Lecture 2: Symbolic Dynamics of MME

Last Time we stated without proof that BHC can carry at most one MME.

Today we will sketch the proof:

(a) We'll describe the symbolic dynamics of f on a BHC

(b) We'll calculate the form of the MME in symbolic coordinates

(c) We'll deduce uniqueness (and much more).

Markov Shifts

Setup: Suppose we're given

- a countable set of "states" S
- a collection of directed edges $E \subseteq S \times S$ denoted by $a \to b$ $\quad ((a,b) \in E)$.
- $\mathcal{G} :=$ directed graph with vertices S, edges E.

Non-Degeneracy Assumption: We assume throughout that $\forall a \in S \ \exists \xi, \eta \in S$ s.t. $\xi \to a$, $a \to \eta$.

Markov Shift: $\Sigma(\mathcal{G}) := \{ (x_i)_{i \in \mathbb{Z}} \in S^{\mathbb{Z}} : \forall i \ \ x_i \to x_{i+1} \}$

with the metric $d(\underline{x}, \underline{y}) := \exp\left[-\min\{ |i| : x_i \neq y_i \} \right]$

and the action of the left-shift map $\sigma : \Sigma(\mathcal{G}) \to \Sigma(\mathcal{G})$,

$\sigma[(x_i)_{i \in \mathbb{Z}}] = (x_{i+1})_{i \in \mathbb{Z}}$.

Exercise: The topology of $\Sigma(\mathcal{G})$ is generated by the base of (non-empty) cylinder sets

$${}_m[a_0, \dots, a_n] := \{ \underline{x} \in \Sigma(\mathcal{G}) : x_{i+m} = a_i \ \ (i = 0, \dots, n) \}$$

Exercise:

(1) $\Sigma(\mathcal{G})$ is compact iff $|S| < \infty$

(2) $\Sigma(\mathcal{G})$ is locally compact iff $\mathcal{G}$ is locally finite:

$$\forall a \in S, \begin{pmatrix} \deg_{in}(a) := \#\{\xi : \xi \to a\} < \infty \\ \deg_{out}(a) := \#\{\xi : a \to \xi\} < \infty \end{pmatrix}.$$

Exercise:

(1) The left shift map is a <u>homeomorphism</u>.

(2) It's <u>topologically transitive</u> iff $\forall a, b \in S$ $\exists$ non-empty cylinder ${}_0[a, \xi_1, \cdots, \xi_{\ell-1}, b]$, $\ell \geq 1$.

In this case we write $a \xrightarrow{\ell} b$

(3)* It's <u>topologically mixing</u> if it's top transitive, and for some $a \in S$ $\gcd\{\ell : a \xrightarrow{\ell} a\} = 1$.

Exercise: Let $\Sigma^{\#}(\mathcal{G}) := \left\{ (x_i)_{i \in \mathbb{Z}} : \begin{array}{l} (x_i)_{i \geq 0}, (x_i)_{i \leq 0} \\ \text{contain constant} \\ \text{subsequences} \end{array} \right\}$.

Prove that $\mu[\Sigma^{\#}(\mathcal{G})] = 1$ for all inv probability measures μ.

Symbolic Dynamics of Diffeos

The Coding Theorem Suppose $f \in \mathrm{Diff}^{1+\varepsilon}(M)$, $\dim M \geq 2$, M compact. Let M be a BHC. For any $\chi > 0$, there's a Markov shift $\Sigma_\chi = \Sigma(\mathcal{G})$ and a Hölder map $\pi : \Sigma \to M$ s.t.

$$\begin{tikzcd} \Sigma_\chi \arrow[r, "\sigma"] \arrow[d, "\pi"'] & \Sigma_\chi \arrow[d, "\pi"] \\ M \arrow[r, "f"'] & M \end{tikzcd}$$

commutes, and

(1) Σ_χ is topologically transitive. Moreover, $\mathcal{G}$ is locally finite, and every cylinder is compact.

