TEMPORAL DISTRIBUTIONAL LIMIT THEOREMS FOR DYNAMICAL SYSTEMS

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ABSTRACT. Suppose $\{T^t\}$ is a Borel flow on a complete separable metric space X, $f : X \to \mathbb{R}$ is Borel, and $x \in X$. A *temporal distributional limit theorem* is a scaling limit for the distributions of the random variables $X_T := \int_0^t f(T^s x) ds$, where t is chosen randomly uniformly from $[0, T]$, x is fixed, and $T \to \infty$. We discuss such laws for irrational rotations, Anosov flows, and horocycle flows.

Dedicated to D. Ruelle and Y. Sinai on the occasion of their 80th birthdays

1. INTRODUCTION

A surprising discovery in the theory of dynamical systems made in the last century is that ergodic sums of deterministic systems can satisfy the same limit theorems as sums of independent random variables. Ergodic dynamical systems satisfy the strong law of large numbers, and hyperbolic dynamical systems (often) satisfy spatial distributional limit theorems analogous to the central limit theorem.

We recall the definitions. Let T be a Borel map on a complete separable metric space X, and let f be a real valued Borel function on X. The *ergodic sums* of f along the orbit of x are

$$S(n, x) := \sum_{k=0}^{n-1} f(T^k x).$$

Similarly, in continuous time, let $\{T^t\}_{t\in\mathbb{R}}$ be a Borel flow on a complete separable metric space X, and fix a Borel function $f : X \to \mathbb{R}$. The *ergodic integrals* along the orbit of x are

$$I(t, x) := \int_0^t f(T^s x) ds,$$

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whenever the integral makes sense (as will be the case for every x when f is bounded, or for a.e. x when f is absolutely integrable with respect to an invariant measure).

If T (or T^t) preserves an ergodic probability measure μ and $f \in L^1(\mu)$, then the ergodic theorem says that for μ-a.e. x, $S(n,x)/n \to \mu(f)$ (respectively, $I(t,x)/t \to \mu(f)$).

Equivalently, for a.e. x, $S(n,x) = n\mu(f)+o(n)$, $I(t,x) = t\mu(f)+o(t)$. In this paper, we study the behavior of the little oh term.

1.1. **Spatial DLT.** Motivated by the analogy between the ergodic theorem and the strong law of large numbers, it is natural to look for dynamical analogues of the central limit theorem. This leads to the following classical definition:

Definition 1.1. *The ergodic sums of f satisfy a* spatial distributional limit theorem (Spatial DLT) *on a probability space $(X, \mathscr{B}, \mu)$, if there are constants $A_N \in \mathbb{R}, B_N \to \infty$ and a non-constant random variable Y such that for all $a \in \mathbb{R}$ s.t.* $\text{Prob}(Y = a) = 0$,

$$\mu\{x \in X : \frac{S(N,x) - A_N}{B_N} < a\} \xrightarrow[N\to\infty]{} \text{Prob}(Y < a).$$

Equivalently, the random variables $\widehat{X}_N(x) := \frac{S(N,x)-A_N}{B_N}$, obtained by fixing N and choosing x randomly in X according to the measure μ, *converge in distribution* to Y. We write $\frac{S(N,x)-A_N}{B_N} \xrightarrow[N\to\infty]{\text{dist}} Y$ *as* $x \sim \mu$. A similar definition can be made for flows.

There is a vast literature on spatial DLTs for hyperbolic dynamical systems. In many cases one can show that $A_N = N\mu(f)$, $B_N \sim \text{const}\sqrt{N}$, and Y is Gaussian. This goes back to the work of Sinai for the geodesic flow in constant negative curvature [63], followed by the works of Ruelle, Bowen, and Ratner for more general classes of uniformly hyperbolic systems ([8, 9, 10, 57, 59, 64, 65]). For results on other systems, and discussion of different methods of proving spatial DLTs for dynamical systems, see [1, 18, 23, 37, 49].

The case of systems with zero entropy is much less understood. It is clear that the spatial DLTs need not hold in general. The simplest example is the following. Let T be an irrational rotation of the circle $\mathbb{R}/\mathbb{Z}$ and f the function equal to $+1$ on $[0, \frac{1}{2})$ and to (-1) on $[\frac{1}{2}, 1)$. Since f has bounded variation, Denjoy-Koksma inequality [20] implies that there is a subsequence $n_j \to \infty$ such that $\|S(n_j, x)\|_\infty \leq 2$, so the distributional limit points $\frac{S(n_j,x)-A_j}{B_j}$ must be atomic for any sequence $\{A_j, B_j\}$. On the other hand by [42] the Gaussian distribution is also a limit point, so there are several limiting distributions, and the limit does not exist. (Some functions with infinite variation do satisfy a Gaussian Spatial DLT [15, 44]. In fact, for any aperiodic map T any Y can appear as a limiting distribution of $S_f(n,x)/B_n$ for a suitable Borel function f ([69]) so to limit the class of possible limit laws one usually considers more regular functions.) Other examples where the spatial DLT does not hold are certain translation flows [14], horocycle flows ([13], see also Section 5 of the present paper), and random walks in random environment [66, 36, 25].

1.2. **Temporal DLT.** The failure of the spatial DLT for natural dynamical systems of low complexity suggests looking for other types of limit theorems which may hold for such systems.

In fact even if the spatial DLT does exist, there are good reasons to look for other types of asymptotic results. For the spatial DLT to be relevant there should be a possibility to sample a large ensemble of initial conditions. However, in many situations only one orbit of a dynamical systems is observed. Therefore, it is of interest to analyze the stochastic behavior of *a single trajectory* generated by a *specific* initial condition. This problem is an example of the research direction known as *single orbit dynamics* [70]. Single Orbit Limit Theorems is the main subject of the present paper.

The sequence of values of ergodic sums $\{S(n,x)\}_{n\geq 1}$ can be very oscillatory: if $T_f(x,t) = (T(x), t + f(x))$ is ergodic conservative infinite measure preserving map on $(X \times \mathbb{R}, \mathscr{B}(X \times \mathbb{R}), \mu \times dt)$ for some T-invariant probability measure μ, then for μ-a.e. x, $S(n,x)$ will visit the neighborhood of every real number infinitely many times.

Highly oscillatory sequences appear in number theory, and number theorists designed ingenious tools to describe their behavior [68]. It is natural to try to use these tools in the dynamical setup. The following definition is motivated by the Erdös-Kac Theorem [31] on the number theoretic sequence $\omega(n) := \#\{p|n : p \text{ is prime}\}$.

Definition 1.2. *The ergodic sums of f satisfy a* temporal distributional limit theorem (Temporal DLT) *on the orbit of x if there are sequences $A_N(x) \in \mathbb{R}, B_N(x) \to \infty$ and a non-constant random variable Y such that for all $a \in \mathbb{R}$ s.t.* $\mathrm{Prob}(Y = a) = 0$*,*

$$\frac{1}{N}\#\{0 \leq n \leq N-1 : \frac{S(n,x) - A_N(x)}{B_N(x)} < a\} \xrightarrow[N\to\infty]{} \mathrm{Prob}(Y < a).$$

This is equivalent to the convergence in distribution to Y of the random variables $X_N(n) := \frac{S(n,x)-A_N}{B_N}$, obtained *by fixing* x and *choosing* $1 \leq$

$n \le N$ randomly uniformly. We write: $\frac{S(n,x)-A_N}{B_N} \xrightarrow[N\to\infty]{\text{dist}} Y$ *as* $n \sim U\{1,\dots,N\}$.

Definition 1.3. *The ergodic integrals of f satisfy a* temporal DLT *on the orbit of x if there are a real valued random variable non-constant Y and two Borel functions $A_T(x) : (0,\infty) \to \mathbb{R}$, $B_T(x) : (0,\infty) \to (0,\infty)$ with $B(T) \xrightarrow[T\to\infty]{} \infty$ s.t. for every a with* $\text{Prob}(Y = a) = 0$,

$$\frac{1}{T}\textit{Lebesgue}\{0 < t < T : \frac{I(t,x) - A_T(x)}{B_T(x)} < a\} \xrightarrow[T\to\infty]{} \text{Prob}[Y < a].$$

We write $\frac{I(t,x)-A_T}{B_T} \xrightarrow[T\to\infty]{\text{dist}} Y$, as $t \sim U[0,T]$. Here and throughout, $U[a,b]$ denotes the uniform distribution on $[a,b]$.

Temporal DLTs are quite different from spatial DLTs. In a temporal DLT, we fix the initial condition x and randomize the "time" n, and in a spatial DLT we fix the time N and randomize the initial condition x. A temporal DLT is a statement in single orbit dynamics; it describes the oscillations of the ergodic sums along a specific orbit. A spatial DLT is a statement on the diversity of the possible behaviors of $S(n,x)$ for different x. It says nothing on the behavior for a specific x.

To better compare the two types of DLT, it is convenient to introduce the following subtle, but significant variation on definition 1.2:

Definition 1.4. *Let T be a measurable map of a probability space $(X, \mathscr{B}, \mu)$ and f be a measurable function. We say that (T,μ,f) satisfies* strong temporal DLT *for μ-a.e. x, if for μ–a.e. x, $\{S(n,x)\}$ satisfies the temporal DLT and the normalizing sequence B_N can be chosen to be independent of x.*

The centering constants $A_N(x)$ remain free to depend on x. The asymmetry between A_N and B_N is justified by examples, see below.

We shall see below that there are T, f, μ with strong temporal DLT for μ-a.e. x which do not satisfy a spatial DLT with $x \sim \mu$, and there are T, f with spatial DLT for $x \sim \mu$ for which there is no strong temporal DLT for μ-a.e. x.

The term *strong* in the above definition refers to the fact that we have less freedom in the choice of normalization than in Definition 1.2. However, the strong DLT does not tells us anything for a fixed x, it only applies to a.e. x.

All temporal DLTs discussed in this paper will be strong DLTs, and they will hold for *all* x, not just almost everywhere. In each of these theorems, the centering constants A_N will depend on x.

1.3. **Almost sure temporal DLT.** In our definition of temporal DLT we require that the time is uniformly distributed on a large interval. This is because we want to capture behavior of ergodic sums or integrals at a typical moment of time. But from the point of view of a single orbit dynamics there is no reason to prefer uniform distribution over other distributions. One common choice is described below.

We discuss flows, but the case of maps is similar. Let $\mathrm{Log}[1,T]$ denote the distribution with probability density $\frac{1}{\ln T}\frac{dt}{t}$ on $[1,T]$.

Definition 1.5. [12, 33, 60] *The Ergodic integrals of f satisfy an* almost sure DLT *on a probability space $(X,\mathscr{B},\mu)$, if there are functions $A(t), B(t)$, and a non-constant random variable Y s.t. for μ-a.e. x,*

$$\frac{I(t,x)-A(t)}{B(t)} \xrightarrow[T\to\infty]{dist} Y, \text{ as } t\sim \mathrm{Log}[1,T].$$

The temporal DLT and the almost sure DLT are both single orbit results in the sense that the distribution of Y can be ascertained from looking at a single orbit, but there are two important differences which we would like to emphasize:

(1) In the temporal DLT, we scale by *deterministic* constants A_T, B_T, whereas in the almost sure case we normalize by *random variables* $A(t)$ and $B(t)$ ($t\sim \mathrm{Log}[1,T]$).
(2) In the temporal DLT the scaling constants are allowed to depend on x (and, in fact, the centering terms A_N depend on x in all the examples we know). But in the almost sure DLT $A(t)$ and $B(t)$ are the same for almost all orbits (otherwise we would have too much freedom and can get rid of $I(t,x)$ completely).

We refer the reader to [17] for a comprehensive list of sufficient conditions for almost sure DLT with stable laws as limiting variables. These results go in the direction of showing that if the dynamical system is sufficiently mixing then an almost sure DLT holds. However, the mixing required here is much weaker than usual. Namely, usually it is required that $T^{t_1}x$ and $T^{t_2}x$ are weakly dependent if $|t_1-t_2|$ is large, but the almost sure DLT seems only to require that $T^{t_1}x$ and $T^{t_2}x$ are weakly dependent if t_1 and t_2 are of different orders of magnitude. This opens the way for some zero entropy systems to satisfy such laws [13, 50]. In fact, it seems not obvious how to construct a system where the almost sure DLT does not hold. We will see later that circle rotations and horocycle windings are such systems.

In the present paper we discuss the three types of the limit theorems described above (spatial DLT, temporal DLT, and almost sure DLT) for three classical examples of smooth systems: Anosov systems, circle rotations, whose rotation number is quadratic irrational, and horocycle flows.

The reader may notice that the orbits of the linear flow considered in section 4 (as a tool for analyzing irrational rotations), and the orbits of the horocycle flow analyzed in section 5 are unstable manifolds for Anosov systems. This is not an accident: the proofs are based on the fact that our (entropy zero) systems are renormalized by a Anosov systems. The temporal DLT for the zero entropy system is a consequence of the spatial DLT for the chaotic map which renormalizes it.

2. Basic properties.

We record here for the future use the fact, that the spatial and temporal DLTs do not change if we modify the function by a coboundary.

Suppose T is a Borel map on a standard probability space $(X, \mathscr{B}, \mu)$, and $f : X \to \mathbb{R}$ is Borel.

Lemma 2.1. *Suppose T preserves an invariant probability measure μ.*

(a) If $f(x) = R(Tx) - R(x)$ for a measurable function R and μ is mixing, then $S_f(n, x)$ converges in distribution as $x \sim \mu$, $n \to \infty$.
(b) If $S_f(n, x)$ converges in distribution as $x \sim \mu$, $n \to \infty$ then $f(x) = R(Tx) - R(x)$ for a measurable function R.
(c) If $f_2(x) = f_1(x) + R(Tx) - R(x)$ for a measurable function R then $S_{f_1}(n, x)$ satisfies a spatial DLT as $x \sim \mu$ (with $B_n \to \infty$!) iff $S_{f_2}(n, x)$ satisfies a spatial DLT as $x \sim \mu$.

Similar statements hold for continuous time.

Proof. We consider maps, the proofs for flows are similar.

(a) Recall that mixing implies that for any bounded measurable function $\mathcal{R} : X \times X \to \mathbb{R}$ we have

$$\int \mathcal{R}(x, T^n x) d\mu(x) \to \iint \mathcal{R}(x_1, x_2) d\mu(x_1) d\mu(x_2)$$

(this equality can be obtained by approximating $\mathcal{R}$ by linear combinations of products $\sum_j A_j(x_1) B_j(x_2)$). Applying this to functions of the form $\mathcal{R}(x_1, x_2) = \phi(R(x_2) - R(x_1))$ where ϕ is a continuous test function we obtain (a).

(b) is proven in [61] (see also [3] for a more general result).

(c) follows from the fact that the sequence $\{R(T^n x)\}$ and hence $\{R(T^n x) - R(x)\}$ is tight and so $\frac{R(T^n x) - R(x)}{B_n} \xrightarrow[n \to \infty]{\text{dist}} 0$ as $n \to \infty$. □

Lemma 2.2.

