f dm_A = \frac{1}{m(A)} \int (1_A g) \circ T^n \cdot (f 1_A) dm = \int g T_n f dm_A$$

whence $T_n = \sum_{k=1}^{n} R_k T_{n-k}$. The other equality is proved in the same way. This and the fact that $R(1)$ is the transfer operator of T_A are left to the reader. $\square$

4. Proof of theorem 1

4.1. **First Main Lemma.** Let $\|\cdot\|$ denote the strong operator norm in $Hom(\mathcal{L},\mathcal{L})$. The purpose of this section is to prove

Lemma 1 (First Main Lemma). *Under the assumptions of theorem* 1*,*

$$\sum_{n=1}^{\infty}\|T_{n+1}-T_n\|<\infty.$$

We start with some general remarks on α-Hölder continuous functions. Let B be some Banach space (in the application that interests us, $B=Hom(\mathcal{L},\mathcal{L})$). Fix some open bounded convex $U\subseteq\mathbb{C}$ and some $0<\alpha<1$. For every $F:U\to B$ set $\|F\|_\infty:=\sup_{z\in U}\|F(z)\|$ and

$$D_\alpha F:=\sup\{\|F(z)-F(w)\|/|z-w|^\alpha : z,w\in U \text{ and } 0<|z-w|<1\}.$$

If F is strongly differentiable r times, set

$$\|F\|_{r+\alpha}:=\sum_{k=0}^{r}\frac{1}{k!}\|F^{(k)}\|_\infty+\frac{1}{(r+1)!}D_\alpha F^{(r)}.$$

Let $C^{r+\alpha}(U)$ denote the collection of all $F:U\to B$ which are differentiable r-times in U, and for which $\|F\|_{r+\alpha}<\infty$.

Lemma 2. $(C^{r+\alpha}(U),\|\cdot\|_{r+\alpha})$ *is a Banach algebra.*

Proof. The proof the $C^{r+\alpha}(U)$ is complete is done by induction on r using standard techniques. We show that it is a Banach algebra. We need the following two inequalities:

$$\begin{array}{rcll} D_\alpha(FG) & \le & \|F\|_\infty D_\alpha G+D_\alpha F\|G\|_\infty & \text{for all} \quad F,G\in C^\alpha(U),\\ D_\alpha F & \le & \|F'\|_\infty & \text{for all} \quad F\in C^{1+\alpha}(U). \end{array} \tag{3}$$

The first inequality is trivial. The second can be proved as follows: Choose some $z,w\in U$, and set $\gamma(t)=tz+(1-t)w$. Since U is convex $\gamma[0,1]\subseteq U$, so $\|F(z)-F(w)\|=\|\int_\gamma F'\|\le\|F'\|_\infty|z-w|$. The definition of D_α limits us to the consideration of z and w such that $|z-w|<1$. Since $0<\alpha<1$, this implies that $\|F(z)-F(w)\|\le\|F'\|_\infty|z-w|^\alpha$.

Set $c_k(F):=\|F^{(k)}\|_\infty$ for $k=0,\ldots,r$ and $c_{r+1}(F):=D_\alpha F^{(r)}$. Using the Leibnitz derivation formula for $k\le r$ and (3) for $k=r+1$ it is not difficult to check that for all $F,G\in C^{1+\alpha}(U)$ and every $k=0,\ldots,r+1$,

$$c_k(FG)\le\sum_{i=0}^{k}\binom{k}{i}c_i(F)c_{k-i}(G).$$

Therefore,

$$\|FG\|_{r+\alpha} \quad \equiv \quad \sum_{k=0}^{r+1}\frac{c_k(FG)}{k!}\le\sum_{k=0}^{r+1}\sum_{i=0}^{k}\frac{c_i(F)}{i!}\frac{c_{k-i}(G)}{(k-i)!}\le\|F\|_{r+\alpha}\|G\|_{r+\alpha}.$$

as required. ∎

Lemma 3. *Let F_n be a vectors in a Banach space B for which $F(z)=\sum_{n\ge1}z^nF_n$ converges in B for all $|z|<1$, and assume that $F(z)$ is strongly differentiable in $\mathbb{D}$. If $F\in C^{1+\alpha}(\mathbb{D})$ for some $\alpha\in(0,1)$, then $\|F_n\|=O(1/n^{1+\alpha})$.*

Proof. The proof is based on the following well-known result from Fourier analysis ([**Z**], §2.21): Let $f : \mathbb{R} \to \mathbb{C}$ be a continuous function such that $f(\theta) = f(\theta + 2\pi)$ for every $\theta \in \mathbb{R}$, and let $\omega_f(\delta) := \sup\{|f(\theta_1) - f(\theta_2)| : |\theta_1 - \theta_2| < \delta\}$ be its modulus of continuity. Under these assumptions, the Fourier coefficients of f satisfy

$$|\widehat{f}(n)| \leq \frac{1}{2}\omega_f\left(\frac{\pi}{|n|}\right) \text{ for all } n \neq 0.$$

For every $r \in (0,1)$ $\|F_n\| = o(\frac{1}{r^n})$, because $\sum r^n F_n$ converges. Fix some $0 < r < 1$ and let $\varphi \in B^*$ be some arbitrary bounded linear functional with norm $\|\varphi\| = 1$. Define $g_r : \mathbb{R} \to \mathbb{C}$ by $g_r(\theta) = \varphi(F'(re^{i\theta}))$. Since φ is bounded, $g_r(\theta) = \sum_{n\geq 1} n\varphi(F_n)r^{n-1}e^{i(n-1)\theta}$, and this sum converges absolutely and uniformly. Consequently, $\widehat{g}_r(n) = (n+1)\varphi(F_{n+1})r^n$.

Since $\|\varphi\| = 1$, $|g_r(\theta_1) - g_r(\theta_2)| \leq \|F'(re^{i\theta_1}) - F'(re^{i\theta_2})\| \leq Cr^\alpha|\theta_1 - \theta_2|^\alpha$, where C is a constant such that $\|F'(z) - F'(w)\| \leq C|z - w|^\alpha$ for all $z, w \in \mathbb{D}$. Therefore, $\omega_{g_r}(\delta) \leq Cr^\alpha\delta^\alpha$, and so for all $n > 1$,

$$\varphi(F_n) = \frac{\widehat{g}_r(n-1)}{nr^{n-1}} \leq \frac{Cr^\alpha\pi^\alpha}{2(n-1)^\alpha nr^{n-1}} \xrightarrow[r\to 1^-]{} \frac{C\pi^\alpha}{2(n-1)^\alpha n} \leq \frac{1}{2}C\pi^\alpha(n-1)^{-(1+\alpha)}.$$

Passing to the supremum over all bounded linear functionals φ with $\|\varphi\| = 1$ gives $\|F_n\| \leq \frac{1}{2}C\pi^\alpha(n-1)^{-(1+\alpha)} = O(1/n^{1+\alpha})$. $\square$

Set $S(z) := [I - R(z)]/(1 - z)$. The renewal equation implies that

$$S(z)^{-1} = I + \sum_{n=1}^{\infty} z^n(T_n - T_{n-1}).$$

We will show that $S(z)^{-1} \in C^{1+\alpha}(\mathbb{D})$ for some $0 < \alpha < 1$, and the first main lemma will then follow from lemma 3 (see below for details).

Lemma 4. *Set $U_\epsilon := \{z \in \mathbb{D} : \Re(z) < 1 - \epsilon\}$. Under the assumptions of theorem 1, $S(z)^{-1}$ is in $C^{1+\alpha}(U_\epsilon)$ for every $\epsilon > 0$ and all $0 < \alpha < 1$.*

Proof. Fix ϵ and let $\|\cdot\|_\infty$ and $\|\cdot\|_{1+\alpha}$ denote the norms relative to U_ϵ. Our assumptions on $\sum_{k>n}\|R_k\|$ imply that $\sum n^2\|R_n\| < \infty$, so $R(z)$ is continuous in $\overline{\mathbb{D}}$, analytic in $\mathbb{D}$ and $\|R'\|_\infty, \|R''\|_\infty < \infty$.