(2) $\pi : \Sigma_\chi^\# \to M$ is finite-to-one

(3) $\pi(\Sigma_\chi^\#) = M \pmod{\mu}$ for each

- ergodic inv χ-hyperbolic measure on M
- projection of any σ-inv measure on Σ_χ

(4) Hyperbolicity: For some $\chi_0 > 0$, $\forall \underline{x} \in \Sigma_\chi$ there's a splitting $T_{\pi(\underline{x})} M = E^s(\underline{x}) \oplus E^u(\underline{x})$ s.t.

- $\limsup_{n \to \infty} \frac{1}{n} \log \| Df^n|_{E^s(\underline{x})} \| < -\chi_0$
- $\limsup_{n \to \infty} \frac{1}{n} \log \| Df^{-n}|_{E^u(\underline{x})} \| < -\chi_0$
- $\underline{x} \mapsto E^s(\underline{x})$, $\underline{x} \mapsto E^u(\underline{x})$ are Hölder continuous

(5) Bornological Property: $\exists \varepsilon > 0$ $\forall (\chi, \varepsilon)$-Pesin block $\Lambda \subseteq M$, $\exists$ cylinders $A_1, \dots, A_N \subseteq \Sigma$ s.t. $\pi^{-1}(\Lambda) \subseteq \bigcup_{i=1}^N A_i \pmod{\mu}$ for <u>any</u> σ-inv measure μ on Σ_χ.

We omit the proof.

We explore some of the properties of the coding.

Projection Lemma: Any ergodic σ-invariant measure $\hat{\mu}$ on Σ projects to an f-invariant measure on H given by

$$\mu = \pi_*(\hat{\mu}) := \hat{\mu} \circ \pi^{-1}$$

(1) If $\hat{\mu}$ is ergodic, μ is ergodic

(2) μ is χ_0-hyperbolic

(3) $h_\mu(f) = h_{\hat{\mu}}(\sigma)$

Sketch of Proof: Part (1) is trivial. Part (2) is by the "hyperbolicity property" of π. Part (3) is a consequence of finiteness-to-one, and the following general fact:

Abramov Thm: $\boxed{\text{countable-to-one}}$ factors preserve entropy. $\square$

Lifting Lemma: Any ergodic f-inv χ-hyperbolic measure μ on H is the projection of some σ-inv ergodic measure $\hat{\mu}$ on Σ_χ, with the same entropy.

Wrong Proof: $\hat{\mu} := \mu \circ \pi$. This doesn't work because π is not injective, so $\mu \circ \pi$ is not σ-additive.

Correct Proof (Sketch): Take a.e. ergodic component of

$$\tilde{\mu} := \int_H \frac{1}{|\Sigma^\# \cap \pi^{-1}(y)|} \Big(\sum_{\substack{\pi(\underline{x}) = y \\ \underline{x} \in \Sigma^\#}} \delta_{\underline{x}} \Big) \, d\mu(y)$$

using $\boxed{\text{finiteness-to-one}}$ to guarantee the existence of the integral. (Measurability issues are non-trivial.) $\square$

Corollary: Any $\underline{\chi\text{-hyperbolic}}$ MME μ_{max} lifts to the MME $\hat{\mu}_{max}$ of the symbolic model $\sigma : \Sigma_\chi \to \Sigma_\chi$ of its BHC $H(\mu_{max})$.

Proof: Firstly, $h_{\hat{\mu}_{max}}(\sigma) = h_{\mu_{max}}(f) = h_{top}(f)$.
If there were another σ-inv $\hat{\mu}$ s.t. $h_{\hat{\mu}}(\sigma) > h_{top}(f)$, then its projection $\mu = \pi_\chi(\hat{\mu})$ would be an f-inv prob measure with entropy $> h_{top}(f)$, which contradicts the variational principle.
$\Longrightarrow \hat{\mu}_{max}$ is an MME of $\sigma : \Sigma_\chi \to \Sigma_\chi$. $\square$

Next, we'll show that transitive Markov shifts can have at most one MME, and identify it. This will

- Imply that any BHC can carry at most one MME μ_{max} and will complete the proof that in dim 2, the number of ergodic MME is finite

- Give us an explicit formula for $\hat{\mu}_{max}$, opening the way for the proof that $\hat{\mu}_{max}$, whence μ_{max}, is Bernoulli up to a period.