(a) *Suppose μ is a T-invariant probability measure. If $f = R \circ T - R$ for a measurable function R then for μ a.e. x, $S_f(n, x)$ converges in distribution as $n \sim U\{1, \dots, N\}$, $N \to \infty$.*
(b) *Suppose μ is a T-invariant probability measure. If $S_f(n, x)$ converges in distribution as $n \sim U\{1, \dots, N\}$, $N \to \infty$ for μ a.e. x, then $f(x) = R(Tx) - R(x)$ with R measurable.*
(c) *If $f_2(x) = f_1(x) + R(Tx) - R(x)$ for a bounded measurable function R then $S_{f_1}(n, x)$ satisfies a temporal DLT for some x (with $B_n \to \infty$!) iff $S_{f_2}(n, x)$ satisfies a temporal DLT for that x.*
(d) *Suppose μ is a T-invariant probability measure. If $f_2(x) = f_1(x) + R(Tx) - R(x)$ for a measurable function R then $S_{f_1}(n, x)$ satisfies a temporal DLT (with $B_n \to \infty$) for μ a.e. x iff $S_{f_2}(n, x)$ satisfies a temporal DLT for μ a.e. x.*

Similar statements hold for continuous time.

Proof. We consider maps, the proofs for flows are similar.

To prove (a) it suffices to check that for a.e. x,

$$\frac{1}{N} \sum_{n=0}^{N-1} e^{it(R(T^n x) - R(x))} = \frac{e^{-itR(x)}}{N} \sum_{n=0}^{N-1} e^{itR(T^n x)}$$

converges for all $t \in \mathbb{R}$. This follows from the ergodic theorem.

To prove part (b) note that if $S(n, x) \xrightarrow[N\to\infty]{\text{dist}} Y_x$, as $n \sim U\{1, \dots, N\}$ then $S(n, Tx) \xrightarrow[N\to\infty]{\text{dist}} Y_x - f(x)$, as $n \sim U\{1, \dots, N\}$. Given a random variable Y define its *median* as

$$\mathfrak{m}(Y) = \inf \left\{ a : \text{Prob}(Y \geq a) \geq \frac{1}{2} \right\}.$$

Then letting $-R(x) = \mathfrak{m}(Y_x)$ we get $f(x) = R(Tx) - R(x)$.

Part (c) is evident, while part (d) follows from part (a). □

It is interesting to characterize functions such that there is a sequence of measurable functions $A_N(x)$ such that for μ a.e. x, $S_f(n, x) - A_N(x)$ converges in distribution as $n \sim U\{1, \dots, N\}$ and $N \to \infty$.

3. Anosov systems

3.1. Temporal DLT and the Almost Sure Invariance Principle.

Let T^t be a continuous flow on a complete metric space X. Fix $f : X \to \mathbb{R}$ continuous, and let $I(t, x) := \int_0^T f(T^t(x))dt$. Recall [54]:

Definition 3.1. *We say that $I(t, \cdot)$ satisfies the* Almost Sure Invariance Principle (ASIP) *with respect to a Borel probability measure μ on X,* *if there exist real valued functions* $B_t(x,s)$ *on* $X \times [0,1]$ $(t \geq 0)$ *and* $\sigma > 0$ *such that*

(1) $(t,x,s) \mapsto B_t(x,s)$ *is a Borel measurable map* $\mathbb{R}^+ \times X \times [0,1] \to \mathbb{R}$*;*
(2) If (x,s) *is distributed according to* $\mu \times ds$ *on* $X \times [0,1]$*, the paths* $t \mapsto B_t(x,s)$ *are distributed like standard Brownian Motion;*
(3) For $\mu \times ds$ *a.e.* (x,s)*,* $|I(t,x) - B_{\sigma^2 t}(x,s)| = o(\sqrt{t})$ *as* $t \to \infty$*.*

A similar definition can be made for discrete time in which case the last condition has to be replaced by

$$|S(n,x) - B_{\sigma^2 n}(x,s)| = o(\sqrt{n}) \text{ as } n \to \infty.$$

The ASIP has been introduced as a generalization of the functional Central Limit Theorem. In particular, it implies the spatial DLT with normal limiting distribution. On the other hand, it was pointed out in [45] that if $I(t,x)$ satisfies the Almost Sure Invariance Principle, then the almost sure DLT holds and the limiting random variable is normal. Since the ASIP is known for a wide class of hyperbolic systems [22, 23, 46] this gives many examples of systems satisfying both spatial DLT and the almost sure DLT. However, we shall see below that ASIP is incompatible with the temporal DLT.

Theorem 3.2. *Suppose that the ergodic integral* $I(t,x)$ *(or ergodic sum* $S(n,x)$*) satisfies the ASIP with respect to a measure* μ*, then for* μ*-a.e.* $x \in X$*, for every real valued random variable* Y*, there exist* $T_n \uparrow \infty$ *s.t.*

$$I(t,x)/\sqrt{T_n} \xrightarrow[n\to\infty]{dist} Y \text{ where } t \sim U[0,T_n]. \tag{3.1}$$

So no Temporal DLT can hold.

Proof. We give the proof for flows, the proof of maps is similar.

There is no loss of generality in assuming that $\sigma = 1$.

The weak star topology on the space $\mathscr{P}(\mathbb{R})$ of Borel probability measures on $\mathbb{R}$ is generated by a countable basis $\mathscr{N}$ of non-empty neighborhoods of the form

$$N(f_1,\dots,f_n;a_1,\dots,a_n;\varepsilon) := \{\nu \in \mathscr{P}(\mathbb{R}) : \Big|\int f_i d\nu - a_i\Big| < \varepsilon \ (i=1,\dots,n)\},$$

where $n \in \mathbb{N}$, $f_i \in C_c(\mathbb{R})$, $\|f_i\|_\infty = 1$, $\varepsilon > 0$, and $0 < a_i < 1$.

Let $(W, \mathscr{F}, m)$ denote Wiener's measure space: the space of continuous functions $B : [0,\infty) \to \mathbb{R}$ s.t. $B(0) = 0$, equipped with the σ-algebra $\mathscr{F}$ and the measure m which turns $B(\cdot) \in W$ into Brownian Motion. The *occupational measure (up to time* 1*)* of a Brownian path $B(\cdot) \in W$ is the random probability measure on $\mathbb{R}$ $\mu_B := \int_0^1 \delta_{B(t)} dt$, where δ_x :=point mass at t. It is not difficult to see that for every non-empty $N \in \mathscr{N}$, $m\{B \in W : \mu_B \in N\} > 0$.

The *scaling flow* on W is the flow $\Phi : W \to W$,

$$[\Phi^s B](t) = e^{-s/2} B(e^s t).$$

This is an ergodic measure preserving flow ([33]). Since $\mathscr{N}$ is countable,

$$W_0 := \{B \in W : \forall N \in \mathscr{N},\ \exists s_n \uparrow \infty \text{ s.t. } \mu_{\Phi^{s_n}(B)} \in N \text{ for all } n \in \mathbb{N}\}$$

has full measure.

We claim that for every $B \in W_0$, for every $\nu \in \mathscr{P}(\mathbb{R})$, there is a sequence $s_n \to \infty$ such that $\mu_{\Phi^{s_n}[B]} \xrightarrow[n\to\infty]{w^*} \nu$. To see this use the fact that $\mathscr{N}$ is a basis to construct a countable sequence of $N_n \in \mathscr{N}$ such that $N_1 \supset N_2 \supset N_3 \supset \cdots$, with the property that every neighborhood of ν contains some N_n. If $B \in W_0$, then there are $s_1 < s_2 < s_3 < \cdots$ tending to infinity such that $\mu_{\Phi^{s_n}(B)} \in N_n$ for all n. Necessarily $\mu_{\Phi^{s_n}(B)} \xrightarrow[n\to\infty]{w^*} \nu$.

Let $X_0 := \{x \in X : \exists s \in [0,1] \text{ s.t. } B_t(x,s) \in W_0\}$. By the ASIP, X_0 has full μ-measure. Fix $x \in X_0$ and $s \in [0,1]$ s.t. $B(t) := B_t(x,s)$ belongs to W_0, then for every real-valued random variable Y, $\exists s_n \to \infty$ s.t.

$$\mu_{\Phi^{s_n}(B)} \xrightarrow[n\to\infty]{w^*} \nu_Y, \tag{3.2}$$

where ν_Y is the measure such that $\nu_Y(E) = \text{Prob}[Y \in E]$.

Let $T_n := \ln s_n$, then for every test function $G \in C_c(\mathbb{R})$,

$$\begin{aligned}\frac{1}{T_n}\int_0^{T_n} G\left(\tfrac{I(t,x)}{\sqrt{T_n}}\right) dt &= \frac{1}{T_n}\int_0^{T_n} G\left(\tfrac{B_t(x,s)}{\sqrt{T_n}} + o(1)\right) dt \\ &= \frac{1}{T_n}\int_0^{T_n} G\left(\tfrac{B_t(x,s)}{\sqrt{T_n}}\right) dt + o(1) \\ &= \int_0^1 G\left(\tfrac{B_{tT_n}(x,s)}{\sqrt{T_n}}\right) dt + o(1) = \int_0^1 G\left(\Phi^{s_n}(B)(t)\right) dt + o(1) \\ &= \int G d\mu_{\Phi^{s_n}(B)} \xrightarrow[n\to\infty]{} \int G d\nu_Y \equiv \mathbb{E}[G(Y)] \text{ by (3.2).}\end{aligned} \tag{3.3}$$

Since G was arbitrary, $\frac{I(t,x)}{\sqrt{T_n}} \xrightarrow[n\to\infty]{\text{dist}} Y$, as $t \sim U[0,T_n]$. □

Remark 3.3. The almost sure divergence of $\lim \frac{1}{T}\int_0^T 1_{[a,b]}(B(t)/\sqrt{t})dt$ for Brownian paths was proved earlier in [33, Thm 5.9].

Remark 3.4. Theorem 3.2 is a special case of the following general problem. We have a probability space $(\Omega, \mathbb{P})$ (in our case, it is (X,μ)) and a collection of random variables V_T with values in topological space

$\mathcal{V}$ (in our, case $\mathcal{V} = \mathscr{P}(\mathbb{R})$ and V_T is the occupation measure of $\frac{I(tT,x)}{\sqrt{T}}$). The problem is to use the information about the limit distribution of V_T as $T \to \infty$ to get a description of the set $\{V_t(\omega)\}_{t \geq T}$ for a fixed ω. Some general results on this question are described in [58] (see also Lemma 4.7).

3.2. **An application to Anosov systems.** Let T^t be a transitive Anosov flow on a compact smooth manifold X, equipped with an equilibrium measure μ of some Hölder continuous potential (e.g. the geodesic flow on the unit tangent bundle of a compact Riemannian surface with negative curvature, equipped with the Liouville measure).

Suppose $f : X \to \mathbb{R}$ is a Hölder continuous function which is not cohomologous to a constant function and such that $\int f d\mu = 0$. By [22] and [9], the ergodic integrals of f satisfy the ASIP with respect to μ. Therefore for such systems

(1) the spatial DLT holds with respect to μ [57];
(2) for μ-a.e. x, there is an almost sure DLT [33];
(3) but, for μ-a.e. x, there is no temporal DLT along the orbit of x (by Theorem 3.2).

The same phenomena holds in discrete time Anosov diffeomorphism on a compact manifold X, μ and f as above (e.g. hyperbolic toral automorphisms equipped with the Lebesgue measure).

4. Irrational rotations

4.1. **Temporal DLT.** Let $\mathbb{T} := \mathbb{R}/\mathbb{Z}$, and define for $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ the map $T : \mathbb{T} \to \mathbb{T}$ by $T(x) = x + \alpha \bmod 1$. Let

$$f_a(t) := 1_{[0,a)}(\{t\}) - a, \tag{4.1}$$

and

$$S_a(n,x) := \sum_{k=0}^{n-1} f_a(T^n(x)).$$

J. Beck proved a temporal DLT for $S_a(n,x)$, under the assumptions that α is a quadratic irrational (a root of quadratic polynomial with integer coefficients), a is rational, and $x = 0$ [6, 7]. A different proof of Beck's temporal DLT was given in [4] in the special case $a = \frac{1}{2}$. Here we explain how to modify that proof to obtain a temporal DLT for all rational a and all initial conditions $x \in \mathbb{T}$.

Theorem 4.1. *Suppose a is rational, α is a quadratic irrational, then for every $x \in \mathbb{T}$ there are $B = B(a, \alpha) > 0$ independent of x and*

$A_n = A_n(a, x, \alpha)$ such that

$$\frac{S_a(n,x) - A_N}{B\sqrt{\ln N}} \xrightarrow[N\to\infty]{dist} \mathfrak{N}(0,1), \text{ as } n \sim U\{1,\dots,N\}.$$

Here and below $\mathfrak{N}(\mu, \sigma^2)$ =normal distribution with mean μ and variance σ^2.

Let $a = p/q$ where p, q be integers such that $0 < p < q$ and $(p,q) = 1$. $f : \mathbb{T} \to \mathbb{Z}$ be the function

$$f(x) = (q-p) \cdot 1_{[0,\frac{p}{q})} - p \cdot 1_{[\frac{p}{q},1)} = (q-p) f_a.$$

Clearly it is enough to prove the temporal DLT for f.

Consider the cylinder map on $\mathbb{T} \times \mathbb{Z}$ given by

$$\mathcal{T}_f(x,\xi) = (x + \alpha, \xi + f(x)).$$

Then $\mathcal{T}_f^n(x,\xi) = (x+n\alpha, \xi + \widetilde{S}_a(n,x))$ where $\widetilde{S}_a(n,x) = (q-p)S_a(n,x)$ so the temporal DLT for (x, f, T) reduces to counting occupation times for the sets of the form $\mathbb{T} \times B_N I$ under $\mathcal{T}$ where $I \subset \mathbb{R}$ is a fixed subinterval and $B_N = B\sqrt{\ln N}$. In fact, it is convenient to work with a constant suspension of $\mathcal{T}$ because it has many symmetries.

Proposition 4.2. [40, 41] *There exists an (infinite area) translation surface $\widetilde{M}$ such that:*

(1) The linear flow in direction $\theta \neq \pm\frac{\pi}{2}$ on $\widetilde{M}$ has a section with Poincaré map conjugate to $\mathcal{T}_f(x,\xi)$, and constant return time.
(2) There's a finite area translation surface M and a regular cover map $p : \widetilde{M} \to M$ whose group of deck transformations is $\cong \mathbb{Z}$.
(3) There is a finite index subgroup $\Lambda \subset \mathrm{SL}(2,\mathbb{Z})$ such that for every $A \in \Lambda$ there are automorphisms $\widetilde{\psi} : \widetilde{M} \to \widetilde{M}$ and $\psi : M \to M$ with derivative A, which fix the punctures, and satisfy $\psi \circ p = p \circ \widetilde{\psi}$.

Let us describe the construction of M and $\widetilde{M}$. Let $M_0 := \mathbb{R}^2/\mathbb{Z}^2$ and $M := M_0 \setminus \{\bullet, \circ\}$ where $\bullet := \mathbb{Z}^2$ and $\circ := \binom{p/q}{0} + \mathbb{Z}^2$. M is isometric to the identification space of the square with $\bullet, \circ$ removed (see Figure 1).