Let $\rho(z)$ denote the $\mathcal{L}$-spectral radius of $R(z)$. By definition

$$\log\rho(z) = \lim_{n\to\infty}\frac{1}{n}\log\|R(z)^n\|$$

and this is equal to $\inf_{n\geq 1}\frac{1}{n}\log\|R(z)^n\|$, since the sequence $\{\log\|R(z)^n\|\}_{n\geq}$ is sub-additive. This shows that $\rho(z)$ is an upper semi-continuous on $\overline{\mathbb{D}}$, being the infimum of the continuous functions. Upper semi-continuous functions achieve their maximum on compacta so by the aperiodicity assumption of theorem 1,

$$\sup_{z\in\overline{U_\epsilon}}\rho(R(z)) < 1.$$

For every $z \in \overline{U_\epsilon}$ there exists $n = n(z)$ such that $\|R(z)^n\| < \frac{1}{2}$. Since $z \mapsto \|R(z)\|$ is continuous in $\overline{U_\epsilon}$, there is some $r = r(z)$ such that $\|R(w)^n\| < \frac{3}{4}$ for all $w \in B_r(z)$. Since $\overline{U_\epsilon}$ is compact, there exists some N such that for *all* $z \in \overline{U_\epsilon}$, $\|R(z)^N\| < \frac{3}{4}$. Thus, since $\|R(z)\|$ is uniformly bounded on U_ϵ,

$$\|R(z)^n\|_\infty = O(\tau^n) \quad \text{uniformly in } U_\epsilon$$

where $\tau = (\frac{3}{4})^{\frac{1}{N}} \in (0,1)$.

We can now prove the lemma. The sum $\sum R(z)^n$ converges in $\mathcal{L}$ for every $z \in U_\epsilon$, so it is equal to $(I - R(z))^{-1}$ there. Therefore, by the renewal equation, $S(z)^{-1} = (1-z)\sum_{n\geq 1} R(z)^n$. Using the formula $(R^n)' = \sum_{k=0}^{n-1} R^k R' R^{n-k-1}$ and (3) one checks that $\|(R^n)'\|_\infty = O(n\tau^n)$ and $D_\alpha((R^n)') \leq \|(R^n)''\|_\infty = O(n^2\tau^n)$. Therefore, $\|R(z)^n\|_{1+\alpha} = \|R^n\|_\infty + \|(R^n)'\|_\infty + \frac{1}{2}D(R^n)' = O(n^2\tau^n)$. It follows that $\sum R(z)^n$ converges in $C^{1+\alpha}(U_\epsilon)$, and the lemma follows from the completeness of $C^{1+\alpha}(U_\epsilon)$. $\square$

Lemma 5. *Set $V_\epsilon := \{z \in \mathbb{D} : \Re(z) > 1-\epsilon\}$. Under the assumptions of theorem 1, there exists some $0 < \alpha < 1$ and some ϵ such that $S(z)^{-1}$ is in $C^{1+\alpha}(V_\epsilon)$.*

Proof. Set $R_A(z) = R(1) + (z-1)R'(1)$, $D_A := \frac{R-R_A}{1-z}$ and $S_A = \frac{I-R_A}{1-z}$. The proof is based on the following formal identity (later proved to hold in V_ϵ for ϵ small enough):

$$S(z)^{-1} = S_A(z)^{-1}\sum_{k=0}^{\infty}\left(D_A S_A^{-1}\right)^k. \tag{4}$$

We prove that $z \mapsto S_A(z)^{-1}$ has a holomorphic extension in $Hom(\mathcal{L},\mathcal{L})$ to a neighbourhood of 1, and is consequently $C^{1+\alpha}$ there for every $\alpha \in (0,1)$. We then show that $\forall z \in \mathbb{D}, \|D_A(z)\|_{Hom(\mathcal{L},\mathcal{L})} < C|z-1|$ for some $C > 0$. Consequently, there exists some ϵ_1 such that the series in (4) converges in $Hom(\mathcal{L},\mathcal{L})$ for $z \in V_{\epsilon_1}$. Next, we find some $\alpha \in (0,1)$ such that $D_A \in C^{1+\alpha}(\mathbb{D})$ and prove that if $\epsilon < \epsilon_1$ is small enough then $\sum \|(D_A S_A^{-1})^k\|_{C^{1+\alpha}(V_\epsilon)} < \infty$. This implies that the series in (4) converges in $C^{1+\alpha}(V_\epsilon)$ and establishes the lemma.

Step 1. $z \mapsto S_A(z)^{-1}$ has a holomorphic extension in $Hom(\mathcal{L},\mathcal{L})$ to a neighbourhood of 1.

Proof. The assumptions of theorem 1 say that the spectrum of $R(1)$ consists of a a subset of $\{\lambda \in \mathbb{D} : |\lambda| \leq \tau\}$ for some $\tau \in (0,1)$ and a simple isolated eigenvalue at 1 with eigenprojection P. Fix some $\kappa \in (\tau, 1)$. Since $z \mapsto R_A(z)$ is holomorphic, we can use analytic perturbation theory [**Ka**] to deduce that $\exists\delta$ and $P_A : B_\delta(1) \to Hom(\mathcal{L},\mathcal{L})$, $\lambda_A : B_\delta(1) \to \mathbb{C}$ holomorphic such that:

(1) $\lambda_A(z)$ is a simple eigenfunction of $R_A(z)$ with eigenprojection $P_A(z)$;
(2) $|\lambda_A(z)| > \kappa$ and the rest of the spectrum of $R_A(z)$ is contained in $\{\lambda \in \mathbb{C} : |\lambda| \leq \kappa\}$.

Derivate both sides of $R_A P_A = \lambda_A P_A$ and apply P_A on both sides to get $P_A R_A' P_A = \lambda_A' P_A$. Evaluating this at $z = 1$ gives $PR'(1)P = \lambda_A'(1)P$. By assumption $PR'(1)P \neq 0$, so $\mu := \lambda_A'(1) \neq 0$ and

$$\lambda_A(z) = 1 + \mu(z-1) + o(|z-1|) \quad \text{as } z \to 1. \tag{5}$$

This expansion implies that $\lambda_A(z) \neq 1$ for $z \neq 1$, $|z-1| < \delta_0$ and $\delta_0 < \delta$ small enough. Fix such a δ_0. We have the following identity for every $z \neq 1$ in $B_{\delta_0}(1)$:

$$S_A(z)^{-1} \quad = \quad \frac{1-z}{1-\lambda_A(z)}P_A(z) + (1-z)\sum_{n=0}^{\infty} R_A(z)^n\left[I - P_A(z)\right].$$

The sum $\sum R_A^n(I-P_A)$ is convergent in norm and holomorphic in $B_\delta(1)$, because $R_A^n(I-P_A) = [R_A - \lambda_A P_A]^n$ and $\rho(R_A - \lambda_A P_A) \le \kappa < 1$. The term $\frac{1-z}{1-\lambda_A} P_A$ extends to a holomorphic function in $B_{\delta_0}(1)$ because the expansion (5) says that the only zero of $1-\lambda_A(z)$ in $B_{\delta_0}(1)$ is a simple zero located in 1. Step 1 follows.

We note for future reference that this proof actually shows that $S_A(z)^{-1} = \frac{1}{\mu}P + (1-z)A(z)$, where $A(z)$ is holomorphic in $B_{\delta_0}(1)$, P is the eigenprojection of $R(1)$ at 1, and μ is given by $\mu P = PR'(1)P$. We also note for future reference that the only properties of R_A we have used are that it is holomorphic in $\mathbb{C}$ and that $R_A(1) = R(1)$ and $R_A'(1) = R'(1)$.

Step 2. $\|D_A(z)\|_{Hom(\mathcal{L},\mathcal{L})} \le C|z-1|$ for some $C>0$ and all $z \in \mathbb{D}$, and $\exists \alpha \in (0,1)$ such that $D_A(z) \in C^{1+\alpha}(\mathbb{D})$.

Proof. One checks that

$$D_A(z) \quad = \quad (1-z)\sum_{n=1}^{\infty}\left(\sum_{k=0}^{n-1}\frac{1-z^k}{1-z}\right)R_n \quad .$$

If $|z|<1$, then $\left|\sum_{k=0}^{n-1}\frac{1-z^k}{1-z}\right| < \frac{n^2}{2}$, so $\|D_A(z)\|_{Hom(\mathcal{L},\mathcal{L})} \le C|1-z|$ where $C := \sum_{n\ge 1} n^2\|R_n\| < \infty$.