Uniqueness and Structure of MME for Transitive Markov Shifts

Reduction to One-Sided Shifts: Let $\mathcal{G}$ be a countable directed graph with set of vertices ("states") $\mathcal{S}$. The One-sided Markov shift associated to $\mathcal{G}$ is

$$\Sigma^+ = \Sigma^+(\mathcal{G}) = \{(x_i)_{i\geq 0} : x_i \in \mathcal{S},\ \forall i\ x_i \to x_{i+1}\}$$

$$\sigma : \Sigma^+ \to \Sigma^+,\ \sigma : (x_i)_{i\geq 0} \mapsto (x_i)_{i\geq 1},\ d(\underline{x},\underline{y}) = e^{-\min\{i\geq 0 : x_i \neq y_i\}}$$

Exercise: $\sigma : \Sigma^+(\mathcal{G}) \to \Sigma^+(\mathcal{G})$ is top. transitive (resp top. mixing) iff $\sigma : \Sigma(\mathcal{G}) \to \Sigma(\mathcal{G})$ is top. transitive (resp top. mixing).

Let $p : \Sigma \to \Sigma^+$ be the projection $p : (x_i)_{i\in\mathbb{Z}} \longmapsto (x_i)_{i\geq 0}$. Any σ-inv prob measure μ on Σ projects to a σ-inv measure $\mu^+ := \mu \circ p^{-1}$ on Σ^+ given by

$$\mu^+({}_0[a_0, \ldots, a_{n-1}]) = \mu({}_0[a_0, \ldots, a_{n-1}]).$$

Conversely, any σ-inv prob. measure μ^+ on Σ^+ extends uniquely to a σ-inv prob. measure μ on Σ called the natural extension of μ and given by

$$\mu({}_m[a_0, \ldots, a_n]) = \mu^+({}_0[a_0, \ldots, a_{n-1}])$$

Note that $\mu \circ p^{-1} = \mu^+$.

Exercise: $h_\mu(\sigma) = h_{\mu^+}(\sigma)$

Corollary: μ is MME $\Leftrightarrow$ μ^+ is MME. Thus, It's enough to study MMEs of one-sided Markov shifts.

Ruelle's Operator: Fix some $\phi:\Sigma^+(\mathcal{G})\to\mathbb{R}$ Hölder cts (e.g. $\phi\equiv 0$). Ruelle's Operator is

$$(L_\phi f)(x) = \sum_{\sigma y = x} e^{\phi(y)} f(y) \equiv \sum_{a:\, a\to x_0} e^{\phi(a,x_0,x_1,\dots)} f(a,x_0,x_1,\dots)$$

(the domain depends on the context).

We will see that L_ϕ is intimately tied to the variational problem of maximizing $h_m(\sigma)+\int\phi\,dm$.

Let $P_G(\phi) := \sup\{h_m(\sigma)+\int\phi\,dm\}$ (the "Gurevich pressure" of ϕ).

Variational Principle: Let Σ^+ be a top transitive Markov shift with finite top entropy, and let $\phi:\Sigma^+\to\mathbb{R}$ be a (bounded) Hölder function. Then for any $a\in S$, $x_a\in[a]$,

$$\boxed{P_G(\phi) = \varlimsup_{n\to\infty} \frac{1}{n}\log(L_\phi^n 1_{[a]})(x_a)}$$

Exercise (Gurevich Variational Principle): Let Σ^+ be a top transitive Markov shift such that $\sup_m h_m(\sigma)<\infty$.* Then

$\forall a,b\in S$, $\sup\{h_m(\sigma) : m \text{ inv Borel prob. mean.}\} = \varlimsup_{n\to\infty}\frac{1}{n}\log t_{a,b}^{(n)}$,

where $t_{ab}^{(n)}$ are the entries of A^n, and A is the transition matrix of Σ^+, i.e. $A=(t_{a,b})_{S\times S}$, $t_{a,b}=\begin{cases}1 & a\to b\\ 0 & \text{else}\end{cases}$.