The $\mathbb{Z}$-cover $\widetilde{M}$ (see Figure 2) is obtained from a disjoint countable union of copies of this square indexed by $k \in \mathbb{Z}$, and with the side pairing

- $(TL_k, TM_k) \sim (BL_{k+(q-p)}, BM_{k+(q-p)})$
- $(TM_k, TR_k) \sim (BM_{k-p}, BR_{k-p})$
- $(BL_k, TL_k) \sim (BR_k, TR_k)$

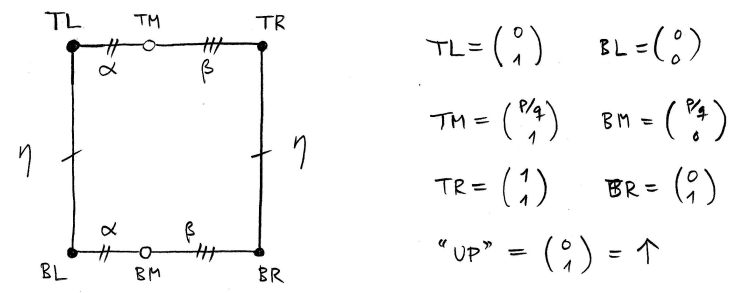

FIGURE 1. M is a twice punctured torus.

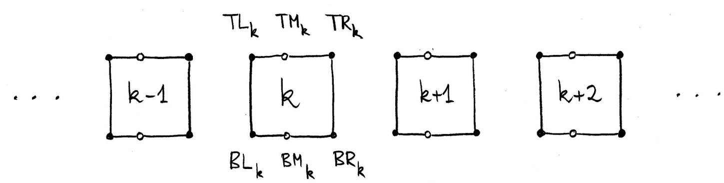

FIGURE 2. $\mathbb{Z}$-cover

This has the effect that as the linear flow exists square k through its top left side, it enters square $k + (q - p)$ through its bottom left side, and as it exists square k through its top right side, it enters square $k - p$ through its bottom right side. So the union of top (left and right) sides is a Poincaré section with constant roof function τ and Poincaré map conjugate to $\mathcal{T}_f$.

Let us now prove Theorem 4.1 postponing the proof of the fact that $M, \widetilde{M}$ have the properties announced in Proposition 4.2 until Appendix A.

We shall use Proposition 4.2 in the following way. Let D be the deck transformation moving the k-th copy of our surface to the $k+1$-st one. Let $\widetilde{M}^*$ denote the regular part of $\widetilde{M}$, that is $\pi^{-1}M$.

Proposition 4.3. *If α is a quadratic irrational, and $\theta = \tan^{-1}\alpha$, then there exists a homeomorphism $\widetilde{\psi} : \widetilde{M} \to \widetilde{M}$ and $0 < \lambda < 1$ with the following properties:*

(1) $\widetilde{\psi}[\varphi_\theta^t(p)] = \varphi_\theta^{\lambda t}[\widetilde{\psi}(p)]$ *for all $t > 0$ and p such that the forward orbit of p does not meet a singularity.*
(2) $\widetilde{\psi} \circ D = D \circ \widetilde{\psi}$ *and $\widetilde{\psi}$ fixes the singularities of $\widetilde{M}$*

(3) $\widetilde{\psi}$ *is differentiable on* $\widetilde{M}^*$, *with constant derivative. The derivative is a hyperbolic matrix in* $\mathrm{SL}(2,\mathbb{Z})$, *with eigenvalues* λ, λ^{-1}. *The direction of* φ_θ *is the contracted eigenvector.*
(4) $\widetilde{\psi}$ *descends to a hyperbolic toral automorphism* ψ *of the punctured torus* M.
(5) $\int_M F_\psi(q)dq = 0$, *where* $F_\psi : M^* \to \mathbb{Z}$, $F_\psi(q) = \xi(\widetilde{\psi}^{-1}(\widetilde{q})) - \xi(\widetilde{q})$ *for some (all)* $\widetilde{q} \in \widetilde{M}$ *which project to* $q \in M$, *and the integration is with respect to the area measure.*

The proof of Proposition 4.3 is similar to the proof of Proposition 2.5 in [4]. Namely, we use Lagrange Theorem to find a hyperbolic matrix A which fixes the direction θ. Replacing A by its inverse if necessary we may assume that this direction is contracted by A. Since Λ has finite index in $\mathrm{SL}(2,\mathbb{Z})$ there exists k such that $A^k \in \Lambda$. We then apply Proposition 4.2 to find the map $\widetilde{\psi}$ with derivative A^k.

Proof of Theorem 4.1. We assume without loss of generality that $\varphi_\theta^t(x,0)$ does not meet any singularity for $t \geq 0$. If it does, say for $t = cn$ $(n \geq 0)$, replace x by $x + (n+1)\alpha \bmod 1$.

Let

$$\mathfrak{a}_N := \xi[\widetilde{\psi}^N(x,0)] = \sum_{j=0}^{N-1} F_\psi(\psi^j(x,0)). \tag{4.2}$$

We will estimate, for given $a_1, a_2 \in \mathbb{R}$ such that $a_1 < a_2$,

$$D_N(a_1,a_2) := \frac{1}{N}\#\left\{1 \leq n \leq N : \frac{\widetilde{S}_a(x,n) - \mathfrak{a}_{N^*}}{\sqrt{N^*}} \in [a_1,a_2]\right\},$$

where $N^* := \lfloor \log_{\lambda^{-1}} N \rfloor$.

Let

$$\Gamma_N := \{\varphi_\theta^t(x,0) : \tau < t < \tau(N+1)\}. \tag{4.3}$$

Till the end of the proof let ℓ denote the one-dimensional Hausdorff measure (in the context we use it, just the length measure along the relevant straight line). By Proposition 4.2(1)

$$D_N(a_1,a_2) = \frac{1}{\ell(\Gamma_N)}\ell\{q \in \Gamma_N : \frac{\xi(q) - \mathfrak{a}_{N^*}}{\sqrt{N^*}} \in [a_1,a_2]\}.$$

By Proposition 4.3 and the choice of $\mathfrak{a}_N$,

$$D_N(a_1,a_2) = \frac{1}{\ell(\gamma_N)}\ell\{q \in \gamma_N : \frac{\xi(\widetilde{\psi}^{-N^*}(q)) - \xi(z_{N^*})}{\sqrt{N^*}} \in [a_1,a_2]\}$$

where $\gamma_N = \widetilde{\psi}^{N^*}(\Gamma_N)$, and $z_{N^*} := \widetilde{\psi}^{N^*}(x,0)$ =beginning point of γ_N. Let $\widehat{\gamma}_N := D^{-\xi(z_{N^*})}(\gamma_N)$. Since $\xi \circ D = \xi + 1$ and $\widetilde{\psi} \circ D = D \circ \widetilde{\psi}$,

$$D_N(a_1, a_2) = \frac{1}{\ell(\widehat{\gamma}_N)} \ell\{q \in \widehat{\gamma}_N : \frac{\xi(\widetilde{\psi}^{-N^*}(q))}{\sqrt{N^*}} \in [a_1, a_2]\}. \tag{4.4}$$

Observe that $\widehat{\gamma}_N$ is a linear segment of length $\lambda^{-N^*} N \asymp 1$, which starts at a point at sheet zero. So $\{\widehat{\gamma}_N\}$ is a pre-compact collection of linear segments.

The remainder of the argument is similar to [4]. We sketch the argument, and refer the reader to that paper for the details.

F_ψ is not smooth but it is sufficiently regular to prove the CLT. Namely given a function F on M_0 let

$$\mathcal{H}(F, d) = \sup_{\mathcal{A}} \|F - \mathbb{E}(F|\mathcal{A})\|_{L^2}$$

where the supremum is taken over all partitions whose diameter is smaller than d. In our case

$$\mathcal{H}(F_\psi, d) = O(d) \tag{4.5}$$

since $|[F - \mathbb{E}(F|\mathcal{A})](p)| \leq 1$ for all p and $[F - \mathbb{E}(F|\mathcal{A})](p) = 0$ unless p belongs to d-neighborhood of the singularity set of F_ψ.

(4.5) allows us to apply [18, Corollary 1.7.(ii)] to

$$\xi(\widetilde{\psi}^{-N^*} q) = \xi(q) + \sum_{j=0}^{N^*-1} F_\psi(\psi^{-j} q)$$

and conclude that if we choose $q \in \widetilde{M}^*$ randomly uniformly in sheet zero, then

$$\frac{\xi[\widetilde{\psi}^{-N^*}(q)]}{\sqrt{N^*}} \xrightarrow[N^* \to \infty]{\text{dist}} \mathfrak{N}(0, \sigma^2)$$

where

$$\sigma^2 = \sum_{n=-\infty}^{\infty} \int_M F_\psi(\psi^{-n} q) F_\psi(q) dq \geq 0. \tag{4.6}$$

We claim that in fact, $\sigma^2 \neq 0$. Postponing the claim till the end of this section let us first finish the proof of the theorem.

By Eagleson's Theorem [28], the convergence in distribution remains true whenever we choose q uniformly according to some probability measure which is absolutely continuous with respect to the area measure on M. In our case, q is sampled according to the one-dimensional Hausdorff measure on $\widehat{\gamma}_N$, which is singular. But there is not much difference between sampling q from $\widehat{\gamma}_N$ and sampling it uniformly from a thickening of $\widehat{\gamma}_N$ in the direction of the expanded eigenvector of $d\psi$ into a solid parallelogram. This is because the direction of thickening is the *stable* direction of ψ^{-1}, and as such its effect on the distributional behavior of $\frac{\xi(\widetilde{\psi}^{-N^*}(q))}{\sqrt{N^*}}$ is negligible.

The pre-compactness of $\{\widehat{\gamma}_N : N \geq 1\}$ is sufficient to push this idea to a proof that the random variables $X_N := \frac{\xi[\widetilde{\psi}^{-N^*}(q)]}{\sqrt{N^*}}$ $(q \sim U[\widehat{\gamma}_N])$ converge in distribution to $\mathfrak{N}(0, \sigma^2)$ as $N \to \infty$. So

$$D_N(a_1, a_2) \xrightarrow[N\to\infty]{} \frac{1}{\sqrt{2\pi\sigma^2}} \int_{a_1}^{a_2} e^{-t^2/2\sigma^2} dt \quad \text{for all} \quad a_1 < a_2.$$

It remains to check that σ is not equal to 0. If it were, then by general spectral theory (see e.g. [53, Prop. 4.12]) we would have that

$$F_\psi(q) = \eta(q) - \eta(\psi(q)) \text{ a.e.} \tag{4.7}$$

for a measurable function η. Using (4.4) we conclude from (4.7) that if x is chosen uniformly in $\mathbb{T}$ then the distribution of $\{\widetilde{S}_a(n, x)\}_{n\in\mathbb{N}}$ is tight. By [3] f_a would be a coboundary which contradicts the ergodicity of $\mathcal{T}$ established in [51],[67] (see also [19], [62], [2]). □

Remark 4.4. If $x = 0$, then the centering constants $A_N = \mathfrak{a}_{N^*}$ in Theorem 4.1 can be taken to be const $\sqrt{\log N}$ [6],[4]. In general, the centering constants can be oscillatory. By (4.2), $A_N = O(N^*) = O(\ln N)$ for all x, and $A_N = o(N^*) = o(\ln N)$ for a.e. x.

For the future use we record the following extension of Theorem 4.1.

Theorem 4.5. *For each $b_1 < b_2$ Theorem 4.1 remains valid if the condition $n \sim U(1, N)$ is replaced by $n \sim U(b_1 N, b_2 N)$.*

The proof of Theorem 4.5 is the same as the proof of Theorem 4.1 except that Γ_n in (4.3) has to be replaced by

$$\widetilde{\Gamma}_N = \{\varphi^t_\theta(x, 0) : \tau b_1 N < t < \tau b_2(N + 1)\}.$$

4.2. **No almost sure DLT for rotations.**

Theorem 4.6. *f_a does not satisfy an almost sure DLT. In fact, for almost every x the following holds. For every random variable $\mathfrak{Y}$ there is a sequence N_m such that*

$$\frac{S_a(n, x)}{\sqrt{\ln n}} \xrightarrow[m\to\infty]{dist} \mathfrak{Y} \oplus \mathfrak{N}(0, B^2) \text{ where } n \sim \mathrm{Log}\{1, \ldots, N_m\} \tag{4.8}$$

Here B is the constant in Theorem 4.1, $\mathrm{Log}\{1, \ldots, N\}$ is the distribution on $\{1, \ldots, N\}$ such that $\mathbb{P}(k) \propto \frac{1}{k}$, and $\oplus$ =independent sum.

Proof. Given $[\varepsilon \ln N] \le L \le [\ln N/\varepsilon]$, let $I_L := (e^{\varepsilon L}, e^{\varepsilon(L+1)}) \cap \{1, \dots, N\}$. If $n \sim \mathrm{Log}\{1, \dots, N\}$, then

(a) $\bigcup I_L$ cover more than $1-\varepsilon$ of the total mass of $\{1, \dots, N\}$;
(b) the conditional density of mass inside I_L is approximately uniform;
(c) the total mass of each I_L is almost the same, $\approx \varepsilon/\ln N$.

Therefore it is enough to prove that for any $0 < \varepsilon < 1$, (4.8) holds if

$$n \sim U([e^{\varepsilon L}], \dots, [e^{\varepsilon(L+1)}]) \quad \text{and} \quad L \sim U\left([\varepsilon \ln N], \dots, [\tfrac{\ln N}{\varepsilon}]\right).$$

Let $\mathbf{P}_{\varepsilon,N}$ and $\mathbf{E}_{\varepsilon,N}$ denote the probability and expectation with respect to this distribution.

Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a continuous function with compact support. Recall the definition of $\mathfrak{a}_N$ from (4.2), and consider the decomposition

$$\Phi\left(\tfrac{S_a(n,x)}{\sqrt{\ln n}}\right) = \Phi\left(\tfrac{\sqrt{\varepsilon L}}{\sqrt{\ln n}}\left(\tfrac{S_a(n,x)-\mathfrak{a}_{\varepsilon L}}{\sqrt{\varepsilon L}} + \tfrac{\mathfrak{a}_{\varepsilon L}}{\sqrt{\varepsilon L}}\right)\right) \quad (n \in I_L).$$

Φ is uniformly continuous, $\frac{\varepsilon L}{\sqrt{\ln n}} = 1 + O(\varepsilon)$ uniformly as $N \to \infty$, and $\frac{S_a(n,x)-\mathfrak{a}_{\varepsilon L}}{\sqrt{\varepsilon L}} \xrightarrow[L\to\infty]{\text{dist}} \mathfrak{N}(0, B^2)$ as $n \sim U(I_L)$ by Theorem 4.5. So

$$\mathbf{E}_{\varepsilon,N}\left(\tfrac{S_a(n,x)}{\sqrt{\ln n}} \middle| n \in I_L\right) = (\mathcal{C}_{\mathfrak{N}(0,B^2)}\Phi)\left(\tfrac{\mathfrak{a}_{\varepsilon L}}{\sqrt{\varepsilon L}}\right) + o(1),$$

where $\mathcal{C}$ denotes the convolution operator

$$[\mathcal{C}_Y \Phi](z) = \mathbb{E}(\Phi(z+Y)). \tag{4.9}$$

Denoting $K = \ln N$ and passing to the expectation over L, we get

$$\mathbf{E}_{\varepsilon,N}\left(\Phi\left(\tfrac{S_a(n,x)}{\sqrt{\ln n}}\right)\right) = \tag{4.10}$$

$$\mathbf{E}_{\varepsilon,N}\left(\frac{1}{((1/\varepsilon)-\varepsilon)K} \sum_{L=\varepsilon K}^{K/\varepsilon} \left[\mathcal{C}_{\mathfrak{N}(0,B^2)}\Phi\right]\left(\tfrac{\mathfrak{a}_{\varepsilon L}}{\sqrt{\varepsilon L}}\right)\right) + o(1).$$

We claim that for each random variable $\mathfrak{Y}$ there exists a sequence $\{N_m\}$ such that the occupation measure of $\{\frac{\mathfrak{a}_{\varepsilon L}}{\varepsilon L}\}$ converges to the distribution of $\mathfrak{Y}$. That is, for any continuous test function Ψ of compact support

$$\lim_{m\to\infty} \mathbf{E}_{\varepsilon,N_m}\left(\sum_{L+\varepsilon K}^{K/\varepsilon} \Psi\left(\frac{\mathfrak{a}_{\varepsilon L}}{\varepsilon L}\right)\right) = \mathbb{E}(\Psi(\mathfrak{Y})). \tag{4.11}$$

Combining (4.9), (4.10) and (4.11) we obtain

$$\lim_{m\to\infty} \mathbf{E}_{\varepsilon,N_m}\left(\Phi\left(\tfrac{S_a(n,x)}{\sqrt{\ln n}}\right)\right) = \mathbb{E}(\Phi(\mathfrak{Y} \oplus \mathfrak{N}(0,B^2)))$$

as claimed.