If $g_n(z) = \sum_{k=0}^{n-1}(1-z^k)$, then $D_A(z) = \sum_{n\ge1} g_n(z)R_n$. It is easy to check that if $|z|<1$, then $|g_n(z)| \le 2n$, $|g_n'(z)| \le n^2/2$, and $|g_n''(z)| \le n^3/3$. By the mean value theorem, $\forall z,w \in \mathbb{D}$ $|g_n'(z) - g_n'(w)| \le \min\{n^2, n^3|z-w|\}$. Fix some $\alpha \in (0,1)$. If $n^3|z-w| < 1$, then $|g_n'(z)-g_n'(w)| \le n^3|z-w| \le n^{3\alpha}|z-w|^\alpha$. If $n^3|z-w| \ge 1$, then $|g_n'(z)-g_n'(w)| \le n^2 \le n^{2+3\alpha}|z-w|^\alpha$. It follows that $D_\alpha(g_n'(z)R_n) \le n^{2+3\alpha}\|R_n\|$, whence $\|g_n(z)R_n\|_{1+\alpha} = O(n^{2+3\alpha}\|R_n\|)$.

Since $\sum_{k>n}\|R_k\| = O(1/n^\beta)$, $\sum n^\gamma \|R_n\|$ converges for all $\gamma < \beta$. Therefore, if $0 < \alpha < \frac{\beta-2}{3}$, then $\sum \|g_n R_n\|_{C^{1+\alpha}(\mathbb{D})} < \infty$, so $D_A = \sum g_n R_n \in C^{1+\alpha}(\mathbb{D})$.

Step 3. There exists ϵ such that $\sum \|(D_A S_A^{-1})^k\|_{C^{1+\alpha}(V_\epsilon)}$ converges. Consequently, (4) holds and $S(z)^{-1} \in C^{1+\alpha}(V_\epsilon)$.

Proof. One checks by induction the identity $(F^n)' = \sum_{k=0}^{n-1} F^k F' F^{n-k-1}$. It follows that $\|(F^n)'\|_\infty \le n\|F\|_\infty^{n-1}\|F'\|_\infty$, and by (3), $D_\alpha F^n \le n\|F\|_\infty^{n-1}\|F'\|_\infty$. Using these estimations it is standard to check that $D_\alpha(F^n)' \le n(n+1)\|F\|_\infty^{n-2}\|F\|_{1+\alpha}^2$ for every $n \ge 2$. Consequently,

$$\|F^n\|_{1+\alpha} \le \binom{n+2}{2}\|F\|_\infty^{n-2}\|F\|_{1+\alpha}^2.$$

Choose some ϵ_1 such that S_A^{-1} is holomorphic in $B_{2\epsilon_1}(1)$ and set

$$M := \sup_{z\in B_{\epsilon_1}(1)} \|S_A(z)^{-1}\|_{Hom(\mathcal{L},\mathcal{L})}.$$

For every $0<\epsilon<\epsilon_1$ and $z \in B_\epsilon(1)$, $\|D_A(z)S_A(z)^{-1}\|_{Hom(\mathcal{L},\mathcal{L})} < CM\epsilon$. If $\epsilon < \frac{1}{2MC}$, this is less than $\frac{1}{2}$, and the previous discussion shows that $\|D_A S_A^{-1}\|_{C^{1+\alpha}(V_\epsilon)} = O(\frac{n^2}{2^n})$. Step 3 follows, and with it the lemma. □

Proof of the First Main Lemma. The renewal equation implies that

$$S(z)^{-1} = I + \sum_{n=1}^{\infty} z^n (T_n - T_{n-1}).$$

Lemma 5 shows that $\exists\epsilon$ and $\alpha \in (0,1)$ such that $S(z)^{-1}$ is in $C^{1+\alpha}(V_\epsilon)$. Lemma 4 shows that $S(z)^{-1} \in C^{1+\alpha}(U_{\epsilon/2})$. Since $\mathbb{D} = V_\epsilon \cup U_{\epsilon/2}$, $S(z)^{-1} \in C^{1+\alpha}(\mathbb{D})$. By lemma 3, $\|T_n - T_{n-1}\| = O(1/n^{1+\alpha})$ whence $\sum \|T_n - T_{n-1}\| < \infty$. $\square$

4.2. **Second Main Lemma.** The purpose of this subsection is to prove

Lemma 6 (Second Main Lemma). *Under the assumptions of theorem* 1*, if P is the eigenprojection of $R(1)$ at* 1 *and μ is given by $PR'(1)P = \mu P$, then there exists $R_B : \mathbb{C} \to Hom(\mathcal{L}, \mathcal{L})$ with the following properties:*

1. *R_B is holomorphic, $R_B(1) = R(1)$, and $R_B'(1) = R'(1)$.*
2. *$\frac{R(1)-R_B}{1-z}$ and $\frac{1}{1-z}\left[\frac{R(1)-R_B}{1-z} - R'(1)\right]$ are polynomials in z.*
3. *$I - R_B(z)$ has a bounded inverse in $Hom(\mathcal{L}, \mathcal{L})$ for every $z \in \overline{\mathbb{D}} \setminus \{1\}$.*
4. *$\forall z \in \mathbb{D}$, $\left(\frac{I-R_B}{1-z}\right)^{-1} = \frac{1}{\mu}P + (1-z)\sum_{n\geq 0} z^n A_n$, where $\|A_n\| = O(\kappa^n)$ for some $0 < \kappa < 1$.*

Proof. The first Main Lemma says that $M := 1 + \sum_{n\geq 1}\|T_n - T_{n-1}\|$ is finite. Choose N such that $\sum_{n>N} n\|R_n\| < \frac{1}{2M}$. Set

$$R_B(z) := \sum_{n=1}^{N} z^n R_n + \sum_{n=N+1}^{\infty} R_n + (z-1)\sum_{n=N+1}^{\infty} nR_n.$$

It is clear that R_B is holomorphic, $R_B(1) = R(1)$, and $R_B'(1) = R'(1)$. It is standard to check that $\frac{R(1)-R_B}{1-z}$ and $\frac{1}{1-z}\left[\frac{R(1)-R_B}{1-z} - R'(1)\right]$ are polynomials.

We claim that $S(z)^{-1} = I + \sum_{n\geq 1} z^n(T_n - T_{n-1})$ for all $z \in \overline{\mathbb{D}} \setminus \{1\}$ (and not just for $z \in \mathbb{D}$). Set $F(z) := I + \sum_{n\geq 1} z^n(T_n - T_{n-1})$ and fix some $z_0 \in \overline{\mathbb{D}} \setminus \{1\}$. Let $z_n \in \mathbb{D}$ be some sequence tending to z_0. Since the renewal equation holds in $\mathbb{D}$, $F(z_n)(I - R(z_n)) = (1 - z_n)I$. Since both $F(z)$ and $I - R(z)$ are continuous on $\overline{\mathbb{D}} \setminus \{1\}$, the first by the first main lemma, and the second by the assumption $\sum \|R_n\| < \infty$, we can pass to the limit on both sides of the equation and deduce that $F(z)(I - R(z)) = (1-z)I$. By the same token $(I - R(z))F(z) = (1-z)I$, and consequently $\left(\frac{I-R(z)}{1-z}\right)^{-1} = F(z)$ for every $z \in \overline{\mathbb{D}} \setminus \{1\}$.

It follows that for every $z \in \overline{\mathbb{D}} \setminus \{1\}$, $\left\|\left(\frac{I-R(z)}{1-z}\right)^{-1}\right\| \leq M$. One checks

$$\frac{R - R_B}{1-z} = \sum_{n=N+1}^{\infty}\left(\sum_{k=0}^{n-1}(1 - z^k)\right) R_n,$$

whence $\left\|\frac{R-R_B}{1-z}\right\| \leq 2\sum_{n>N} n\|R_n\| < \frac{1}{M}$. Therefore, $\|(R - R_B)(I - R)^{-1}\| < 1$, which implies that $(I-R)^{-1}[I + (R - R_B)(I - R)^{-1}]^{-1}$ is well-defined and bounded for all $z \in \overline{\mathbb{D}} \setminus \{1\}$. This is the bounded inverse of $I - R_B$.

Set $S_B(z) := \frac{I-R_B}{1-z}$. We show that S_B^{-1} admits the required holomorphic extension. The argument is similar in spirit to the one we used to prove the first main lemma.

Step 1. Every $z_0 \in \overline{\mathbb{D}} \setminus \{1\}$ has a neighbourhood where $S_B(z)^{-1}$ is well-defined and holomorphic.