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* Exercise*: Remove the condition on the finiteness of $\sup_m h_m(\sigma)$.

Generalized "Ruelle's Perron-Frobenius Thm": Let $\lambda = \exp P_G(\phi)$.

Under the prev. assump., if $\exists a \in S, \exists x_a \in [a]$ s.t. $\sum_{n=1}^{\infty} \lambda^{-n} (L_\phi^n 1_{[a]})(x_a) = \infty$ ("ϕ is recurrent"), then $\exists$ Borel measure ν and $\exists$ positive continuous function h s.t. $L_\phi^* \nu = \lambda \nu$ and $L_\phi h = \lambda h$. Moreover,

(1) ν is finite and positive on cylinders, and without wandering sets of positive measure (but perhaps $\nu(\Sigma^+) = \infty$)

(2) $\log h$, $(\log h)\circ\sigma$ are uniformly Hölder on cylinders. But perhaps $\int h \, d\nu = \infty$.

(3) If $\phi = \text{const}$, h is constant on cylinders (i.e. $h(x) = h(x_0)$).

Structure Thm for Equilibrium Measures: Suppose

- (Σ^+, σ) is a top. transitive one-sided Markov shift with finite top entropy ($\sup\{h_\mu(\sigma)\} < \infty$).
- $\phi : \Sigma^+ \to \mathbb{R}$ is a bounded Hölder continuous function
- m_ϕ is an equilibrium measure:

$$h_{m_\phi}(\sigma) + \int \phi \, dm_\phi = \sup\left\{ h_\mu(\sigma) + \int \phi \, d\mu : \mu \ \sigma\text{-inv} \right\}$$

Then:

(1) Structure: $m_\phi = h \cdot d\nu$ where $h \geq 0$ is cts, ν is a Borel measure finite and positive on cylinders, and

$$L_\phi h = \lambda h, \quad L_\phi^* \nu = \lambda \nu, \quad \int h \, d\nu = 1, \quad \lambda = \exp(P_G(\phi))$$

(2) Ergodic Properties: The equilibrium is ergodic and it's natural extension is Bernoulli up to a period. In the special case of MMEs ($\phi \equiv 0$), it's Markov.

(3) Uniqueness: There's at most one equilibrium measure.

——— " ——— MME.

The proof requires some preparation.

Sub-Eigenfunctions: For any Hölder continuous ϕ $\exists h$ positive continuous s.t. $L_\phi h \le \lambda h$, $\lambda = \exp P_G(\phi)$.

Sketch of Proof:

(1) If ϕ is "recurrent" ($\sum_{n=1}^{\infty} \lambda^{-n}(L_\phi^n 1_{[a]})(x_a) = \infty$ for some $a \in S$, $x_a \in [a]$), then we can apply the "generalized Ruelle's Perron-Frobenius theorem" above.

(2) Otherwise $\exists a \in S$, $\exists x_a \in [a]$ s.t. $\sum_{n=1}^{\infty} \lambda^{-n}(L_\phi^n 1_{[a]})(x_a) < \infty$.

By Hölder continuity $\forall x \in [a]$, $\sum_{n=1}^{\infty} \lambda^{-n}(L_\phi^n 1_{[a]})(x) < \infty$.

By top transitivity $\forall x \in \Sigma^+$, $\sum_{n=1}^{\infty} \lambda^{-n}(L_\phi^n 1_{[a]})(x) < \infty$.