It remains to prove (4.11). Recalling (4.2) let

$$W_n^x = \frac{\xi(\widetilde{\psi}^{[tn]}(x,0))}{\sqrt{tn}}.$$

Lemma 4.7. *Let $g : [\varepsilon, 1/\varepsilon] \to \mathbb{R}$ be a continuous function. For a.e. x, g is a limit point of W_n^x as $n \to \infty$.*

Lemma 4.7 implies (4.11) since for each $\mathfrak{Y}$ it is simple to find a sequence $\{g_n\}$ so that the occupation measures of $\{g_n\}$ converge to the distribution of $\mathfrak{Y}$. (Some non continuous functions can appear as limits of W_n^x as well but it does not concern us here since continuous functions are enough to obtain (4.11).) □

Proof of Lemma 4.7. It suffices to show that for a.e. x and each $\widehat{\varepsilon} > 0$, W_n^x visits the set $\mathcal{N}_\varepsilon = \{h : ||h - g|| < \widehat{\varepsilon}\}$ for some n. Using a weak invariance principle proven in [18] we conclude that there is $\delta > 0$ such that if q is uniformly distributed on sheet 0 and $W_n^q(t) = \frac{\xi(\widetilde{\psi}^{[nt]}(t)) - \xi(q)}{\sqrt{tn}}$ then for large n

$$\mathbb{P}(W_n^q \in \mathcal{N}_{\widehat{\varepsilon}/2}) \geq \delta.$$

Arguing as in the proof of Theorem 4.1 we conclude that the same is true is q is uniformly distributed on a segment Γ which is uniformly transversal to the stable direction e_s of ψ. Moreover n can be chosen uniformly over the set of segments such that

$$\frac{1}{K} \leq |\Gamma| \leq K, \quad \angle(\Gamma, e_s) \geq \frac{1}{K}.$$

We want to apply this to $\Gamma = \{(x, 0)\}_{x \in \mathbb{T}}$. Take a large n_1. The set where $W_{n_1}^q \notin \mathcal{N}_\varepsilon$ is a finite union of segments. Let $\widehat{\Gamma}$ be one such segment. Take m_1 so that

$$\frac{1}{K} \leq \left|\widetilde{\psi}^{m_1}\widehat{\Gamma}\right| \leq K.$$

By the foregoing discussion if $\widetilde{n}_2$ is large enough then

$$\mathbb{P}_{\widehat{\Gamma}}\left(W_{\widetilde{n}_2}^{\widetilde{\psi}^{n_1+m_1}q} \in \mathcal{N}_{\widehat{\varepsilon}/2}\right) \geq \delta.$$

On the other hand letting $n_2 = \widetilde{n}_2 + m_1 + n_1$ we have

$$\left\|W_{n_2}^q - W_{\widetilde{n}_2}^{\widetilde{\psi}^{n_1+m_1}q}\right\| \leq C\frac{n_1 + m_1}{\sqrt{\widetilde{n}_2}}$$

which can be made smaller than $\widehat{\varepsilon}/2$ if n_2 is large enough. Hence we defined a function $n_2(q)$ on the set where $W_{n_1}^q \notin \mathcal{N}_{\widehat{\varepsilon}}$ so that

$$\mathbb{P}\left(W_{n_2}^q \notin \mathcal{N}_{\widehat{\varepsilon}} | W_{n_1}^q \notin \mathcal{N}_{\widehat{\varepsilon}}\right) \leq (1 - \delta).$$

Continuing this procedure we can define functions $n_k(q)$ on the set where

$$W^q_{n_j} \notin \mathcal{N}_{\widehat{\varepsilon}} \text{ for } j < k$$

so that

$$\mathbb{P}\left(W^q_{n_k} \notin \mathcal{N}_{\widehat{\varepsilon}} | W^q_{n_j} \notin \mathcal{N}_{\widehat{\varepsilon}} \text{ for } j < k\right) \leq (1-\delta)$$

so the set of points avoiding $\mathcal{N}_{\widehat{\varepsilon}}$ is a Cantor set of zero measure. □

5. Horocycle flows

5.1. **Horocycle windings.** Let M be a compact connected orientable Riemannian surface. Assume that M is *hyperbolic*, that is, every $p \in M$ has a neighborhood which is isometric to a neighborhood of i in $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$, equipped with the metric $|dz|/\mathrm{Im}(z)$. Let $T^1M := \{\vec{v} \in T_xM : x \in M, \|\vec{v}\| = 1\}$, and $\pi : T^1M \to M$ the projection which sends a tangent vector to its base point. Let m denote the Riemannian area measure on T^1M.

The *geodesic flow* $g^t : T^1M \to T^1M$ moves a unit tangent vector $\vec{v}$ at unit speed along its geodesic, in the direction of $\vec{v}$.

The *stable horocycle flow* $h^t : T^1M \to T^1M$ moves a unit tangent vector $\vec{v}$ at unit speed and in the positive direction, along its stable horocycle $\mathrm{Hor}(\vec{v}) := W^{ss}(\vec{v}) = \{\vec{u} \in T^1M : \mathrm{dist}(g^t(\vec{v}), g^t(\vec{u})) \xrightarrow[t\to\infty]{} 0\}$. The "positive direction" is the direction $\vec{w} \in T_{\pi(\vec{v})}[W^{ss}(\vec{v})]$ such that the ordered basis $\langle \vec{w}, \vec{v}\rangle$ has positive orientation in $T_{\pi(\vec{v})}M$.

We are interested in the way these flows wind around M (the definition of "winding" is below). The winding of the geodesic flow was analyzed in [39], [47],[48],[30], [29], and [5]. We will analyze the horocycle flow.

Given a closed form ω and a curve Γ in M we can define winding $W(\Gamma, \omega) = \int_\Gamma \omega$. If ω_1 and ω_2 belong to the same cohomology class, $\omega_2 = \omega_1 + dB$ then

$$W(\Gamma, \omega_1) = W(\Gamma, \omega_2) + B(\mathrm{end}(\Gamma)) - B(\mathrm{beginning}(\Gamma)). \tag{5.1}$$

Below we consider sequences of curves whose length tend to infinity. Then (5.1) shows that the asymptotic winding depends only on cohomology class of ω. By the Hodge Theorem every cohomology class contains a harmonic form, i.e. a closed form ω such that $\omega^*(v) = \omega(Rv)$ is closed where R denotes the rotation by $-\pi/2$. Therefore it is enough to consider *harmonic* 1-forms.

Let us fix a harmonic 1-form ω. Let $G_t(\vec{v})$ and $H_t(\vec{v})$ denote the projections to M of the pieces of geodesics of length t and of stable horocycle of length t starting at $\vec{v}$. Let

$$\mathcal{W}_g(\omega,\vec{v},t) = W(G_t(\vec{v}),\omega), \quad \mathcal{W}_h(\omega,\vec{v},t) = W(H_t(\vec{v}),\omega).$$

Theorem 5.1. *For each $\vec{v}$,*

$$\frac{\mathcal{W}_h(\omega,\vec{v},t) - \mathcal{W}_g(\omega,\vec{v},\ln T)}{\sqrt{\ln T}} \xrightarrow[T\to\infty]{dist} \mathfrak{N}(0,\sigma^2) \text{ as } t \sim U(0,T) \tag{5.2}$$

where

$$\sigma^2 = \frac{4}{\text{Volume}(T^1M)} \int_{T^1M} |\omega(\vec{\xi})|^2 dm(\vec{\xi}). \tag{5.3}$$

Proof. The quadrilateral with sides

$$H_t(\vec{v}), \quad G_{\ln T}(h^t\vec{v}), \quad -H_{t/T}(g^{\ln T}\vec{v}), \quad G_{\ln T}(-g^{\ln T}\vec{v})$$

is contractible (decrease t to 0, and T to 1), therefore homologous to zero. By the Stokes Theorem,

$$\mathcal{W}_h(\omega,\vec{v},t) - \mathcal{W}_g(\omega,\vec{v},\ln T) = \mathcal{W}_g(\omega,\vec{w},\ln T) - \mathcal{W}_h(\omega, g^{\ln T}\vec{v},u) \tag{5.4}$$

where $u = t/T$, $\vec{w} = -h^u g^{\ln T}\vec{v}$.

The second term on the RHS of (5.4) is $O(1)$ since $u \in [0,1]$, so the random variable on the left of (5.2) has the same distributional limit as the random variable $\frac{\mathcal{W}_g(\omega,\vec{w},t)}{\sqrt{\ln T}}$ with $t \sim U[0,1]$, as $T \to \infty$.

Notice that when $t \sim U[0,T]$, $-\vec{w}$ is uniformly distributed on the horocyclic arc $H_1(g^{\ln T}\vec{v})$. The measure on T^1M which describes this distribution is supported inside a one-dimensional curve, and is therefore singular with respect to the volume measure on T^1M. However similarly to Theorem 4.1 we can smoothen this measure by thickening its support in stable and neutral direction and show that if $-\vec{w}$ is uniformly distributed on $H_1(g^{\ln T}\vec{v})$ then $\frac{\mathcal{W}_g(\omega,\vec{w},t)}{\sqrt{\ln T}}$ converges to $\mathfrak{N}(0,\sigma^2)$ where σ^2 is the same as when $\vec{w}$ is uniformly distributed on T^1M. By [48, pages 164-5] and the assumption that ω is harmonic, σ^2 is given by (5.3). □

Corollary 5.2. *Suppose $\vec{v}$ generates a closed geodesic of length $\ell(\gamma)$ and homology class $[\gamma]$, then*

$$\frac{\mathcal{W}_h(\omega,\vec{v},t) - A\ln T}{\sqrt{\ln T}} \xrightarrow[T\to\infty]{dist} \mathfrak{N}(0,\sigma^2) \text{ as } t \sim U[0,T],$$

where $A = \frac{1}{\ell(\gamma)}\int_\gamma \omega$ and σ is as before.

Remark 5.3. Every homotopy class contains a closed geodesic γ. Fix some $\vec{v}$ tangent to γ. If the homotopy class is homologous to zero, then $A = 0$ for every harmonic 1-form ω. If the homotopy class is not homologous to zero, then the de-Rahm and Hodge theorems provide an harmonic 1-form ω such that $A \neq 0$ for all $\vec{v}$ tangent to γ.

Thus, even though the horocycle flow is uniquely ergodic and all its orbits satisfy $(1/t)\mathcal{W}_h(\omega, \vec{v}, t) \xrightarrow[t\to\infty]{} 0$ uniformly on T^1M, the behavior on the scale $\ln t$ is sensitive to $\vec{v}$, oscillatory, and may reveal a bias of order $\ln T$.

Remark 5.4. One can also consider the case where our surface has finite area but is not compact. In this case one can no longer claim that the second term in the right hand side of (5.4) is small since the short curves can wind many times around the cusp if it is located sufficiently high. In fact we show in [27] that the main contribution to "windings around cusps" comes exactly from such long excursions. As a result the temporal DLT does not hold for cusp windings, however the set of possible limit points is much smaller than in Theorem 3.2 and is described explicitly in [27].

5.2. **More general ergodic integrals.** In order to understand the windings of horocycles one needs to study ergodic integrals $\int_0^T f(h^t\vec{v})dt$ where $f(q, v) = \omega(q)(Rv)$, ω is a closed 1 form on M and R is rotation by $-\pi/2$. In this subsection we review the results of [34, 13] concerning ergodic integrals of more general functions.

Consider the splitting $L^2(T^1M) = \mathcal{L}_p \oplus \mathcal{L}_c \oplus \mathcal{L}_d$ corresponding to principal, complimentary and discrete components of the induced representation of $SL_2(\mathbb{R})$ on $L^2(T^1M)$.

Letting $\mathcal{H}^s$ denote the Sobolev space of index s on T^1M we have a splitting $\mathcal{H}^s = \mathcal{H}^s_p + \mathcal{H}^s_c + \mathcal{H}^s_d$, where $\mathcal{H}^s_* = \mathcal{H}^s \cap \mathcal{L}_*$ for $s \geq 0$, and $\mathcal{H}^s$ is the space of distributions vanishing on the orthogonal complement of $\mathcal{H}^{-s}_*$ for $s < 0$.

A distribution $D \in (C^\infty(T^1M))^*$ is called *horocycle invariant* if

$$D\left(\left.\frac{\partial}{\partial t}\right|_{t=0}(f\circ h^t)\right) = 0 \text{ for all } f \in C^\infty(M).$$

Let $\mathcal{I}^s$ be the space of invariant distributions in $\mathcal{H}^{-s}$ and $\mathcal{I}^s_* = \mathcal{I}^s \cap \mathcal{H}^{-s}_*$. Theorem 1.1 in [34] gives the following information on $\mathcal{I}^s_*$:

- $\mathcal{I}^s_c$ is finite dimensional. If the smallest non zero eigenvalue of the Laplacian on M is greater than $\frac{1}{4}$, this dimension is equal to zero.
- $\mathcal{I}^s_p$ has infinite dimension.
- $\mathcal{I}^s_d$ has finite dimension for each s, and for each $\varepsilon > 0$,

$$\dim(\mathcal{I}^{1+\varepsilon}_d) = 2\,\text{genus}(M).$$

Let $\mathbb{T}^\infty$ and $\mathbb{R}^\infty$ denote the products of countably many copies of $\mathbb{T}$ and $\mathbb{R}$ respectively, with the product topology.

Given a 1-form ω let A_ω denote a function of the form

$$A_\omega(q,v)=\omega(v).$$

The results of [34, 13] can be summarized as follows.