Proof. Fix some $z_0 \in \overline{\mathbb{D}} \setminus \{1\}$. $S_B(z_0)^{-1}$ is bounded, because $I - R_B$ is invertible. Choose some $\delta = \delta(z_0)$ such that $\|S_B(z) - S_B(z_0)\| < (2\|S_B(z_0)^{-1}\|)^{-1}$ whenever $|z - z_0| < \delta_0$. This is possible, since $z \mapsto S_B(z)$ is continuous in $\mathbb{C} \setminus \{1\}$. If $|z - z_0| < \delta_0$, then $S_B(z)$ is invertible, and its inverse is

$$S_B(z_0)^{-1}[I + (S_B - S_B(z_0))S_B(z_0)^{-1}]^{-1}.$$

It is also clear that $\forall z \in B_{\delta_0}(z_0)$, $\|S_B(z)^{-1}\| \le 2\|S_B(z_0)^{-1}\|$.

We show that $z \mapsto S_B(z)^{-1}$ is holomorphic in $U = B_{\delta_0}(z_0)$. This follows from the following well-known result: if B is a Banach space, $U \subseteq \mathbb{C}$ is open and $F : U \to B$ is holomorphic such $\exists M\ \forall z \in U$, $\|F(z)^{-1}\| < M$, then $z \mapsto F(z)^{-1}$ is holomorphic in U. (The proof follows from the identity $F(z)^{-1} - F(w)^{-1} = -F(w)^{-1}[F(z) - F(w)]F(z)^{-1}$ which implies that $z \mapsto F(z)^{-1}$ is continuous, and that $\frac{1}{h}[F(z+h)^{-1} - F(z)^{-1}]$ converges in norm to $-F(z)F'(z)F(z)$.)

Step 2. $S_B(z)^{-1}$ has a holomorphic extension to some neighbourhood V of $z = 1$, of the form $\frac{1}{\mu}P + (1-z)A(z)$ where $A(z)$ is holomorphic in V.

Proof. The proof is the same as the proof of step 1 in lemma 5 with R_B, S_B, P_B and λ_B replacing R_A, S_A, P_A and λ_A.

Step 3. For every $z \in \mathbb{D}$, $S_B^{-1} = \frac{1}{\mu}P + (1-z)\sum_{n\ge1} z^n A_n$ where $\|A_n\| = O(\kappa^n)$ for some $0 < \kappa < 1$.

Proof. Set $A(z) := \frac{1}{1-z}(S_B^{-1} - \frac{1}{\mu}P)$. Steps 1 and 2 show every $z \in \overline{\mathbb{D}}$ has a neighbourhood where A is holomorphic. Therefore, there exists some $r_0 > 1$ such that $A(z)$ is holomorphic in $\{z \in \mathbb{C} : |z| < r_0\}$. Step 2 follows with $1/r_0 < \kappa < 1$ from the following well-known general result: If B is a Banach space and $F : \{z : |z| < r_0\} \to B$ is holomorphic, then $F(z) = \sum_{n\ge1} z^n F_n$ where $\|F_n\| = O(r^{-n})$ for every $0 < r < r_0$. □

4.3. **Proof of theorem 1.** Let $S = \frac{I-R}{1-z}$ where $R = R(z)$, let R_B be as in the second main lemma, and set $S_B = \frac{I-R_B}{1-z}$. The proof of the theorem is based on the following identity, which holds for every $z \in \mathbb{D}$:

$$T(z) = \frac{1}{1-z}S_B^{-1} + \frac{1}{1-z}S_B^{-1}(S_B - S)S_B^{-1} + \frac{1}{1-z}[S_B^{-1}(S_B - S)]^2 S^{-1}. \tag{6}$$

The idea of the proof is to expand both sides into power series in z, and to equate coefficients. The theorem is then deduced from the following lemma:

Lemma 7. *Under the assumptions of theorem 1, if P is the eigenprojection of $R(1)$ at 1, and μ is given by $PR'(1)P = \mu P$, then*

(1) $\frac{1}{1-z}S_B^{-1} = \frac{1}{\mu}\sum_{n\ge0} z^n(P + \epsilon_n)$ *where $\|\epsilon_n\| = O(\kappa^n)$ for some $0 < \kappa < 1$.*

(2) $\frac{1}{1-z}S_B^{-1}(S_B-S)S_B^{-1}=\frac{1}{\mu^2}\sum_{n\geq 0}z^n(\sum_{k>n}P_k+\epsilon_n')$ *where* $\|\epsilon_n'\|=O(1/n^\beta)$ *and* $P_n=\sum_{\ell>n}PR_\ell P$.
(3) $\frac{1}{1-z}[S_B^{-1}(S_B-S)]^2S^{-1}=\sum_{n\geq 0}z^nE_n$ *where* $\|E_n\|=O(1/n^{\lfloor\beta\rfloor})$.

Proof. We use the following notation, adapted from [**R1**] and [**R2**]. For every sequence of positive real numbers $\{c_n\}_{n\geq 0}$ such that $\limsup_{n\to\infty}\sqrt[k]{c_k}\leq 1$, set

$$\begin{aligned}\Re(\{c_n\}) &:= \{\sum_{n=1}^{\infty}z^nF_n : F_n\in Hom(\mathcal{L},\mathcal{L}) \text{ and } \|F_n\|=O(c_n)\}\\ \Re_\infty &:= \{\sum_{n=1}^{\infty}z^nF_n : F_n\in Hom(\mathcal{L},\mathcal{L}) \text{ and } \sum\|F_n\|<\infty\}.\end{aligned}$$

Abusing notation, we sometime write $\Re(\frac{1}{n^\alpha})$ for $\Re(\{\frac{1}{(n+1)^\alpha}\}_{n\geq 0})$.

Recall that the convolution of two sequences $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 0}$ is the sequence $\{c_n\}=\{a_n\}*\{b_n\}$ given by $c_n=a_0b_n+a_1b_{n-1}+\ldots+a_nb_0$. Set

$$\Re(\{a_n\})*\Re(\{b_n\}):=\{F(z)G(z):F(z)\in\Re(a_n),G(z)\in\Re(b_n)\}.$$

It is easy to check that whenever $\{a_n\}$ and $\{b_n\}$ are non-negative sequences such that $\sum a_n,\sum b_n<\infty$, $\Re(\{a_n\})*\Re(\{b_n\}=\Re(\{a_n\}*\{b_n\})$. Consequently,

$$\begin{array}{lrcl}\forall\alpha_1,\alpha_2>1 & \Re(\frac{1}{n^{\alpha_1}})*\Re(\frac{1}{n^{\alpha_2}}) &=& \Re(1/n^{\min\{\alpha_1,\alpha_2\}})\\ \forall\kappa_1,\kappa_2\in(0,1) & \Re(\kappa_1^n)*\Re(\kappa_2^n) &\subseteq& \Re(n\max\{\kappa_1,\kappa_2\}^n)\end{array}\tag{7}$$

It is also easy to verify that $\Re_\infty*\Re_\infty\subseteq\Re_\infty$.

Step 1. $S(z)^{-1}\in\Re(\frac{1}{n^{\lfloor\beta\rfloor}})$, where β is as in the statement of the theorem.

Proof. The proof is based on the following identity, which holds in $\mathbb{D}$:

$$S^{-1}=S_B^{-1}(I+(S-S_B)S_B^{-1})^{-1}.$$

(To check this, start from the identity $I+(S-S_B)S_B^{-1}=SS_B^{-1}$, recall from the two main lemmas that S and S_B are both invertible in $\mathbb{D}$, and pass to the inverses on both sides.) The second main lemma says that $S_B^{-1}\in\Re(\kappa^n)$ for some $0<\kappa<1$, and consequently $S_B^{-1}\in\Re(\frac{1}{n^{\lfloor\beta\rfloor}})$. It is therefore enough to prove that $(I+(S-S_B)S_B^{-1})^{-1}\in\Re(\frac{1}{n^{\lfloor\beta\rfloor}})$, because (7) will then imply that S^{-1} belongs to $\Re(\kappa^n)*\Re(\frac{1}{n^{\lfloor\beta\rfloor}})=\Re(\frac{1}{n^{\lfloor\beta\rfloor}})$.

We begin by noting that $S-S_B\in\Re(\frac{1}{n^\beta})$. This is because both terms in the representation

$$S-S_B=\frac{R(1)-R}{1-z}-\frac{R(1)-R_B}{1-z},$$

belong to $\Re(\frac{1}{n^\beta})$: The first is equal to $\sum z^k\sum_{n\geq k+1}R_n$ and is in $\Re(\frac{1}{n^\beta})$ by our assumptions; and the second is a polynomial in z by the second main lemma.