Moreover the partial sums are uniformly bounded and equicontinuous on partition sets.

$$\Rightarrow \quad h := \sum_{n=1}^{\infty} \lambda^{-n} L_\phi^n 1_{[a]} \text{ is positive and continuous.}$$

One checks that $L_\phi h \le \lambda h$. $\square$

The Measure $m\circ\sigma$: Let m be a σ-inv prob. measure on Σ^+. Let $\alpha = \{[a] : a\in S\}$ denote the natural partition. Define a Borel measure $m\circ\sigma$ via

$$(m\circ\sigma)(E) := \sum_{A\in\alpha} m(\sigma(E\cap A))$$

Explanation: $\forall A\in\alpha$, $\sigma|_A : A\to\sigma(A)$ is injective. But $\sigma:\Sigma^+\to\Sigma^+$ is not. The measure $m\circ\sigma$ measures the size of $\sigma(E)$ "with multiplicity."

Example: If $\Sigma^+ = \{0,1\}^{\mathbb{N}}$ then $\sigma([i]) = \Sigma^+$ but $(m\circ\sigma)([i]) = m(\sigma[1]) + m(\sigma[2]) = 2\cdot m(\Sigma^+) = 2$.

Exercise:

(1) We cannot define a measure by $\tilde{m}(E) = m(\sigma(E))$, because (sometimes) $E\mapsto m(\sigma(E))$ is not σ-additive.

(2) If $f:\Sigma\to\mathbb{R}^+$ is bounded and measurable, then

$$\int_{\Sigma^+} f\, dm\circ\sigma = \sum_{a\in S}\int_{\sigma[a]} f(ax)\, dm(x) \qquad (ax := (a, x_0, x_1, \ldots))$$

(3) If $m\circ\sigma^{-1} = m$, then $m\circ\sigma \ll m$

Sol$^{\underline{n}}$ of Exercise:

(1) If $\Sigma^+ = \{0,1\}^{\mathbb{N}}$, then $\sigma[0] = \sigma[1] = \Sigma^+$ so

$\tilde{m}([0]) + \tilde{m}([1]) = 2 \cdot m(\Sigma^+) = 2$,

But $\tilde{m}([0] \cup [1]) = \tilde{m}(\Sigma^+) = m(\sigma(\Sigma^+)) = m(\Sigma^+) = 1$.

(2) It's enough to check the identity for indicators of Borel sets:

$$\int 1_E \, dm\circ\sigma = (m\circ\sigma)(E) = \sum_{a \in S} m(\sigma(E \cap [a]))$$

$$= \sum_{a\in S} \int 1_{\sigma(E\cap[a])}(y)\, dm(y)$$

$$\overset{!}{=} \sum_{a\in S} \int_{\sigma[a]} 1_E(ay)\, dm(y).$$

$y \in \sigma(E \cap [a])$

$\Leftrightarrow \sigma|_{[a]}^{-1}(y) \in E \cap [a]$

$\Leftrightarrow ay \in E \cap [a]$

$\Leftrightarrow y \in \sigma[a],\ ay \in E$

(3) We are asked to show that for every Borel set E,

$(m\circ\sigma)(E) = 0 \Rightarrow m(E) = 0$. Indeed,

$$(m\circ\sigma)(E) = 0 \Rightarrow \sum_{a\in S} m(\sigma(E\cap[a])) = 0$$

$$\Rightarrow \forall a \in S \quad m(\sigma(E\cap[a])) = 0$$

$m\circ\sigma^{-1} = m$

$$\Rightarrow \forall a \in S \quad (m\circ\sigma^{-1})(\sigma(E\cap[a])) = 0$$

$$\Rightarrow \forall a\in S \quad m(E\cap[a]) \le (m\circ\sigma^{-1})(\sigma(E\cap[a])) = 0$$

$$\Rightarrow m(E) = \sum_{a\in S} m(E\cap[a]) = 0. \qquad \square$$

The Jacobian: The Jacobian of an invariant measure m is the Radon-Nikodym derivative

$$g_m = \frac{dm}{dm\circ\sigma} \qquad (m\circ\sigma\text{ - a.e.})$$

(I.e. the function g_m s.t. $\int f g_m \, dm\circ\sigma = \int f \, dm$.)