Theorem 5.5. *Let $f\in\mathcal{H}^s$ with $s>\frac{11}{2}$.*
(a) $D(f)=0$ for all $D\in\mathcal{I}^s_c\oplus\mathcal{I}^s_p$ iff for a.e. $\vec v$ we have

$$\int_0^T f(h^t\vec v)dt=\int_0^T A_\omega(h^t\vec v)dt+R(h^T\vec v)-R(\vec v) \tag{5.5}$$

for some harmonic form ω and an L^2 function R. Moreover, for each $\bar s<1$, $R\in\mathcal{H}^{\bar s}$ and $\|R\|_{\mathcal{H}^{\bar s}}\le K||f||_{\mathcal{H}^s}$.

If $\frac14$ is not an eigenvalue of the Laplacian on M then:

(b) If $D(f)\neq 0$ for some $D\in\mathcal{I}^s_c$ then there is $\alpha>\frac12>\delta>0$ and a Hölder continuous non zero function $\Phi: T^1M\to\mathbb{R}$ of zero mean such that for all $\vec v$ we have

$$\int_0^T f(h^t\vec v)dt=T^\alpha\Phi(g^{\ln T}\vec v)+O\left(T^{\alpha-\delta}\right). \tag{5.6}$$

(c) If $D(f)=0$ for all $D\in\mathcal{I}^s_c$ but $D(f)\neq 0$ for some $D\in\mathcal{I}^s_p$ then there is a Hölder continuous non zero function $\Phi: T^1M\times\mathbb{T}^\infty\to\mathbb{R}$ such that $\int\Phi(\vec v,\theta)d\mu(\vec v)=0$ for each θ and a vector $u\in\mathbb{R}^\infty$ such that for all $\vec v$ we have

$$\int_0^T f(h^t\vec v)dt=\sqrt{T}\Phi(g^{\ln T}\ \vec v,u\ln T)+O\left(\ln T\right). \tag{5.7}$$

If $\frac14$ belongs to the spectrum of the Laplacian on M then parts (b) and (c) are modified as follows. There are two additional distributions $D^\pm\in\mathcal{I}^s_c$ such that if $D^-(f)\neq 0$ but $D(f)=0$ for all $D^\pm\neq D\in\mathcal{I}^s$ then (5.6) has to be replaced by

$$\int_0^T f(h^t\vec v)dt=\sqrt{T}\ln T\ \Phi(g_{\ln T}\ \vec v)+O\left(\sqrt{T}\right) \tag{5.8}$$

while if D^+ is the only distribution in $\mathcal{I}^s_c$ such that $D(f)\neq 0$ then (5.7) holds.

(d) If $D(f)=0$ for all $D\in\mathcal{I}^s_c\oplus\mathcal{I}^s_p$ then

$$\left|\int_0^T f(h^t\vec v)dt\right|\le C\ ||f||_{\mathcal{H}^s}\ \ln T.$$

Part (a) of Theorem 5.5 follows from [34, Theorem 1.2], the fact that the codimension of the space of coboundaries in $\mathcal{H}^1_d$ is $2g$ by [34, Theorem 1.1(4)] and the fact that the function $A_\omega(\vec{v})$ above is not a coboundary unless ω^* is exact. The last fact can be seen from the proof of Theorem 1.1 in [34] or from Theorem 5.1 and Lemma 2.2. Parts (b), (c) and (d) follow from Theorem 1.2 and Corollary 1.3 of [13] (for part (d) see also [34, Theorem 1.5]).

We now record several corollaries of Theorem 5.5 pertaining to single orbit DLTs. In what follows we assume that $f \in \mathcal{H}^s$ with $s > 11/2$.

Corollary 5.6. *If $D(f) = 0$ for all $D \in \mathcal{I}^s_c \oplus \mathcal{I}^s_p$, then f satisfies (5.5) and for a.e. $\vec{v}$,*

$$\frac{\int_0^t f(h^s\vec{v})ds - \mathcal{W}_g(\omega^*, \vec{v}, \ln T)}{\sqrt{\ln T}} \xrightarrow[T\to\infty]{dist} \mathfrak{N}(0, \sigma^2(\omega^*)), \text{ as } t \sim U[0,T],$$

where $\sigma^2(\cdot)$ is given by (5.3).

Corollary 5.7. *If $D(f) \neq 0$ for some $D \in \mathcal{I}^s_c \oplus \mathcal{I}^s_p$, then f does not satisfy a strong temporal DLT.*

Corollary 5.8.

(a) *If $D(f) \neq 0$ for some $D \in \mathcal{I}^s_c \oplus \mathcal{I}^s_p$, then $\int_0^T f(h^t(\vec{v}))dt$ satisfies an almost sure DLT.*

(b) *If $D(f) = 0$ for all $D \in \mathcal{I}^s_c \oplus \mathcal{I}^s_p$, then $\int_0^T f(h^t\vec{v})dt$ does not satisfy an almost sure DLT.*

Corollary 5.6 follows easily from Theorem 5.1 and Lemma 2.2. Corollary 5.7 is proved in §5.5. Corollary 5.8 is proved in §5.4.

5.3. **Temporal Large deviations.** Theorem 5.1 gives an additional information comparing to Theorem 5.5(d). It shows that for typical $\vec{v}$ most of the time the windings are of order $\sqrt{\ln T}$, not $\ln T$. However, Theorem 5.1 does not rule out that windings of typical orbits can be as large as $O(\ln T)$ during a density zero set of times. In fact, it turns out that the estimate of Theorem 5.5(d) is sharp.

Recall that given a homology class $[\gamma] \in H_1(M)$ its *stable norm* is defined as

$$||\gamma||_s = \inf_{\sum_j [r_j\gamma_j]=[\gamma]} \sum_j |r_j| \text{ length}(\gamma_j)$$

where γ_j are smooth closed curves.

Given a smooth closed 1-form ω the dual stable norm is

$$||\omega||_s = \sup_{\sigma\neq 0} \frac{|\omega(\sigma)|}{||\sigma||_s}.$$

The following statement holds

Theorem 5.9. *For every* $\vec{v}$

$$\liminf_{T\to\infty} \frac{\max_{t\leq T} |\mathcal{W}_h(\omega,\vec{v},t)|}{\ln T} \geq ||\omega||_s. \tag{5.9}$$

In fact, for almost every $\vec{v}$

$$\lim_{T\to\infty} \frac{\max_{t\leq T} |\mathcal{W}_h(\omega,\vec{v},t)|}{\ln T} = ||\omega||_s. \tag{5.10}$$

Note by contrast, that for *geodesics windings* the maximum grows as $\sqrt{T \ln\ln T}$ in view of the ASIP and the law of iterated logarithm [22] for almost every orbit. However, there are exceptional orbits with much slower growth. For example, [55] shows that there are many periodic orbits with zero winding number.

Theorem 5.9 demonstrates another interesting direction in single orbit dynamics–Single Orbit Large Deviations ([32, 16, 21]). Its proof is given in Appendix B since it is not directly related to the main theme of our paper.

5.4. **Almost sure DLTs.** In the proofs given here and in §5.5 we assume that $\frac{1}{4}$ does not belong to the spectrum of Laplacian so that either (5.6) or (5.7) hold. The case where (5.8) holds requires minor modifications which are left to the reader.

Lemma 5.10. $\mathcal{W}_h(\omega,\vec{v},t)$ *does not satisfy an almost sure DLT. In fact, for almost every* $\vec{v}$ *the following holds. For every random variable* $\mathfrak{Y}$ *there is a sequence* T_n *such that*

$$\frac{\mathcal{W}_h(\omega,\vec{v},t)}{\sqrt{\ln t}} \xrightarrow[n\to\infty]{dist} \mathfrak{Y} \oplus N(0,\sigma^2(\omega)) \text{ where } t \sim \mathrm{Log}[1,T_n].$$

Lemma 5.10 is a consequence of the following more general result.

Theorem 5.11. *Let* $\{T^t\}_{t\in\mathbb{R}}$ *be a continuous flow on a complete separable metric space* X*, preserving a probability measure* μ *and fix a continuous function* $f : X \to \mathbb{R}$*. Let* $I(t,x)$ *be the ergodic integral of* f*. Suppose that there exist a Borel function* $W : \mathbb{R}\times X \to \mathbb{R}$ *and a random variable* $\mathcal{Z}$ *such that*

(i) for each c*, the random variable* $\frac{I(t,x)-W(T,x)}{\sqrt{T}} \xrightarrow[T\to\infty]{dist} \mathcal{Z}$ *where* $t \sim U[ce^T, e^T]$;

(ii) The function $W(T,x)$ *satisfies the ASIP as* $x \sim \mu$.

Then for μ *almost every* x *the following holds.*

For every random variable $\mathfrak{Y}$ *there is a sequence* T_n *such that*

$$\frac{I(t,x)}{\sqrt{\ln t}} \xrightarrow[n\to\infty]{dist} \mathfrak{Y} \oplus \mathcal{Z} \quad \text{where} \quad t \sim \mathrm{Log}[1,T_n].$$

The proof of this theorem is similar to the proof of Theorems 3.2 and 4.6, so we omit it.

Proof of Corollary 5.8*(a).* Under the assumptions of the corollary, (5.6) or (5.7) hold. We give the proof in case (5.7), the other case is easier.

Assume (5.7), then there are $u \in \mathbb{T}^\infty$ and a continuous $\Phi : \mathbb{T}^\infty \to \mathbb{R}$ such that $I(t, \vec{v}) := \int_0^t f(h^\tau \vec{v}) d\tau = \sqrt{t}\Phi(g^{\ln t}(\vec{v}), u \ln t) + O(\ln t)$, so the almost sure DLT reduces to a distributional limit as $T \to \infty$ for the random variables $\Phi(g^{\ln t}(\vec{v}), u \ln t)$ where $t \sim \text{Log}[1, T]$.

Since $t \sim \text{Log}[1, T] \Rightarrow \ln t \sim U[0, \ln T]$, this is equivalent to a distributional limit as $T \to \infty$ for the random variables

$$\Phi(g^t(\vec{v}), ut) \text{ where } t \sim U[0, \ln T].$$

Thus it suffices to check that for a.e. $\vec{v}$, for every bounded continuous $\phi : \mathbb{R} \to \mathbb{R}$ the following limit exists and is independent of $\vec{v}$:

$$\lim_{T\to\infty} \left(\frac{1}{\ln T} \int_0^{\ln T} \phi(\Phi(g^t \vec{v}, ut)) dt \right). \tag{5.11}$$

Let $\tau^t(\theta) = \theta + tu$ denote the translation flow on $\mathbb{T}^\infty$ in "direction" $u \in \mathbb{T}^\infty$. The orbit closure of 0 is compact abelian subgroup $G \subset \mathbb{T}^\infty$, and the Haar measure m_G on G is ergodic and invariant for τ^t.

Consider $\Omega := T^1 M \times G$, equipped with the product measure

$$\mu_\Omega := (\text{normalized volume measure on } T^1 M) \times m_G.$$

Since $g^t : T^1 M \to T^1 M$ is mixing and $\tau^t : G \to G$ is ergodic, the product $g^t \times \tau^t : \Omega \to \Omega$ is ergodic. Therefore μ-a.e. $(\vec{v}, \theta) \in T^1 M \times G$ is a *generic point* for μ_Ω:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \psi(g^t \vec{v}, \theta + ut) dt = \int_\Omega \psi d\mu_\Omega \text{ for all } \phi \in C(\Omega).$$

It is easy to see that $(\vec{v}, \theta)$ is μ_Ω-generic iff $(\vec{v}, 0)$ is μ_Ω-generic. Therefore for a.e. $\vec{v}$ the limit in (5.11) exists for all bounded continuous $\phi : \mathbb{R} \to \mathbb{R}$, and equals $\int_\Omega \phi \circ \Phi d\mu_\Omega$.

This proves the almost sure DLT with the limiting random variable $Y = \Phi(\vec{v}, \theta)$, $(\vec{v}, \theta) \sim \mu_\Omega$. □

The argument presented above also proves the following more general result. Let $\{T^t\}_{t\in\mathbb{R}}$ be a continuous flow on a complete separable metric space X, preserving a probability measure μ and fix a continuous function $f : X \to \mathbb{R}$. Let $I(t, x)$ be the ergodic integral of f. Suppose that X can be embedded into a complete separable metric space Y and that there is a continuous function $F : Y \to \mathbb{R}$ which is not identically zero and a continuous (*renormalization*) flow R^t on Y preserving an ergodic measure ν such that for $x \in X$

$$I(e^t, x) = e^{\lambda t} F(R^t x) + o\left(e^{\lambda t}\right).$$

Let $\mathcal{Y}$ be the distribution of $F(y)$ then $y \sim \nu$.

Theorem 5.12. *If μ almost every x is ν-typical then $I(t,x)$ satisfies an almost sure DLT. That is*

$$\frac{I(t,x)}{t^\lambda} \xrightarrow[T\to\infty]{dist} \mathcal{Y} \quad \textit{when} \quad t \sim \mathrm{Log}[1,T].$$

Proof of Corollary 5.8*(b).* The corollary follows from Lemma 5.10 and Lemma 5.13 below.

Lemma 5.13. *If $R \in L^2(T^1M, \mathscr{B}, m)$ then for a.e. $\vec{v}$,*

$$\frac{R(h^t(\vec{v}))}{\sqrt{\ln T}} \xrightarrow[T\to\infty]{dist} 0 \quad \textit{when} \quad t \sim \mathrm{Log}[1,T].$$

Proof. It suffices to show that for a.e. $\vec{v}$

$$\frac{R(h^t\,\vec{v})}{\sqrt{\ln T_N}} \xrightarrow[N\to\infty]{\text{dist}} 0 \text{ where } t \sim \mathrm{Log}[1,T_N] \text{ and } T_N = e^N.$$

To this end it is enough to show that for each $\varepsilon_1 > 0$ and a.e. $\vec{v}$

$$\mathbb{P}_N(|R(h^t\,\vec{v})| \geq \varepsilon_1\sqrt{N}) \to 0, \tag{5.12}$$

where the index N in $\mathbb{P}_N$ indicates that we are sampling $t \sim \mathrm{Log}[1,T_N]$. To prove (5.12) note that

$$\int \mathbb{P}_N(|R(h^t\,\vec{v})| > \varepsilon_1\sqrt{N})dm(\vec{v}) = \int \mathbb{E}_N(1_{[R(h^t\,\vec{v})^2 > \varepsilon_1^2 N]})dm(\vec{v})$$
$$= \mathbb{E}_N\left(\int 1_{[R(h^t\,\vec{v})^2 > \varepsilon_1^2 N]}dm(\vec{v})\right)$$

and so by Markov inequality and the fact that $\{h^t\}$ preserves m

$$m\{\vec{v} : \mathbb{P}_N(|R(h^t\,\vec{v})| > \varepsilon_1\sqrt{N}) \geq \varepsilon_2\} \leq \frac{1}{\varepsilon_2}\int 1_{[R^2(\vec{v}) > \varepsilon_1^2 N]}dm(\vec{v}).$$

Since $R^2 \in L^1(m)$, $\sum_N \int 1_{[R^2(\vec{v})>\varepsilon_1^2 N]}dm(\vec{v}) < \infty$. By the Borel-Cantelli Lemma, for each $\varepsilon_1, \varepsilon_2$, for a.e. $\vec{v}$, $\mathbb{P}_N(|R(h^t\,\vec{v})| > \varepsilon_1\sqrt{N}) < \varepsilon_2$ for all N large enough. $\square$

5.5. **Temporal DLTs.** Here we prove Corollary 5.7. We suppose that f satisfies a strong temporal DLT a.e. with respect to the volume measure, and obtain a contradiction. We consider the case where f satisfies (5.6). The case where (5.7) is satisfied requires minor modifications which are described at the end of this section.