Next, we note that $(I+(S-S_B)S_B^{-1})^{-1}\in\Re_\infty$. This is because $S^{-1}\in\Re_\infty$ (first main lemma), so

$$(I+(S-S_B)S_B^{-1})^{-1} \;=\; S_BS^{-1}=I+(S_B-S)S^{-1}\in\Re(\frac{1}{n^\beta})*\Re_\infty\subseteq\Re_\infty.$$

We now appeal to the following general result for Banach spaces:

Let B be a Banach space and suppose that $F(z)=\sum z^kF_k$ where $\|F_k\|=O(\frac{1}{n^\beta})$

for some $\beta > 1$.[1] Suppose further that for every $z \in \mathbb{D}$, $I + F(z)$ is invertible, and that $(I + F(z))^{-1} = \sum z^k G_k$. If $\sum \|G_k\| < \infty$, then $\|G_k\| = O(\frac{1}{k^{\lfloor\beta\rfloor}})$ when $\beta \neq 2$, and $\|G_k\| = O(\frac{\log^2 k}{k^2})$ when $\beta = 2$.

Choose some $p > \min\{\frac{1}{\beta-1}, 1\}$ and let q be its conjugate, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. Set $c_n := \sum_{i+k=n} \|G_i\| \cdot \|G_k\|$.

Equating coefficients in $\left[(I+F)^{-1}\right]' = -(I+F)^{-1}F'(I+F)^{-1}$ gives

$$\begin{aligned} n\|G_n\| &\leq \sum_{i+j+k=n} \|G_i\| \cdot j\|F_j\| \cdot \|G_k\| \equiv \sum_{j=0}^{n} j\|F_j\| c_{n-j} \\ &\leq \left(\sum_{j=0}^{n} j^p \|F_j\|^p\right)^{\frac{1}{p}} \left(\sum_{j=0}^{n} c_{n-j}^q\right)^{\frac{1}{q}} \leq \left(\sum_{j=0}^{\infty} j^p \|F_j\|^p\right)^{\frac{1}{p}} \left(\sum_{j=0}^{\infty} c_n^q\right)^{\frac{1}{q}} \end{aligned}$$

By definition $\sum c_n = (\sum \|G_i\|)^2$, so $\{c_n\}_{n\geq 0}$ is summable, whence $\sum c_n^q < \infty$. By assumption, $j\|F_j\| = O(\frac{1}{n^{\beta-1}})$ and $p > \frac{1}{\beta-1}$, so $\sum j^p\|F_j\|^p < \infty$. Therefore, $n\|G_n\|$ is uniformly bounded, and consequently $\|G_n\| = O(\frac{1}{n})$.

If $\beta \in (1,2)$, we're done. If $\beta = 2$, for every n

$$\begin{aligned} n\|G_n\| &\leq \sum_{0\leq j<\frac{n}{2}} j\|F_j\| c_{n-j} + \sum_{\frac{n}{2}\leq j\leq n} j\|F_j\| c_{n-j} \\ &\leq \max_{0\leq j<\frac{n}{2}} c_{n-j} \cdot \sum_{j=0}^{\lfloor n/2\rfloor} j\|F_j\| + \max_{\frac{n}{2}\leq j\leq n} j\|F_j\| \cdot \sum_{n=0}^{\infty} c_n. \end{aligned}$$

$\{c_n\}$ is the convolution of $\|G_n\|$ with itself, and $\|G_n\| = O(\frac{1}{n})$. Consequently $c_n = O(\frac{\log n}{n})$, so $\|G_n\| = \frac{1}{n}[O(\frac{\log^2 n}{n}) + O(\frac{1}{n})] = O(\frac{\log^2 n}{n^2})$.

Assume $\beta > 2$. We follow Rogozin ([**R2**], page 919). Let $i := \max\{k \in \mathbb{N}\cup\{\infty\} : \|G_n\| = O(\frac{1}{n^i})\}$. This is a proper definition, since we already proved that the range of the maximum is non-empty. For every n

$$\begin{aligned} n\|G_n\| &\leq \sum_{0\leq j<\frac{n}{2}} j\|F_j\| c_{n-j} + \sum_{\frac{n}{2}\leq j\leq n} j\|F_j\| c_{n-j} \\ &\leq \max_{0\leq j<\frac{n}{2}} c_{n-j} \cdot \sum_{j=0}^{\infty} j\|F_j\| + \max_{\frac{n}{2}\leq j\leq n} j\|F_j\| \cdot \sum_{n=0}^{\infty} c_n. \end{aligned}$$

Since[2] $c_n = O(\frac{1}{n^i})$, we have $n\|G_n\| = O(\frac{1}{n^i}) + O(\frac{1}{n^{\beta-1}})$. Therefore $\|G_n\| = O(\frac{1}{n^{i+1}}) + O(\frac{1}{n^\beta})$. If β were larger than or equal to $i+1$, we would have had $\|G_n\| = O(\frac{1}{n^{i+1}})$ in contradiction to the maximality of i. Consequently, $\beta < i+1$, whence $i \geq \lfloor\beta\rfloor$.

Step 2. $\frac{1}{1-z}S_B^{-1} = \frac{1}{\mu}\sum_{n\geq 0} z^n(P + \epsilon_n)$ where P is as in the statement of the lemma and $\|\epsilon_n\| = O(\kappa^n)$ for some $0 < \kappa < 1$.

Proof. This is part (4) in the second main lemma.

[1] We only need this lemma for $\beta > 2$. The case $\beta \in (1,2]$ is treated here for completeness.

[2] because $c_n = \sum_{k+j} \|G_k\| \cdot \|G_j\| \leq 2\max\{\|G_k\| : \frac{n}{2} \leq k \leq n\} \sum \|G_j\|$.

Step 3. $\frac{1}{1-z}S_B^{-1}(S_B - S)S_B^{-1} = \frac{1}{\mu^2}\sum_{n\geq 0} z^n(P_n + \epsilon'_n)$ where P_n are as in the statement of the lemma, and $\|\epsilon'_n\| = O(1/n^\beta)$.

Proof. We write $\frac{1}{1-z}S_B^{-1}(S_B - S)S_B^{-1} = I + II$ where

$$\begin{aligned} I &:= \frac{1}{1-z}S_B^{-1}\left(\frac{R(1)-R_B}{1-z} - R'(1)\right)S_B^{-1} \\ II &:= \frac{1}{1-z}S_B^{-1}\left(R'(1) - \frac{R(1)-R}{1-z}\right)S_B^{-1} \end{aligned}$$

and analyze each of the summands separately.

Analysis of I. The second main lemma says that $\frac{1}{1-z}\left[\frac{R(1)-R_B}{1-z} - R'(1)\right]$ is a polynomial, and that $S_B^{-1} \in \Re(\kappa^n)$ for some $0 < \kappa < 1$. Therefore, by (7), $I \in \Re(\kappa^n)$.

Analysis of II. Set $F(z) = R'(1) - \frac{R(1)-R}{1-z}$. We have already remarked that $\frac{R(1)-R}{1-z} = \sum z^k \sum_{n\geq k+1} R_n$ is in $\Re(\frac{1}{n^\beta})$ and consequently, $F \in \Re(\frac{1}{n^\beta})$ as well. The second main lemma says that $S_B^{-1} = \frac{1}{\mu}P + (1-z)A$ where $A = A(z) \in \Re(\kappa^n)$ for some $0 < \kappa < 1$. Therefore,

$$\begin{aligned} II &= \frac{1}{1-z}\left[\frac{1}{\mu}P + (1-z)A\right]F(z)\left[\frac{1}{\mu}P + (1-z)A\right] \\ &= \frac{1}{\mu^2}P\left(\frac{F}{1-z}\right)P + \frac{1}{\mu}PFA + \frac{1}{\mu}AFP + (1-z)AFA. \end{aligned}$$

It follows that $II \in \frac{1}{\mu^2}P\left(\frac{F(z)}{1-z}\right)P + \Re(\frac{1}{n^\beta})$ (because $\Re(\frac{1}{n^\beta}) * \Re(\kappa^n) = \Re(\frac{1}{n^\beta})$). Now,

$$\begin{aligned} PF(z)P &= PR'(1)P - \sum_{k=0}^\infty z^k\left(\sum_{n=k+1}^\infty PR_nP\right) \\ &= \left(\sum_{n=1}^\infty nPR_nP - \sum_{k=0}^\infty z^k \sum_{n=k+1}^\infty PR_nP\right) \\ &= \sum_{k=0}^\infty (1-z^k)\sum_{n=k+1}^\infty PR_nP \end{aligned}$$

Therefore,

$$II \in \frac{1}{\mu^2}\sum_{n=0}^\infty z^n\left(\sum_{k=n+1}^\infty \sum_{\ell=k+1}^\infty PR_\ell P\right) + \Re(\frac{1}{n^\beta}).$$

This together with $I \in \Re(\kappa^n)$ gives the result.