Explanation: The "ordinary" Jacobian is the function which appears in the change of variables formula.
The Jacobian g_m is the "Jacobian" of the "substitution" $x = ay = (a, y_0, y_1, \dots)$ for $x \in [a]$:

Fact:
$$\boxed{\int_{[a]} f \, dm = \int_{\sigma[a]} f(ax)\left(\frac{dm}{dm\circ\sigma}\right)(ax)\, dm(x)}$$

Proof:
$$\int_{[a]} f\,dm = \int_{\Sigma^+}\left(1_{[a]}\cdot f\cdot \frac{dm}{dm\circ\sigma}\right)\cdot dm\circ\sigma$$

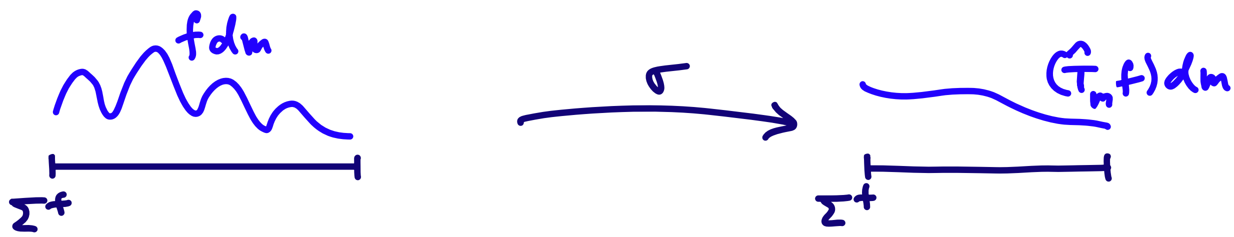
$$= \sum_{b\in S}\int_{\sigma[b]}\left(1_{[a]}\cdot f\cdot\frac{dm}{dm\circ\sigma}\right)(bx)\,dm(x) \qquad \text{(prev. exercise)}$$

$$= \int_{\sigma[a]}\left(1_{[a]}\cdot f\cdot\frac{dm}{dm\circ\sigma}\right)(ax)\,dm(x) \qquad \text{(all other terms vanish)}$$

$$= \int_{\sigma[a]} f(ax)\left(\frac{dm}{dm\circ\sigma}\right)(ax)\,dm(x).$$

The Transfer Operator: Suppose m is a σ-invariant prob measure. The transfer operator of m is $\hat{T}_m : L^1(m) \to L^1(m)$ s.t.

$$\sigma_*(f\,dm) = (\hat{T}_m f)\,dm$$



Lemma: $(\hat{T}_m f)(x) = \sum_{\sigma y = x} g_m(y) f(y)$, $\quad g_m = \dfrac{dm}{dm\circ\sigma}$

Proof: For each $\varphi \in L^\infty(m)$,

$$\int \varphi \cdot \hat{T}_m f\,dm \equiv \int \varphi \; \sigma_*(f\,dm) \equiv \int (\varphi\circ\sigma)\cdot f\,dm$$

$$= \sum_{a\in S} \int_{[a]} \varphi(\sigma x) f(x)\,dm$$

$$= \sum_{a\in S} \int_{\sigma[a]} \varphi(\sigma(ax)) f(ax)\, g_m(ax)\,dm(x)$$

$$\left(\text{because } \int_{[a]} \psi\,dm = \int_{\sigma[a]} \psi(ax)\left(\frac{dm}{dm\circ\sigma}\right)(ax)\,dm(x)\right)$$

$$= \int \varphi(x)\left(\sum_{a\in S} 1_{\sigma[a]}(x)\, f(ax)\, g_m(ax)\right) dm(x)$$

$$= \int \varphi(x)\left(\sum_{\sigma y = x} g_m(y) f(y)\right) dm(x).$$

Since this holds for all $\varphi \in L^\infty$, $\hat{T}_m f = \sum_{\sigma y = x} g_m(y) f(y)$ in L^1. $\square$

Properties of the Transfer Operator

Let m be a σ-inv prob measure, then:

(1) $\forall \varphi \in L^\infty(m)$, $f \in L^1(m)$, $\int \varphi \hat{T}_m f \, dm = \int \varphi \circ \sigma \, f \, dm$

(2) $\mathbb{E}_m(f \mid \alpha_1^\infty) = (\hat{T}_m f) \circ \sigma$, $\alpha := \{[a] : a \in S\}$,

$\alpha_1^\infty := \begin{pmatrix} \sigma\text{-algebra generated} \\ \text{by } \bigcup_{i=1}^\infty \sigma^{-i}\alpha \end{pmatrix}$

<u>Proof</u>: The first part follows from the previous proof.