Let $a(\vec{v}, T)$ be the centering terms (which are allowed to depend on $\vec{v}$) and B_T be a normalizing constants (which are not allowed to depend on $\vec{v}$). Write $I(t, \vec{v}) = \int_0^t f(h^s \vec{v}) ds$ and denote

$$B_T = T^\alpha \, c(T), \quad a(\vec{v}, t) = T^\alpha \, \widetilde{a}(\vec{v}, T).$$

We claim that $c(T)$ is uniformly bounded from above and below. This is because the limiting random variable is non-constant, so there are two disjoint closed intervals J_1 and J_2 and a number $p > 0$ such that for large T

$$\mathrm{Leb}(t \in [0, T] : I(t, \vec{v}) \in a(\vec{v}, T) + B_T J_j) \geq pT \quad \text{for } j = 1, 2,$$

which is incompatible with $c(T)$ being either too large or too small in view of (5.6).

Since $c(T)$ is bounded the limiting distribution $\mathfrak{Y}$ has compact support, in particular all its moments are finite. Fix $\vec{v}$. Since Φ is uniformly bounded, $I(uT, \vec{v})/B_T$ is bounded, and therefore by the bounded convergence theorem, if $u \sim U[0, 1]$ then

$$\lim_{T \to \infty} \frac{\mathrm{Var}_u(I(uT, \vec{v}))}{c^2(T) T^{2\alpha}} = \mathrm{Var}(\mathfrak{Y}) < \infty \tag{5.13}$$

for almost all $\vec{v}$ (where Var_u = variance, when $u \sim U[0, 1]$).

Let $\eta_{\vec{v},t}$ denote the random variable $\eta_{\vec{v},t} := e^{-\alpha\tau} \Phi(g^{t-\tau} \vec{v})$ where τ has exponential distribution with mean 1 (i.e. $\tau > 0$ with density $e^{-\tau} d\tau$). Then τ has the same distribution as $-\ln u$, where $u \sim U[0, 1]$, and consequently by (5.6), the fraction in (5.13) is asymptotic to $\frac{\mathrm{Var}_\tau(\eta_{\vec{v},\ln T})}{c^2(T)}$. It follows that for a.e. $\vec{v}$,

$$\lim_{T \to \infty} \frac{\mathrm{Var}_\tau(\eta_{\vec{v},\ln T})}{c^2(T)} = \mathrm{Var}(\mathfrak{Y}).$$

Integrating over $\vec{v}$ with respect to the volume measure on $T^1 M$, we obtain (by the invariance of this measure with respect to the geodesic flow and uniform boundedness of $\eta_{\vec{v},t}$) that $c(T) \xrightarrow[T \to \infty]{}$ non-zero constant. So $B_T / T^\alpha \xrightarrow[T \to \infty]{}$ non-zero constant. Without loss of generality, $B_T = T^\alpha$.

By choice of $\widetilde{a}(\vec{v}, \ln T)$, $\eta_{\vec{v},\ln T} - \widetilde{a}(\vec{v}, T)$ converges in distribution to $\mathfrak{Y}$. By the uniform boundedness of $\eta_{\vec{v},\ln T}$,

$$\widetilde{a}(\vec{v}, T) = \mathbb{E}_\tau(\eta_{\vec{v},\ln T}) - \mathbb{E}(\mathfrak{Y}) + o(1).$$

Accordingly we may assume that $\widetilde{a}(\vec{v}, T) = \mathbb{E}_\tau(\eta_{\vec{v},\ln T})$. Since the distribution of $\eta_{\vec{v},\ln T}$ equals the distribution of $\eta_{g^{\ln T}\vec{v},0}$, $\widetilde{a}(\vec{v}, T)$ takes the form $A(g^{\ln T}\vec{v})$.

It follows that for every $a \in \mathbb{R}$, there is a function $F_a : T^1M \to [0, 1]$ such that $\mathbb{P}(\eta_{\vec{v},\ln T} - \widetilde{a}(\vec{v}, T) < a) = F_a(g^{\ln T}(\vec{v}))$. By convergence in distribution, for a.e. $\vec{v}$, for every a such that $\mathbb{P}(\mathfrak{Y} = a) = 0$, $F_a(g^{\ln T}\vec{v}) \xrightarrow[T\to\infty]{} \mathbb{P}(\mathfrak{Y} < a)$. By ergodicity, $F_a = \text{const}$ a.e. for all a, which implies that the distribution of $\eta_{\vec{v},T} - A(g^{\ln T}\vec{v})$ does not depend on $\vec{v}$ or T.

Lemma 5.14. *If there exist a function A such that the distribution of $\eta_{\vec{v},\ln T} - A(g^{\ln T}\vec{v})$ is independent of $\vec{v}$ and T then Φ is constant.*

Proof. Let $\phi(u) := \mathbb{E}(e^{iu(\eta_{\vec{v},t} - A(g^t\vec{v}))})$, the characteristic function of $\eta_{\vec{v},t} - A(g^t\vec{v})$. For a function $\mathcal{A}$ let us temporarily abbreviate $\mathcal{A}_t = \mathcal{A}(g^t\vec{v})$. Then for every u,

$$\phi(u)e^{iuA_t} = \int_{-\infty}^{t} \exp\left(iu\Phi_s\, e^{\alpha(s-t)}\right) e^{s-t} ds.$$

The right hand side is differentiable in t, and this can be used to prove that $t \mapsto e^{iuA_t}$ is differentiable for every u. This in turn implies that $t \mapsto A_t$ is differentiable. Differentiating, we obtain

$$\begin{aligned} iu\phi(u)e^{iuA_t}\dot{A}_t =& e^{iu\Phi_t} - \int_{-\infty}^{t} \exp\left(iu\Phi_s e^{\alpha(s-t)}\right) e^{s-t} ds \\ & - i\alpha u \int_{-\infty}^{t} \exp\left(iu\Phi_s e^{\alpha(s-t)}\right) \Phi_s e^{(\alpha+1)(s-t)} ds. \\ =& e^{iu\Phi_t} - \phi(u)e^{iuA_t} - \partial_u(\phi(u)e^{iuA_t}). \end{aligned} \tag{5.14}$$

Denoting $\Psi = \Phi - A$ we obtain

$$e^{iu\Psi_t} = \left[\phi(u)\left(1 + iu\dot{A}_t + iA_t\right) + \phi'(u)\right]. \tag{5.15}$$

Squaring (5.15) for u and comparing it with (5.15) for $2u$ we get

$$\begin{aligned} &\left[\phi(u)\left(1 + iu\dot{A}_t + iA_t\right) + \phi'(u)\right]^2 = \\ &\left[\phi(2u)\left(1 + i2u\dot{A}_t + iA_t\right) + \phi'(2u)\right]. \end{aligned} \tag{5.16}$$

Thus A_t satisfies a one parameter family of autonomous ODEs. However we will show that even one such ODE implies that A is constant.

So pick a small non zero u. Then the term $u^2\phi^2(u)$ in front of $(\dot{A})^2$ is non zero since $\phi(u)$ is close to 1. (5.16) defines a parabola on $(\dot{A}, A)$ plane. In fact, for fixed u, one can solve (5.16) for $\dot{A}(t)$ and obtain a relation of the form

$$\dot{A}(t) = c_1 A(t) + c_2 \pm \sqrt{c_3 A(t) + c_4}. \tag{5.17}$$

We now show that A is constant in two steps.

Step 1. We show that in fact the $\pm$ term in (5.17) has always a definite sign. If $c_3 = 0$ then this claim follows from the continuity of $\dot{A}(t)$ (see (5.14)). Therefore we assume for the rest of Step 1 that $c_3 \neq 0$. Replacing A by A+const if necessary we may assume that $c_4 = 0$ so that (5.17) takes form

$$\dot{A}(t) = c_1 A(t) + c_2 \pm \sqrt{c_3 A(t)}. \tag{5.18}$$

We claim that one of the three cases below occurs.

(i) $A \equiv 0$ for all t; or
(ii) $\dot{A}(t) = c_1 A + c_2 + \sqrt{c_3 A(t)}$ and $A(t) \neq 0$ for all t; or
(iii) $\dot{A}(t) = c_1 A + c_2 - \sqrt{c_3 A(t)}$ and $A(t) \neq 0$ for all t.

Due to ergodicity, it is enough to show that one of the alternatives (i)-(iii) occurs for all sufficiently large t. Note that due to continuity of $\dot{A}$ (see (5.14)) the transition between two branches of (5.18) is only possible when $A = 0$. Also depending on the sign of c_3 we always have either $A \geq 0$ or $A \leq 0$. This implies that if $c_2 \neq 0$ then A can not pass through 0 since in that case the solution would be strictly monotone near 0 and so it would have to change sign. It remains to consider the case $c_2 = 0$. Since $\sqrt{|A|} \gg |A|$ for $|A| \ll 1$, the solutions to $\dot{A} = \sqrt{c_3 A} + c_1 A$ have $|A|$ increasing near $A = 0$ while solutions to $\dot{A} = -\sqrt{c_3 A} + c_1 A$ have $|A|$ decreasing near $A = 0$. Therefore if $A(t_0) = 0$ for some t_0 then either $A(t) = 0$ for all $t \geq t_0$ or $\dot{A} = \sqrt{c_3 A} + c_1 A$ for all $t \neq t_0$. This completes Step 1.

Step 2. A is constant. By Step 1 A satisfies an autonomous ODE

$$\dot{A} = \mathfrak{f}(A) \tag{5.19}$$

where $\mathfrak{f}$ is a smooth function on the image of A. Namely

$$\mathfrak{f}(A) = \begin{cases} c_1 A + \bar{c}_2 & \text{if } c_3 = 0, \\ 0 & \text{in case } (i), \\ c_1 A + c_2 + \sqrt{c_3 A(t)} & \text{in case } (ii), \\ c_1 A + c_2 - \sqrt{c_3 A(t)} & \text{in case } (iii) \end{cases}$$

where $\bar{c}_2 = c_2 + \sqrt{c_4}$ or $\bar{c}_2 = c_2 - \sqrt{c_4}$. (Note that the last two expressions are smooth for $\{A \neq 0\}$.) Thus the solution to (5.19) should be monotone (or constant), since A can not pass through an equilibrium point of $\mathfrak{f}$ due to the uniqueness of ODE with smooth (Lipschitz) $\mathfrak{f}$. By ergodicity A must be constant.

Now (5.15) shows that Φ is constant. □

The analysis in case (5.7) holds is similar. Now instead of g^t we have to work with the flow $T^t(x,\theta) = (g^t x, \theta + tu)$. Thus we can not conclude that $c(T)$ is constant since the distribution of $\eta_{t,x}$ as $x \sim \mu$ depends on t. To overcome this problem let $\widetilde{\eta}_{\vec{v},\theta} = e^{-\tau/2}\Phi(T^{-\tau}(\vec{v},\theta))$ where τ has exponential distribution with parameter 1 and

$$\bar{c}(\theta) = \sqrt{\int_{T^1M} \mathrm{Var}_\tau(\widetilde{\eta}_{\vec{v},\theta}) d\mu(\vec{v})}.$$

Arguing as in the first part of the proof in case (5.6) holds we conclude that we can choose $c(T) = \bar{c}(u \ln T)$. Next we use Lemma 5.14 to conclude that $\frac{\Phi(\vec{v},\theta)}{\bar{c}(\theta)}$ is constant. In particular, Φ does not depend on $\vec{v}$. Since Φ has zero mean in $\vec{v}$ it must be zero, giving a contradiction.

6. Spatio-temporal theorems

For each of the limit properties discussed in the introduction, we gave examples of systems which do not satisfy this property. It is therefore of interest to consider weaker properties which apply to wider classes of systems. One possibility is described below.

Definition 6.1. *We say that* (f, T^t, μ) *satisfy a* spatio-temporal distributional limit theorem *if there are functions* A_T, B_T *and a non trivial random variable* Y *such that*

$$\frac{I(t,x) - A_T}{B_T} \xrightarrow[T\to\infty]{dist} Y \quad \textit{when} \quad (x,t) \sim \mu \times U(0,T).$$

As always, $I(t,x) := \int_0^T f(T^t x)dt$. A similar definition can be made for discrete time systems.

First, we show that the spatio-temporal DLT is weaker than the spatial DLT. Recall that a Borel function $R : \mathbb{R}^+ \to \mathbb{R}^+$ is called *regularly varying (of index α)* if for every $u > 0$, $R(uT)/R(u) \xrightarrow[T\to\infty]{} u^\alpha$. It is well-known that in this case the convergence is uniform on compact subsets of $u \in (0,\infty)$.

Lemma 6.2. *If there is a regularly varying function* $R(T)$ *of index* α *such that* $\frac{I(T,x)}{R(T)}$ *converges in distribution as* $x \sim \mu$ *and* $T \to \infty$ *then* (f, T^t, μ) *satisfies a spatio-temporal DLT. A similar statement holds for discrete time.*

Proof. We give a proof for flows, the case of maps is similar.

Let $t = uT$. For every continuous function Φ of compact support

$$\iint \Phi\left(\frac{I(uT,x)}{R(T)}\right) d\mu(x)du = \iint \Phi\left(\frac{R(uT)}{R(T)}\frac{I(uT,x)}{R(uT)}\right) d\mu(x)du.$$

Since R is regularly varying, Φ is bounded and uniformly continuous, and $I(T,x)/R(T)$ is tight, the last expression equals to

$$\iint \left[\Phi\left(u^\alpha \frac{I(uT,x)}{R(uT)}\right) + o(1)\right] d\mu(x)du = \iint \Phi_u\left(\frac{I(uT,x)}{R(uT)}\right) d\mu(x)du + o(1)$$

where $\Phi_u(z) = \Phi(u^\alpha z)$. By the spatial DLT for each fixed $u \neq 0$, $\int \Phi_u\left(\frac{I(uT,x)}{R(uT)}\right) d\mu(x)$ converges as $T\to\infty$ to $\mathbb{E}(\Phi_u(Y))$. Hence by the Dominated Convergence Theorem,

$$\lim_{T\to\infty}\iint \Phi\left(\frac{I(uT,x)}{R(T)}\right) d\mu(x)du = \mathbb{E}(\Phi(\mathfrak{U}^\alpha Y))$$

where $\mathfrak{U}$ is a random variable having uniform distribution on $[0,1]$ and independent of Y. This completes the proof. □

Corollary 6.3. *The following systems satisfy spatio-temporal DLTs:*

(*a*) *Anosov systems and Hölder continuous observables f which are not cohomologous to a constant;*
(*b*) *Rotations by angles α and observables $f_a(t) = 1_{[0,a)}(\{t\}) - a$ where α is a quadratic irrational and a is rational, see* §4*;*
(*c*) *horocycle flows and observables satisfying the assumptions of Theorem* 5.5 (*a*)*;*
(*d*) *horocycle flows in satisfying the assumptions of Theorem* 5.5 (*b*)*;.*

Proof. (a) and (d) follow from Lemma 6.2. The proofs of (b) and (c) are similar, so let us prove (b) as an example.