Step 3. $\frac{1}{1-z}[S_B^{-1}(S_B - S)]^2 S^{-1} = \sum_{n\geq 0} z^n E_n$ where $\|E_n\| = O(1/n^{\lfloor\beta\rfloor})$.

Proof. Set $G = S_B^{-1}(S_B - S)$. We have already seen that $S_B^{-1} \in \Re(\kappa^n)$ and $S_B - S \in \Re(\frac{1}{n^\beta})$. Therefore, in the expansion $G = \sum z^k G_k$, $\|G_k\| = O(\frac{1}{n^\beta})$.

We claim that $\sum G_k = 0$. There exists $M > 0$ such that $\|S_B(z)^{-1}\| < M$ for every $z \in \mathbb{D}$, because $S_B^{-1} \in \Re_\infty$. Therefore, $\forall z \in \mathbb{D}$

$$\|G(z)\| \leq M\|S_B(z) - S(z)\| \leq M \sum_{n=N+1}^{\infty} \left| n - \frac{1-z^n}{1-z} \right| \cdot \|R_n\| \xrightarrow[z\to 1]{} 0$$

where we've used the summability of $\sum n\|R_n\|$ to insert the limit into the sum. It follows that $\sum G_k = 0$.

As a corollary, we get that $\frac{1}{1-z}G(z) = \frac{G(z)-G(1)}{1-z} = -\sum z^n \sum_{k>n} G_k$, and consequently, $\frac{G(z)}{1-z} \in \Re\left(\frac{1}{n^{\beta-1}}\right)$. It is also clear that $G' \in \Re(\frac{1}{n^{\beta-1}})$, and step 1 shows that $(S^{-1})' \in \Re(\frac{1}{n^{\lfloor\beta\rfloor-1}})$. Derivating the identity $E = \frac{1}{1-z}G^2 S^{-1}$ we get

$$\begin{aligned} E' &= \frac{1}{(1-z)^2}G^2S^{-1} + \frac{1}{1-z}(GG' + G'G)S^{-1} + \frac{1}{1-z}G^2(S^{-1})' \\ &= \left(\frac{G}{1-z}\right)^2 S^{-1} + \left[\left(\frac{G}{1-z}\right)G' + G'\left(\frac{G}{1-z}\right)\right]S^{-1} + \left(\frac{G}{1-z}\right)(S^{-1})' \end{aligned}$$

and so $E' \in \Re(\frac{1}{n^{\lfloor\beta\rfloor-1}})$. This implies that $\|E_n\| = O(\frac{1}{n^{\lfloor\beta\rfloor}})$.

The lemma, and with it theorem 1, are proved. $\square$

5. Proof of theorem 2

5.1. **Preliminaries.** It is well known (see e.g. [**A**], proposition 4.2.3) that Markov maps are measure algebra conjugate to countable Markov shifts[3], so it is enough to prove the theorem in the case when (X, T) is a countable Markov shift. We remind the reader that these are defined in the following way: let S be a countable set, $\mathbb{A} = (t_{ij})_{S\times S}$ a matrix of zeroes and ones, $X := \{(x_0, x_1, \ldots) \in S^{\mathbb{N}\cup\{0\}} : t_{x_i,x_{i+1}} = 1 \text{ for all } i \geq 0\}$, and define $T : X \to X$ by $(Tx)_i = x_{i+1}$. The cylinder sets for this map are given by

$$[a_0, \ldots, a_{n-1}] := \{x \in X : x_i = a_i \text{ for all } i = 0, \ldots, n-1\}.$$

Endow X with the topology generated by the base of cylinders, and let $\mathcal{B}$ be the completion of the corresponding Borel σ-algebra.

We call S the collection of *states*, $\mathbb{A}$ the *transition matrix*, and the elements of $\alpha := \{[a] : a \in S\}$ the *partition sets*. We remind the reader that a Markov map admits such a representation with $S = \alpha$ and $\mathbb{A} = (t_{ab})_{S\times S}$ where $t_{ab} = 1$ iff $m(a \cap T^{-n}b) > 0$, the conjugacy being given by $x \mapsto (a_0, a_1, \ldots)$ where a_i are given by $T^i x \in a_i$. Henceforth we assume that (X, T) is a topological Markov shift.

Given a partition set $[a]$, set $\varphi_a(x) := 1_{[a]}(x) \inf\{n \geq 1 : T^n x \in [a]\}$. By our assumptions, $\exists a$ s.t. $g.c.d.\{\varphi_a(x) - \varphi_a(y) : 0 < \varphi_a(x), \varphi_a(y) < \infty\} = 1$, so $g.c.d.\{\varphi_a(x) : 0 < \varphi_a(x) < \infty\} = 1$. This together with the irreducibility assumption implies that $\forall a, b \in S\, \exists N_{ab}$ s.t. $\forall n \geq N_{ab}$, $[a] \cap T^{-n}[b] \neq \varnothing$, or equivalently, that (X, T) is topologically mixing.

The induced map $T_a = T^{\varphi_a}$ on $\{x \in [a] : x_i = a \text{ infinitely often }\}$ is again a countable Markov shift and can be coded symbolically as follows: Set

$$\overline{S} := \{[a, \xi_1, \ldots, \xi_{n-1}] : [a, \xi_1, \ldots, \xi_{n-1}, a] \neq \varnothing, \text{ and } \forall i, \xi_i \neq a\},$$

[3] Actually, since we assumed that $(X, \mathcal{B}, m)$ is a Lebesgue space, this conjugacy induces a measure theoretic isomorphism.

let $\overline{X} := \overline{S}^{\mathbb{N}\cup\{0\}}$ and let $\overline{T} : \overline{X} \to \overline{X}$ be the left shift. The countable Markov shift $(\overline{X}, \overline{T})$ is a natural symbolic representation of the induced map T^{φ_a}. Indeed, if $\pi : \overline{X} \to [a]$ is $\pi([\underline{a}_1], [\underline{a}_2], \ldots) := (\underline{a}_1, \underline{a}_2, \ldots)$, then $T^{\varphi_a} \circ \pi = \pi \circ \overline{T}$.

Fix some probability measure m as in the statement of the theorem 2, and let $[a]$ be a partition set such that $\log g_{m_a}$ has a (T_a, α_a)-locally Hölder continuous version. Let $\theta \in (0,1)$ be the exponent of Hölder continuity. Note that if $\phi := \log g_m$, then $\log g_{m_a} = \sum_{i=0}^{\varphi_a - 1} \phi \circ T^i$ and the (T_a, α_a)-local Hölder continuity of $\log g_{m_a}$ is equivalent to saying that

$$\overline{\phi} := \left(\sum_{i=0}^{\varphi_a - 1} \phi \circ T^i \right) \circ \pi$$

is a $(\overline{T}, \overline{\alpha})$-locally Hölder continuous where $\overline{\alpha} = \{[[\underline{a}]] : [\underline{a}] \in \overline{S}\}$.

Note that in this notation $D_a f := Lip_\theta(f \circ \pi)$ where

$$Lip_\theta(f) := \sup_{\overline{x} \neq \overline{y}} |f(\overline{x}) - f(\overline{y})| / \theta^{\min\{i : \overline{x}_i \neq \overline{y}_i\}}.$$

By Poincare's recurrence theorem, $m\big([a] \setminus \pi(\overline{X})\big) = 0$. Therefore, for our purposes there is no damage in redefining $\mathcal{L}$ to be the Banach space

$$\mathcal{L} := \{f : \pi(\overline{X}) \to \mathbb{C} : \|f\|_{\mathcal{L}} := \|f\|_\infty + Lip_\theta(f \circ \pi) < \infty\}.$$

Set $\phi_n = \phi + \phi \circ T + \ldots \phi \circ T^{n-1}$. For every $[\underline{b}] = [b_0, \ldots, b_{n-1}]$ such that $b_0 = a$ and $[\underline{b}, a] \neq \varnothing$ define an operator acting on $\mathcal{L}$ by $(M_{\underline{b}} f)(x) = e^{\phi_n(\underline{b}, x)} f(\underline{b}, x)$, where n is the length of $\underline{b}$. This is well defined, because $x_0 = a$. We need the following norm estimate:

Lemma 8. *There exist $B, K > 0$ independent of the choice of $\underline{b}$ such that for all $f \in \mathcal{L}$, $\|M_{\underline{b}} f\|_{\mathcal{L}} \leq B m[\underline{b}, a] \left(\theta^k \|f\|_{\mathcal{L}} + \frac{1}{m[\underline{b}, a]} \int_{[\underline{b}, a]} |f| dm \right)$ where k is the number of times a appears in $\underline{b}$.*

Proof. Write $\underline{b} = (\underline{b}_0, \ldots, \underline{b}_{k-1})$, $x = \pi([\underline{x}_1], [\underline{x}_2], \ldots)$ and $y = \pi([\underline{y}_1], [\underline{y}_2], \ldots)$ where $[\underline{b}_i], [\underline{x}_i], [\underline{y}_i] \in \overline{S}$. Since $\overline{\phi} \equiv \phi_{\varphi_a} \circ \pi$ is locally Hölder continuous, there exist some A such that $v_n(\overline{\phi}) < A\theta^n$ for all $n \geq 1$ (where v_n are the $(\overline{T}, \overline{\alpha})$-variations). Therefore, for $\overline{t}(x, y) = \inf\{i \geq 0 : \underline{x}_i \neq \underline{y}_i\}$ and $\overline{\phi}_k = \overline{\phi} + \overline{\phi} \circ \overline{T} + \ldots + \overline{\phi} \circ \overline{T}^{k-1}$,

$$\begin{aligned} |\phi_n(\underline{b}, x) - \phi_n(\underline{b}, y)| &= \big|\overline{\phi}_k\big([\underline{b}_0], \ldots, [\underline{b}_{k-1}], [\underline{x}_1], \ldots\big) - \overline{\phi}_k\big([\underline{b}_0], \ldots, [\underline{b}_{k-1}], [\underline{y}_1], \ldots\big)\big| \\ &\leq A(\theta^{k + \overline{t}(x,y)} + \ldots \theta^{\overline{t}(x,y)}) \leq \frac{A\theta^{\overline{t}(x,y)}}{1 - \theta}. \end{aligned}$$

Choose M such that $\left|e^{\pm A\theta^i/(1-\theta)} - 1\right| \leq M\theta^i$ for all $i \geq 1$, and set $K = M + 1$. A standard argument shows that

$$\|M_{\underline{b}} f\|_{\mathcal{L}} \leq \|e^{\phi_n(\underline{b}, x)}\|_\infty \left(\theta^k \|f\|_{\mathcal{L}} + K \sup_{y \in [\underline{b}, a]} |f(y)| \right).$$

If $B_1 = \exp \frac{A}{1-\theta}$, then $\|e^{\phi_n(\underline{b}, x)}\|_\infty \leq B_1 R(1)^k 1_{[\underline{b}, a]}$. Integrating this over $[a]$ gives $\|e^{\phi_n(\underline{b}, x)}\|_\infty \leq B_1 \frac{m[\underline{b}, a]}{m[a]}$. A similar argument shows that $\sup_{y \in [\underline{b}, a]} |f(y)| \leq \frac{1}{m[\underline{b}, a]} \int_{[\underline{b}, a]} |f| dm + \|f\|_{\mathcal{L}} \theta^k$. The lemma follows with $B = (K+1)B_1/m[a]$. □

5.2. **The proof.** Let $L_\phi f(x) = \sum_{Ty=x} e^{\phi(y)} f(y)$ be *Ruelle's operator* [**Rue**].[4] Set

$$T_n f := 1_{[a]} L_\phi^n (f 1_{[a]}) \ \text{ and } \ R_n f := 1_{[a]} L_\phi^n (f 1_{[\varphi_a = n]}).$$

Step 1 (Renewal equation). T_n and R_n are bounded linear operators on $\mathcal{L}$, $\|T_n\| = O(1)$, $\sum \|R_n\| < \infty$ and $T(z) = (I - R(z))^{-1}$.

Proof. Let $\Lambda_n := \{(b_0, \dots, b_{n-1}) : [b_0, \dots, b_{n-1}] \in \overline{S}\}$. By definition, $[\varphi_a = n] = \biguplus_{\underline{b} \in \Lambda_n} [\underline{b}, a]$ and $R_n = \sum_{\underline{b} \in \Lambda_n} M_{\underline{b}}$. Lemma 8 now shows that

$$\|R_n\| \leq B(1+\theta) m[\varphi_a = n]. \tag{8}$$

It follows that $R_n \in Hom(\mathcal{L}, \mathcal{L})$ and that $\sum \|R_n\| < \infty$. The boundness of T_n is proved in a similar way, by showing that $\|T_n\| = O\big(m([a] \cap T^{-n}[a])\big) = O(1)$.

To see that the renewal equation holds note that every $f \in \mathcal{L}$ is absolutely integrable, so by proposition 1 for every $z \in \mathbb{D}$ $T(z)\big(I - R(z)\big) f = f$ m-almost everywhere. Since both sides of this equation are continuous functions on $\pi(\overline{X})$, and since m is positive on every cylinder, this equality must be true everywhere in $\pi(\overline{X})$, whence $T(z)\big(I - R(z)\big) = I$. The identity $\big(I - R(z)\big) T(z) = I$ is proved in a similar way.

Step 2 (Spectral Gap). The operator $R(1) : \mathcal{L} \to \mathcal{L}$ satisfies:

(1) $R(1)$ has a simple eigenvalue at 1, and its eigenprojection is the operator P_a given by $P_a f \equiv \left(\frac{1}{m[a]} \int_{[a]} f dm \right)$;
(2) The spectral radius of $R(1) - P_a$ is strictly smaller than one.

Proof. Let $([a], \mathcal{B} \cap [a], m_a, T_a)$ be the induced map of T on $[a]$. This system is measure preserving, since m is T-invariant (see e.g. [**A**]). It follows that $R(1)1 \equiv 1$ whence $R(1)P_a = P_a R(1) = P_a$. It follows that 1 is an eigenvalue of $R(1)$. The remaining spectral properties (simplicity, the identification of P_a as the eigenprojection, and the spectral gap) will follow once we prove that the spectral radius of $R(1) - P_a$ is strictly less than one.

The collection $\alpha_a := \{[\underline{b}, a] : \underline{b} \in \overline{S}\}$ is a Markov partition for this system, and m_a is a Gibbs measure with respect to this partition. Indeed, there exists some $M > 1$ such that for all (T_a, α_a)-cylinders $\bigcap_{i=0}^{n-1} T_a^{-i} [\underline{a}_i]$ and for all $x \in \bigcap_{i=0}^{n-1} T_a^{-i} [\underline{a}_i]$

$$M^{-1} \exp \sum_{k=0}^{n-1} \psi(T_a^k x) \leq m_a \left(\bigcap_{i=0}^{n-1} T_a^{-i} [\underline{a}_i] \right) \leq M \exp \sum_{k=0}^{n-1} \psi(T_a^k x) \tag{9}$$

where $\psi(x) = \sum_{i=0}^{\varphi_a(x)-1} \phi(T^i x)$. By assumption, ψ is (T_a, α_a)-locally Hölder continuous. To prove (9), use $L_\psi^* m_a = m_a$ to prove that $m[\underline{b}, a] \asymp e^{\psi_n(x)}$ for every (T_a, α_a)-cylinder $[\underline{b}] \subset [a]$ of length n which contains x.

The symbolic representation of T_a with respect to the partition $\overline{S}$ is that of a full shift (i.e. all entries of the corresponding transition matrix are equal to one), so $T[\overline{a}] = \overline{X}$ for every $\overline{a} \in \overline{S}$. Therefore, since m_a is a Gibbs measure with respect

[4] The difference between L_ϕ and $\widehat{T}$ is that L_ϕ acts on functions, whereas $\widehat{T}$ acts on L^1 equivalence classes of functions.

to T_a, theorem 4.7.7 in [**A**] applies, so $\exists C > 0$ and $0 < \kappa < 1$ such that $\forall f \in \mathcal{L}$, $\|\widehat{T}_a^n f - \int f dm_a\|_{\mathcal{L}} < C\|f\|_{\mathcal{L}}\kappa^n$, where $\widehat{T}_a$ is the transfer operator of T_a.[5]

Step 3 (Aperiodicity). If $g.c.d.\{\varphi_a(x) - \varphi_a(y) : 0 < \varphi_a(x), \varphi_a(y) < \infty\} = 1$, then the spectral radius of $R(z) : \mathcal{L} \to \mathcal{L}$ is strictly less than one for all $z \in \overline{\mathbb{D}} \setminus \{1\}$.