Split

$$\frac{\widetilde{S}_a(n,x)}{\sqrt{N^*}} = \frac{\widetilde{S}_a(n,x) - \xi(\psi^{N^*}(x,0))}{\sqrt{N^*}} + \frac{\xi(\psi^{N^*}(x,0))}{\sqrt{N^*}}$$

Both terms here converge to $\mathfrak{N}(0,\sigma^2)$, where σ^2 is given by (4.6), the first term by Theorem 4.1 and the second one by the CLT for Anosov diffeomorphisms. Moreover those terms are asymptotically independent since the second term depends only on x but not on n while the distribution of the first term is asymptotically independent of x. It follows that

$$\frac{\widetilde{S}_a(n,x)}{\sqrt{N^*}} \xrightarrow[T\to\infty]{\text{dist}} \mathfrak{N}(0,2\sigma^2) \text{ as } (x,n) \sim \text{Lebesgue} \times U(1,N). \quad \square$$

7. Conclusion

In this paper we discussed four types of limit theorems for three classical types of systems: Anosov systems, rotations, and horocycle flows. The state of art is summarized in the table below. Here the second column refers to Anosov diffeomorphisms and flows with respect to smooth observables, the third column refers to rotation by a quadratic irrational with observable f_a given by (4.1) with a rational, and the remaining columns refer to horocycle flows on compact hyperbolic surfaces with smooth observables satisfying cases (a),(b) or (c) in Theorem 5.5.

DLT	Anosov Systems	Irrational Rotations	Horocycle (a)	Horocycle (b)	Horocycle (c)
Spatial	Yes	No	Unknown	Yes	Unlikely
Strong temporal	No	Yes	Yes	No	No
Almost sure Temporal	Yes	No	No	Yes	Yes
Spatio -Temporal	Yes	Yes	Yes	Yes	Unlikely

Let us make a few comments on this table.

Concerning the spatial DLT for horocycle integrals in case (c) we note that the mixing of the geodesic flow implies that for large T the distribution of $\int_0^T f(h^s x)ds$ is close to the distribution of $\Phi(y, u \ln T)$ as $y \sim \mu$. Thus the spatial DLT holds only if the distribution of $\Phi(\cdot, \theta)$ does not depend on $\theta \in \mathbb{T}^\infty$. While to the best of our knowledge such a possibility has not been ruled out, it seems quite unlikely. Similar considerations show that a spatio-temporal DLT is unlikely in case (c).

The fact that rotations do not satisfy the spatial DLT was explained in the introduction. However the argument given there does not rule out that the spatial DLT holds if we discard a small subset of times.

Conjecture 7.1. *Let a, α be as in Theorem 4.1. Then there is a subset $\mathbb{M} \subset \mathbb{N}$ of density 1 such that*

$$\frac{S_a(n,x)}{\sqrt{\mathrm{Var}(S_a(n,x))}} \xrightarrow[\mathbb{M}\ni n\to\infty]{dist} \mathfrak{N}(0,1) \text{ as } x \sim U(0,1).$$

In fact, it is quite possible that the condition $\mathbb{M} \ni n \to \infty$ above can be replaced by $\mathrm{Var}(S_a(n,x)) \xrightarrow[n\to\infty]{} \infty$.

We emphasize that Conjecture 7.1 pertains only to quadratic irrationals. In fact, the results of [43] suggest that for typical rotation number the growth rate of $S_a(n,x)$ is $\ln n$ as opposed to $\sqrt{\ln n}$ for quadratic irrationals. Moreover, for typical rotation numbers the main contribution to growth of ergodic sums comes from the large elements of continued fractions which prevents the spatial DLT since the statistics of the relevant elements of continued fractions changes from one scale to the next. We refer the reader to [24] for comprehensive discussion of limit theorems for toral translations in one and higher dimensions.

Giovanni Forni has told us that a similar picture is expected for horocycle windings:

Conjecture 7.2. *For every $\omega \in H^1(M)$ there is a subset $\mathbb{S} \subset \mathbb{R}$ of density 1 such that*

$$\frac{\mathcal{W}_h(\omega, \vec{v}, t)}{\sqrt{\mathrm{Var}(\mathcal{W}_h(\omega, \vec{v}, t))}} \xrightarrow[\mathbb{S}\ni t\to\infty]{dist} \mathfrak{N}(0,1) \text{ as } x \sim m.$$

In fact, it is quite possible that the spatial DLT holds for horocycle windings, that is, one can take $\mathbb{S} = \mathbb{R}$.

We remark that the statements corresponding to Conjectures 7.1 and 7.2 have been proven in [11] for substitution dynamical systems with eigenvalues of modulus 1. We refer the reader to [52] for a temporal limit theorem in this case.

To summarize, in this paper we considered three classical models in smooth ergodic theory from the point of view of single orbit limit theorems. The examples presented here exhibit the great diversity of possible scenarios. Indeed each of the systems in our table satisfies at least one limit theorem but none satisfy all of them. We note that even for the classical examples considered in our paper we were able to obtain new results.

In this paper we tried to keep the proofs as simple as possible in order to illustrate the underlying ideas. Therefore we did not pursue the most general statements possible. In particular, we considered only rotations with *eventually periodic* continued fraction expansion, and horocycle flows on *compact* hyperbolic surfaces. Extensions of the temporal DLTs to a less restrictive setups are possible, and will be given elsewhere [26, 27].

The discussion of this section shows that there are several interesting open problems left even in the classical setting. It is also interesting to extend our analysis to other zero entropy systems studied in [14, 24, 35, 38] etc. This demonstrates that the subject of Limit Theorems for Deterministic Systems developed to a large extent through to the pioneering efforts of Sinai and Ruelle is still an active research area.

Appendix A. Veech group of the cylinder suspension.

Here we prove Proposition 4.2. We do this in two steps. First we construct many automorphisms $\psi : M \to M$, and then we show using the methods of [41] that they lift to automorphisms $\widetilde{\psi} : \widetilde{M} \to \widetilde{M}$.

A.1. **The automorphisms of M.**

Lemma A.1. *Let $\Gamma(q) := \{A \in \mathrm{SL}(2,\mathbb{Z}) : A = I \bmod q\}$. For every $A \in \Gamma(q)$ there exists a linear automorphism $\psi : M_0 \to M_0$ which fixes $\bullet, \circ$, and has derivative A. Necessarily, $\psi|_M$ is an automorphism of M.*

Proof. Every matrix $A \in \mathrm{SL}(2,\mathbb{Z})$ determines an automorphism $\psi : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ which fixes $\bullet$ and has derivative A: Take $\psi(\binom{x}{y} + \mathbb{Z}^2) = A\binom{x}{y} + \mathbb{Z}^2$. This automorphism fixes $\circ$ iff

$$A\begin{pmatrix} p/q \\ 0 \end{pmatrix} \in \begin{pmatrix} p/q \\ 0 \end{pmatrix} + \mathbb{Z}^2 \ , \ A^{-1}\begin{pmatrix} p/q \\ 0 \end{pmatrix} \in \begin{pmatrix} p/q \\ 0 \end{pmatrix} + \mathbb{Z}^2. \tag{A.1}$$

Write $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $A^{-1} = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$, then (A.1) holds iff $\frac{ap}{q} \in \frac{p}{q} + \mathbb{Z}$, $\frac{cp}{q} \in \mathbb{Z}$, $\frac{dp}{q} \in \frac{p}{q} + \mathbb{Z}$, $\frac{-cp}{q} \in \mathbb{Z}$. This is equivalent to

$$a = 1(\bmod q), \quad d = 1(\bmod q), \quad c = 0(\bmod q).$$

In particular, if $A \in \Gamma(q)$, then $\psi : M_0 \to M_0$ fixes $\bullet, \circ$ and has derivative A. □

A.2. **Lifting the automorphisms of M to $\widetilde{M}$.**

Theorem A.2 (Hooper-Weiss)**.** *Let $M, \widetilde{M}$ be as above, then every automorphism of M fixing $\bullet, \circ$ lifts to an automorphism of $\widetilde{M}$.*

Theorem A.2 can be easily deduced from general results in [41], but we decided to include a self-contained proof for completeness. The following proof, which uses the methods of [41], was explained to us by Pat Hooper and Barak Weiss.

First we give a general criterion for liftability of maps of M to maps of $\widetilde{M}$, and then we check this criterion for automorphisms.

Suppose $\psi : M \to M$ is a homeomorphism which fixes some point x_0, and fix some lift $\widetilde{x}_0 \in \widetilde{M}$. For every $\widetilde{x} \in \widetilde{M}$:

(a) Choose a smooth path $\widetilde{\gamma}_{\widetilde{x}}(t)$ from $\widetilde{x}_0$ to $\widetilde{x}$;
(b) Form the curve $\psi \circ p \circ \widetilde{\gamma}_{\widetilde{x}}$. This is a smooth path from x_0 to $\psi(p(\widetilde{x}))$;
(c) Let $\widetilde{\psi}(\widetilde{x})$:=endpoint of the lift of $\psi \circ p \circ \widetilde{\gamma}_{\widetilde{x}}$ to $\widetilde{M}$ at $\widetilde{x}_0$.

If we can show that $\widetilde{\psi}(\widetilde{x})$ is independent of the choice of $\widetilde{\gamma}_{\widetilde{x}}$, then it will be a simple matter to conclude that $\widetilde{\psi} : \widetilde{M} \to \widetilde{M}$ is a continuous map such that $p \circ \widetilde{\psi} = \psi \circ p$.

To see that this lift is invertible, we repeat the procedure for ψ^{-1}, to obtain a continuous map $\widetilde{\psi^{-1}}$ such that $p \circ \widetilde{\psi^{-1}} = \psi^{-1} \circ p$. So $p \circ \widetilde{\psi^{-1}} \circ \widetilde{\psi} = p$, whence for every $\widetilde{x} \in \widetilde{M}$ there is a $k(\widetilde{x}) \in \mathbb{Z}$ such that $(\widetilde{\psi^{-1}} \circ \widetilde{\psi})(\widetilde{x}) = D^{k(\widetilde{x})}(x)$, where D is a generator for the group of deck transformations. Since $\widetilde{\psi^{-1}} \circ \widetilde{\psi}$ is continuous and $\widetilde{M}$ is connected, $k = \text{const}$. It follows that $D^{-k} \circ \widetilde{\psi^{-1}}$ is the inverse of $\widetilde{\psi}$.

This general discussion reduces the problem of lifting $\psi : M \to M$ to a map on $\widetilde{M}$ to checking that the endpoint of the lift of $\psi \circ p \circ \widetilde{\gamma}_{\widetilde{x}}$ to $\widetilde{M}$ at $\widetilde{x}_0$ is independent of the choice of $\widetilde{\gamma}_{\widetilde{x}}$. Here is an obvious necessary and sufficient condition:

Liftability criterion: *Let γ be a closed smooth loop in M which lifts to a closed loop in $\widetilde{M}$, then $\psi \circ \gamma$, $\psi^{-1} \circ \gamma$ lift to closed loops in $\widetilde{M}$.*

Let α, β denote the linear segments on M connecting $\bullet$ to $\circ$ with parameterizations $\alpha(t) = \binom{t}{1}$ $(0 < t < p/q)$, $\beta(t) = \binom{t}{1}$ $(p/q < t < 1)$.

We call a smooth path γ on M *proper*, if it intersects α, β at most finitely many times, and all these intersections (if any) are transverse.

For such paths we can define the *intersection numbers* $i(\alpha, \gamma), i(\beta, \gamma)$ which count the points in $\gamma \cap \alpha$ (resp. $\gamma \cap \beta$) with signs $+, -$ according to the orientation of the ordered pair $\langle \alpha', \gamma' \rangle$ (resp. $\langle \beta', \gamma' \rangle$).

Lemma A.3. *A closed proper path γ on M lifts to a closed path on $\widetilde{M}$ iff $(q-p) \cdot i(\gamma, \alpha) - p \cdot i(\gamma, \beta) = 0$*

Proof. Let γ be a closed proper path on M, and $\widetilde{\gamma}$ its lift to $\widetilde{M}$. Let $t_1 < \cdots < t_N$ denote the times γ intersects $\alpha \cup \beta$. Let σ_i denote the sign of the intersection of $\gamma, \alpha \cup \beta$ at time t_i.

Every intersection with α *increases* the index of the square containing $\widetilde{\gamma}(t_i)$ by $\sigma_i(q-p)$. Every intersection of γ with β *decreases* the index of the square containing $\widetilde{\gamma}(t_i)$ by $\sigma_i p$. The lifted loop closes iff the total change is zero. □

Lemma A.4. *Let $H_1(M_0, P, \mathbb{Z})$ be the relative homology group, where $P = \{\bullet, \circ\}$. Let ω denote a (non-closed!) smooth path connecting $\bullet$ and $\circ$ (e.g. α, β).*

(a) For all proper loops γ in M, $i(\omega, \gamma)$ only depends on the homology classes $[\![\omega]\!], [\![\gamma]\!] \in H_1(M_0, P, \mathbb{Z})$.

(b) $i(\cdot, \cdot)$ is bilinear on $H_1(M_0, P, \mathbb{Z}) \times H_1(M_0, P, \mathbb{Z})$.

(c) If $\psi : M_0 \to M_0$ is a diffeomorphism which fixes $\bullet, \circ$ and $\gamma, \psi^{\pm 1} \circ \gamma$ are proper, then $i(\psi^{\pm 1} \circ \omega, \psi^{\pm 1} \circ \gamma) = \pm \sigma \cdot i(\omega, \gamma)$ where $\sigma = 1$ when ψ preserves orientation, and $\sigma = -1$ if it doesn't.

Proof. We think of M_0 as of a simplicial complex. Form the space $M^* := M_0 \uplus CP$ where CP is a cone over P. In our case this means that we attach to M a path from $\bullet$ to $\circ$ which does not intersect M. Then α is a part of a loop α^* in M^* and for every proper path $\gamma \subset M$, $i(\alpha, \gamma) = i(\alpha^*, \gamma)$.

Denote the homology classes in $H_1(M^*, \mathbb{Z})$ by $[\![\cdot]\!]$. By [56], page 43, $i(\alpha^*, \gamma)$ only depends on the (absolute) homology classes $[\alpha^*], [\gamma] \in H_1(M^*, \mathbb{Z})$. By [56], page 13, these classes only depend on the *relative* homology classes $[\![\alpha]\!], [\![\gamma]\!] \in H_1(M_0, P, \mathbb{Z})$. This proves part (a). Parts (b) and (c) are immediate. □

Lemma A.5. *Suppose $\psi : M_0 \to M_0$ is an automorphism which fixes $\bullet, \circ$, and let $\psi_* : H_1(M_0, P, \mathbb{Z}) \to H_1(M_0, P, \mathbb{Z})$ denote the homomorphism it induces. Let*

$$[\![\omega]\!] := q(q-p)[\![\alpha]\!] - pq[\![\beta]\!] \in H_1(M_0, P, \mathbb{Z}),$$

then there are $0 \neq m, n \in \mathbb{Z}$ such that $m \cdot \psi_*[\![\omega]\!] = n \cdot [\![\omega]\!]$.

Proof. The *holonomy* of a smooth path γ in M is defined to be the vector $\mathrm{hol}(\gamma) = \binom{\mathrm{hol}_x(\gamma)}{\mathrm{hol}_y(\gamma)} \in \mathbb{R}^2$ given by

$$\mathrm{hol}(\gamma) := \mathrm{endpoint}(\widetilde{\gamma}) - \mathrm{beginning}(\widetilde{\gamma})$$

for some (any) lift $\widetilde{\gamma}$ of γ to $\mathbb{R}^2$.

Two homotopic paths have the same holonomy. Therefore, hol defines a homomorphism $\pi_1(M, x_0) \to \mathbb{Z}$. Since $\mathbb{Z}$ is abelian, $\mathrm{hol}(\gamma)$ defines a homomorphism $\mathrm{hol} : H_1(M, \mathbb{Z}) \to \mathbb{Z}$.

We now work in simplicial homology. If $[\![\gamma]\!] \in H_1(M_0, P, \mathbb{Z})$ equals zero, then $\gamma = \partial c + c_p$ where c is a finite linear combination of 2-cells in M and c_p is zero or a finite linear combination of 1-cells in P. Since $P = \{\bullet, \circ\}$, there are no 1-cells in P, so $c_p = 0$ and γ is homologous to zero. So $\mathrm{hol}(\gamma) = \vec{0}$. We see that $[\![\gamma]\!] = 0$ implies that $\mathrm{hol}(\gamma) = \vec{0}$.

It follows that hol determines a homomorphism $\mathrm{hol} : H_1(M_0, P, \mathbb{Z}) \to \mathbb{Q}$. The range of values is $\mathbb{Q}$, because absolute cycles in M_0 have integral holonomies, and paths connecting $\bullet$ to $\circ$ have rational holonomies.

Calculating, we find that $\mathrm{hol}([\![\omega]\!]) = (q-p) \cdot \frac{p}{q}\vec{e}_1 - p \cdot \frac{q-p}{q}\vec{e}_1 = \vec{0}$, and $\mathrm{hol}(\psi_*[\![\omega]\!]) = d\psi\big(\mathrm{hol}([\![\omega]\!])\big) = \vec{0}$ (here we use the fact that ψ has constant derivative). Thus

$$[\![\omega]\!], \psi_*[\![\omega]\!] \in W := \{[\![\gamma]\!] \in H_1(M_0, P, \mathbb{Z}) : \mathrm{hol}([\![\gamma]\!]) = \vec{0}\}.$$

The plan now is to show that W spans a one-dimensional linear vector space over $\mathbb{Q}$. This implies that $\exists m, n \in \mathbb{Z} \setminus \{0\}$ s.t. $m \cdot [\![\omega]\!] = n \cdot \psi_* [\![\omega]\!]$.

Step 1. $\exists [\![\gamma_1]\!], [\![\gamma_2]\!], [\![\gamma_3]\!]$ s.t. $H_1(M_0, P, \mathbb{Z}) = \mathrm{span}_{\mathbb{Z}}\{[\![\gamma_1]\!], [\![\gamma_2]\!], [\![\gamma_3]\!]\}$.

Proof. The long exact sequence for relative homology states the existence of homomorphisms a, b, c, d such that the following sequence is exact ([56], page 13):

$$\cdots \to H_1(P) \to H_1(M_0) \xrightarrow{a} H_1(M_0, P) \xrightarrow{b} H_0(P) \xrightarrow{c} H_0(M_0) \xrightarrow{d} H_0(M_0, P)$$

In our case $H_1(P) = 0$, $H_1(M_0) = H_1(\mathbb{T}^2) = \mathbb{Z}^2$, $H_0(P) = \mathbb{Z}^2$, $H_0(M_0) = \mathbb{Z}$, and $H_0(M_0, P) = 0$ (see e.g. [56] pages 3, 4, 12, 39), so

$$0 \to \mathbb{Z}^2 \xrightarrow{a} H_1(M_0, P) \xrightarrow{b} \mathbb{Z}^2 \xrightarrow{c} \mathbb{Z} \xrightarrow{d} 0 \text{ is exact.}$$

We now chase arrows. Since $\ker(d) = \mathbb{Z}$, $c : \mathbb{Z}^2 \to \mathbb{Z}$ is onto, so $\ker(c) \cong \mathbb{Z}$. So $\mathrm{Im}(b) \cong \mathbb{Z}$. Choose $[\![\gamma_1]\!] \in H_1(M_0, P)$ such that $b[\![\gamma_1]\!]$ generates $\mathrm{Im}(b)$. Next by exactness, $a : \mathbb{Z}^2 \to H_1(M_0, P)$ is one-to-one so there are $[\![\gamma_2]\!]$, $[\![\gamma_3]\!] \in H_1(M_0, P)$ which generate $\mathrm{Im}(a) = \ker(b)$.

For every $[\![\xi]\!] \in H_1(M_0, P, \mathbb{Z})$ there is $k \in \mathbb{Z}$ s.t. $b[\![\xi]\!] = k \cdot b[\![\gamma_1]\!]$. So $[\![\xi]\!] - k \cdot [\![\gamma_1]\!] \in \ker(b) = \mathrm{span}_{\mathbb{Z}}\{[\![\gamma_2]\!], [\![\gamma_3]\!]\}$, whence $[\![\xi]\!] \in \mathrm{span}_{\mathbb{Z}}\{[\![\gamma_1]\!], [\![\gamma_2]\!], [\![\gamma_3]\!]\}$. Since $[\![\xi]\!]$ was arbitrary, the step is proved.

Step 2. $\dim \mathrm{span}_{\mathbb{Q}}\{\mathrm{hol}([\![\gamma_1]\!]), \mathrm{hol}([\![\gamma_2]\!]), \mathrm{hol}([\![\gamma_3]\!])\} = 2$.

Proof. By step 1, $\mathrm{span}_{\mathbb{Z}}\{\mathrm{hol}([\![\gamma_1]\!]), \mathrm{hol}([\![\gamma_2]\!]), \mathrm{hol}([\![\gamma_3]\!])\} = \mathrm{hol}(H_1(M_0, P, \mathbb{Z}))$. The last set can be easily seen to equal $\mathbb{Z}^2$, so it contains two vectors which are linearly independent over $\mathbb{Z}$, whence also over $\mathbb{Q}$.

Step 3. Completion of the proof.

By step 1, every $[\![\gamma]\!] \in W$ equals $\sum_{i=1}^{3} a_i [\![\gamma_i]\!]$ for some $a_i \in \mathbb{Z}$ which solve

$$a_1 \mathrm{hol}_x([\![\gamma_1]\!]) + a_2 \mathrm{hol}_x([\![\gamma_2]\!]) + a_3 \mathrm{hol}_x([\![\gamma_3]\!]) = 0$$
$$a_1 \mathrm{hol}_y([\![\gamma_1]\!]) + a_2 \mathrm{hol}_y([\![\gamma_2]\!]) + a_3 \mathrm{hol}_y([\![\gamma_3]\!]) = 0$$

This is a system of linear equations with rational coefficients. By step 2, the rank is two. So the space of solutions over $\mathbb{Q}$ is one-dimensional over $\mathbb{Q}$. In particular, $[\![\omega]\!] = \sum_{i=1}^{3} a_i [\![\gamma_i]\!]$ and $\psi_* [\![\omega]\!] = \sum_{i=1}^{3} b_i [\![\gamma_i]\!]$ where (a_1, a_2, a_3) and (b_1, b_2, b_3) are linearly dependent over $\mathbb{Q}$, and $\exists m, n \in \mathbb{Z} \setminus \{0\}$ s.t. $m \cdot [\![\omega]\!] = n \cdot \psi_* [\![\omega]\!]$. □

Proof of Theorem A.2. We check the liftability criterion: Let γ be a smooth loop in M, and suppose γ lifts to a closed loop in $\widetilde{M}$. We show that $\psi^{\pm 1} \circ \gamma$ lift to closed loops in $\widetilde{M}$. Obviously this property only depends on the homotopy class of γ, so there is no loss of generality in assuming that $\gamma, \psi \circ \gamma$, and $\psi^{-1} \circ \gamma$ are proper.

Since γ lifts to a closed loop in $\widetilde{M}$, $i([\![\omega]\!], [\![\gamma]\!]) = 0$ (Lemma A.3). Thus by Lemma A.4(c),

$$i(\psi_*[\![\omega]\!], [\![\psi \circ \gamma]\!]) = i([\![\psi \circ \omega]\!], [\![\psi \circ \gamma]\!]) = \pm i([\![\omega]\!], [\![\gamma]\!]) = 0.$$

By Lemma A.5, there are $m, n \in \mathbb{Z} \setminus \{0\}$ s.t. $m\psi_*[\![\omega]\!] = n[\![\omega]\!]$, so

$$\begin{aligned} 0 &= m \cdot i(\psi_*[\![\omega]\!], [\![\psi \circ \gamma]\!]) = i(m \cdot \psi_*[\![\omega]\!], [\![\psi \circ \gamma]\!]) = i(n \cdot [\![\omega]\!], [\![\psi \circ \gamma]\!]) \\ &= n \cdot i([\![\omega]\!], [\![\psi \circ \gamma]\!]), \text{ whence } i([\![\omega]\!], [\![\psi \circ \gamma]\!]) = 0. \end{aligned}$$

Since $i([\![\omega]\!], [\![\psi \circ \gamma]\!]) = 0$, $\psi \circ \gamma$ lifts to a closed loop in $\widetilde{M}$ (Lemma A.3). A similar argument shows that $\psi^{-1} \circ \gamma$ lifts to a closed loop in $\widetilde{M}$ as well.
We see that ψ satisfies the liftability criterion. By the discussion at the beginning of the section, ψ has an invertible continuous lift to $\widetilde{M}$. $\square$

Appendix B. Maximal growth

To prove Theorem 5.9 we need the following fact.

Proposition B.1.

(a) *If $0 < a < ||\omega||_s$ then there is T_0 such that for $T \geq T_0$ there exists $\vec{v}$ such that $\mathcal{W}_g(\omega, \vec{v}, T) > aT$.*
(b) *If $||\omega||_s < a$ then there is T_0 such that for $T \geq T_0$ for all $\vec{v}$ we have $\mathcal{W}_g(\omega, \vec{v}, T) < aT$.*
(c) *If $0 < a < ||\omega||_s$ then there is T_0 such that for $T \geq T_0$ there exists $\vec{v}$ such that $\mathcal{W}_g(\omega, \vec{v}, T) < -aT$.*
(d) *If $||\omega||_s < a$ then there is T_0 such that for $T \geq T_0$ for all $\vec{v}$ we have $\mathcal{W}_g(\omega, \vec{v}, T) > -aT$.*

Proof. This proposition is well known but we sketch the proof to make the paper self contained.

(a) Fix $\varepsilon > 0$. By the definition of the stable norm there is a finite set of closed curves $\gamma_1, \gamma_2, \ldots, \gamma_m$ such that

$$\sum_{j=1}^{m} r_j \text{length}(\gamma_j) = 1 \quad \text{and} \quad \sum_{j=1}^{m} r_j \omega(\gamma_j) \geq ||\omega||_s - \varepsilon.$$

Since geodesics minimize length in its homotopy class, we may assume, increasing r_j if necessary, that γ_j are geodesics. Using the specification property of geodesic flow we see that there are numbers n_0 and L such that for each T there are numbers t_j and geodesic Γ of length T such that denoting $n_j = [Tr_j]$ we have

$$d(\Gamma(t_j + t), \gamma_j(t)) \le 1 \text{ for } t \in [0, n_j - n_0], \text{ and } |t_j - \sum_{i=1}^{j-1} n_i| \le L.$$

By convexity

$$\int_\Gamma \omega = \sum_j n_j \omega(\gamma_j) + O(1).$$

Since

$$\sum_j n_j \omega(\gamma_j) = T \sum_j r_j \omega(\gamma_j) + O(1)$$

part (a) follows.

(b) Assume by contradiction that for every T_0 there are $T > T_0$ and $\vec{v}$ s.t. $\mathcal{W}_g(\omega, \vec{v}, T) \ge aT$. Let $\tilde{\Gamma} := \{g^t(\vec{v}\}_{0<t<T}$, then $\int_{\tilde{\Gamma}} \omega \ge aT$. By Anosov's Closing Lemma there is a closed geodesic Γ with $|\int_\Gamma \omega - \int_{\tilde{\Gamma}} \omega|$ bounded by a constant independent of T and $\vec{v}$. Thus if T is sufficiently large then $\omega([\Gamma]/\text{length}(\Gamma)) \ge a > ||\omega||_s$ giving a contradiction.

Parts (c),(d) follow from parts (a),(b) by substituting $-\vec{v}$ for $\vec{v}$. □

Proof of Theorem 5.9. Given $T > 0$, $0 < t < T$, let $\vec{w} := -h^{t/T} g^{\ln T} \vec{v}$, then we have by (5.4)

$$\mathcal{W}_h(\omega, \vec{v}, t) = \mathcal{W}_g(\omega, \vec{v}, \ln T) + \mathcal{W}_g(\omega, \vec{w}, \ln T) + O(1). \tag{B.1}$$

Let us assume to fix our notation that $\mathcal{W}_g(\omega, \vec{v}, \ln T) \ge 0$. By Proposition B.1(c) for each ε, if T is sufficiently large then we can find $\vec{u} \in T^1M$ such that

$$\mathcal{W}_g(\omega, \vec{u}, \ln T) \le -(||\omega||_s - \varepsilon) \ln T. \tag{B.2}$$

The vector $-\vec{u}$ does not need to belong to $\text{Hor}(g^{\ln T}\vec{v})$, but since our surface is compact, there exists L such that

$$\exists t \in [0,1], \tilde{t} \in [-L, L], \text{ and } r \in [-L, L] \text{ s.t. } \quad \vec{u} = \tilde{h}^{\tilde{t}} g^r \vec{w} \tag{B.3}$$

where $\vec{w} = h^t(-g^{\ln T}\vec{v})$ and $\tilde{h}$ denotes the stable horocycle flow.

To show (B.3) it suffices to find $\tilde{L}$ such that every pair $\vec{u}, \vec{v}$ can be joined by a three leg path consisting of orbits of g, h and $\tilde{h}$ respectively so that each leg is shorter that $\tilde{L}$. To see this represent T^1M by T^1F where F is a compact subset of $\text{PSL}(2, \mathbb{R})$ and use the NAN^- decomposition. Now apply the geodesic flow to shorten the stable leg.

(B.3) shows that

$$\mathcal{W}_g(\omega, \vec{w}, \ln T) = \mathcal{W}_g(\omega, \vec{u}, \ln T) + O(1).$$

Thus (B.2) tells us that the $\liminf$ in (5.9) is greater than $||\omega||_s - \varepsilon$. Since ε is arbitrary, (5.9) follows.

To prove (5.10) it remains to bound

$$\limsup_{T\to\infty} \max_{t\leq T} \frac{|\mathcal{W}_h(\omega, \vec{v}, t)|}{\ln T}.$$

By Ergodic Theorem for almost all $\vec{v}$, $\lim_{T\to\infty} \dfrac{\mathcal{W}_g(\omega, \vec{v}, \ln T)}{\ln T} = 0$. Thus by (B.1)

$$\limsup_{T\to\infty} \max_{t\leq T} \frac{|\mathcal{W}_h(\omega, \vec{v}, t)|}{\ln T} = \limsup \frac{|\mathcal{W}_g(\omega, \vec{w}, \ln T)|}{\ln T}$$

which is less than $||\omega||_s$ by parts (a) and (c) of Proposition B.1. □

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