Proof. Lemma 8 can be used to show that for all $f \in \mathcal{L}$ and $k \geq 1$,

$$\|R(z)^k f\|_{\mathcal{L}} \leq B|z|^k \left(\theta^k \|f\|_{\mathcal{L}} + \|f\|_1\right) \leq 2B|z|^k \|f\|_{\mathcal{L}} \quad . \tag{10}$$

Therefore $\rho(R(z)) \leq 1$ for all $z \in \overline{\mathbb{D}}$ and $\rho(R(z)) < 1$ when $|z| < 1$. We show that $\rho(z) < 1$ for every $|z| = 1$, $z \neq 1$.

Let $(\Omega, \mathcal{C}, \mu, S)$ be some probability preserving transformation, and let $\mathbb{S}^1 := \{z \in \mathbb{C} : |z| = 1\}$. Recall that $\varphi : \Omega \to \mathbb{R}$ is called S-*aperiodic*, if the only $t \in [0, 2\pi)$ for which there exist $\lambda_0 \in \mathbb{S}^1$ and a measurable function $g : \Omega \to \mathbb{S}^1$ such that $e^{it\varphi} = \lambda_0 g/g \circ S$ almost everywhere is $t = 0$ (see [**Gu**]).

We claim that φ_a is T_a-aperiodic. Assume by way of contradiction that $e^{it\varphi_a} = \lambda_0 g/g \circ T_a$ where $g : [a] \to \mathbb{S}^1$ is measurable, $\lambda_0 \in \mathbb{S}^1$, and $t \in [0, 2\pi)$ is different than zero. The system $([a], \mathcal{B} \cap [a], m_a, T_a)$ is an Gibbs-Markov system in the sense of [**AD**] with respect to the partition α_a. It is clear that φ_a is α_a-measurable, so the conditions of theorem 3.1 in [**AD**] (see also [**Ko**]) apply, and consequently g must be $(\alpha_a)_*$-measurable, where $(\alpha_a)_*$ is the finest partition with the property that for every $A \in \alpha_a$, $T_a A$ is contained in some atom of $(\alpha_a)_*$.

Since $T_a A = [a]$ for every $A \in \alpha_a$, we have that $(\alpha_a)_* = \{[a], \varnothing\}$, and consequently g is constant almost everywhere. It follows that $e^{it\varphi_a} = \lambda_0$ almost everywhere. This identity actually holds *everywhere* on $E_0 = \{x : \varphi_a(x) \in (0, \infty)\}$, because φ_a is continuous on E_0, being locally constant there. It follows that $e^{it[\varphi_a(x) - \varphi_a(y)]} = 1$ for all $(x, y) \in E_0 \times E_0$. This, however, contradicts $t \neq 0$, since $g.c.d.\{\varphi_a(x) - \varphi_a(y) : x, y \in E_0\} = 1$.

Assume that $z = e^{it}$, $0 < t < 2\pi$. Assume by way of contradiction that $\rho(R(z)) = 1$. There exist $f_n \in \mathcal{L}$ of norm $\|f_n\|_{\mathcal{L}} = 1$, such that $\|R(z)^n f_n\|_{\mathcal{L}} \geq \frac{1}{2}$. Note that (10) implies that $\liminf_{n\to\infty} \|f_n\|_1 > 0$.

The family $\{f_n : n \geq 1\}$ is equicontinuous and uniformly bounded, so by the Arzela-Ascoli theorem, there is a subsequence $\{f_{n_k}\}$ and a function $f \in \mathcal{L}$ such that $f_{n_k}(x) \to f(x)$ for every x. Since $m[a] < \infty$ and since $\|f_{n_k}\|_\infty \leq 1$, $f_{n_k} \to f$ in $L^1(m|_{[a]})$. Since $\|f_n\|_1 \not\to 0$, $\|f\|_1 > 0$.

We now proceed as in the proof of theorem 4.5 in [**PP**]. Consider the operator $V : L^\infty[a] \to L^\infty[a]$ given by $Vw = e^{it\varphi} w \circ T_a$, and set $E := \bigcap_{n\geq 1} V^n(L^2[a])$. Let $g_n := sgn[R(z)^n f_n]$, where $sgn(0) := 0$. E is closed under weak convergence in L^2, and therefore contains all weak L^2-limit points of $\{g_{n_k}\}_{k\geq 1}$. These limit points are not trivial, because their inner product with f is non-zero, since

$$\begin{aligned} 0 \not\leftarrow \int g_{n_k} R(z)^{n_k} f_{n_k} &= \int g_{n_k} R(1)^{n_k} (e^{it\varphi_{n_k}} f_{n_k}) \\ &= \int V^{n_k} g_{n_k} f_{n_k} = \int V^{n_k} g_{n_k} f + o(1), \qquad (k \to \infty). \end{aligned}$$

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[5]Theorem 4.7.7 in [**A**] is formulated in terms of another norm $\|\cdot\|_L$, but this norm satisfies $\|\cdot\|_L \leq \|\cdot\|_{\mathcal{L}} \leq 2\|\cdot\|_L$ because of the remark on line 9 of page 165 there.

This shows that $E \neq 0$. Choose some $w \in E \setminus \{0\}$. By definition, $Vw \in E$ as well, so there exist w_n and w'_n measurable such that $w = e^{it\varphi_n} w_n \circ T_a^n$ and $Vw = e^{it\varphi_n} w'_n \circ T_a^n$.

By assumption $\log g_{m_a}$ is locally Hölder continuous, so by theorem 4.4.7 in [**A**], m_a is T_a-exact. The exactness of T_a means that $|w|$ is constant almost everywhere, so Vw/w is well defined a.e. and equal to $(w'_n/w_n) \circ T_a^n$. This means that Vw/w is $\bigcap_{n\geq 1} T_a^{-n}(\mathcal{B} \cap [a])$-invariant, so it must be constant a.e., whence $Vw = \lambda w$ for some λ. This implies that $e^{it\varphi} = \lambda w / w \circ T_a$ almost everywhere. Passing to absolute values we see that $|\lambda| = |w| \equiv 1$. But φ is T_a-aperiodic, so $t = 0$. We arrive at a contradiction, since $0 < t < 2\pi$.

Step 4. (1) holds.

Proof. By (8), $\|R_n\| = O(m[\varphi_a = n])$ and consequently, $\sum_{k>n} \|R_k\| = O(1/n^\beta)$ for some $\beta > 2$. Step 2 says that $Pf = \frac{1}{m[a]} \int_{[a]} f dm \cdot 1_{[a]}$. It is not difficult to use this to check that, $PR_nP = \frac{m[\varphi_a = n]}{m[a]} P$, and consequently $PR'(1)P = \frac{1}{m[a]} \int_{[a]} \varphi_a dm \cdot P = \frac{1}{m[a]} P$ by the Kac formula. Consequently $\mu = \frac{1}{m[a]} \neq 0$. This, together with the previous lemmas, shows that the conditions of theorem 1 are satisfied, and consequently there exist $E_n \in Hom(\mathcal{L}, \mathcal{L})$ with norms $\|E_n\| = O(1/n^{\lfloor\beta\rfloor})$ such that for all $f \in \mathcal{L}$ (recall that such functions are supported inside $[a]$):

$$1_{[a]} \widehat{T}^n f = 1_{[a]} \left(\int f dm + \sum_{k=n+1}^{\infty} m[\varphi_a > n] \int f dm + E_n f \right)$$

Multiplying by an arbitrary $g \in L^\infty(X, \mathcal{B}, m)$ which is supported inside $[a]$, we have by the definition of the transfer operator

$$\int fg \circ T^n dm = \int f \int g + \sum_{k=n+1}^{\infty} m[\varphi_a > k] \int f \int g + \int g E_n f dm$$

Since $\|\cdot\|_\infty \leq \|\cdot\|_{\mathcal{L}}$, the absolute value of the last term is bounded by $\|g\|_\infty \|f\|_{\mathcal{L}} \|E\|_{\mathcal{L}}$ so the theorem is proved. □

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Department of Mathematics, University of Warwick, Coventry CV4 7AL, England
E-mail address: `sarig@maths.warwick.ac.uk`