To see the second part we recall the def^n of the definition of the <u>conditional expectation</u> $\mathbb{E}_m(f \mid \alpha_1^\infty)$: It's the unique $L^1(m)$ element s.t.

(a) $\mathbb{E}(f \mid \alpha_1^\infty)$ is α_1^∞-measurable;

(b) $\forall \varphi \in L^\infty$ α_1^∞-measurable, $\int \varphi \, \mathbb{E}(f \mid \alpha_1^\infty) \, dm = \int \varphi f \, dm$.

We need the following exercise: $\varphi \in L^\infty$ is α_1^∞-measurable iff $\varphi = \psi \circ \sigma$ where ψ is Borel.

<u>Exercise in Measure Theory</u>: Suppose φ is bounded, then ψ is α_1^∞-measurable iff $\varphi = \psi \circ \sigma$ for ψ Borel.

By the exercise,

(a) $(\hat{T}_m f) \circ \sigma$ is α_1^∞-measurable, and

(b) $\forall \varphi \in L^\infty$ α_1^∞-meas, $\varphi = \psi \circ \sigma$ with ψ Borel meas., and

$$\int \varphi (\hat{T}_m f) \circ \sigma \, dm = \int \psi \circ \sigma \, (\hat{T}_m f) \circ \sigma \, dm = \int \psi \hat{T}_m f \, dm$$
$$= \int \psi \circ \sigma \cdot f \, dm = \int \varphi f \, dm.$$

So $(\hat{T}_m f) \circ \sigma = \mathbb{E}_m(f \mid \alpha_1^\infty)$ a.e.

<u>Solution of Exercise:</u> The Borel σ-algebra $\mathcal{B}$ is generated by the cylinders, therefore $\mathcal{B} = \alpha_0^\infty$. It follows that $\sigma^{-1}\mathcal{B} = \alpha_1^\infty$.
Thus φ is α_1^∞-measurable iff φ is $\sigma^{-1}\mathcal{B}$-measurable.

Any $\varphi = \phi \circ \sigma$ with ϕ Borel is $\sigma^{-1}(\mathcal{B})$-measurable:

$$\{x : \varphi(x) > t\} = \{x : \phi(\sigma x) > t\} = \sigma^{-1}\{y : \phi(y) > t\} \in \mathcal{B}.$$

Conversely, if φ is $T^{-1}\mathcal{B}$-measurable, then $\forall\, |k| < N+1$
$\left\{x : \frac{k}{N} \leq \varphi(x) < \frac{k+1}{N}\right\} \in T^{-1}\mathcal{B}$, therefore $\exists\, A_{k,N} \in \mathcal{B}$ s.t.
$\left\{x : \frac{k}{N} \leq \varphi(x) < \frac{k+1}{N}\right\} = \sigma^{-1}A_{k,N}$, whence

$$\varphi(x) \equiv \lim_{N\to\infty} \sum_{k=-\|\varphi\|_\infty N-1}^{\|\varphi\|_\infty N+1} \frac{k}{N} 1_{\left[\frac{k}{N} \leq \varphi < \frac{k+1}{N}\right]} \quad \text{(uniform convergence)}$$

$$= \lim_{N\to\infty} \sum_{k=-\|\varphi\|_\infty N-1}^{\|\varphi\|_\infty N+1} \frac{k}{N} 1_{A_{k,N}} \circ \sigma = \phi \circ \sigma$$

for the Borel function $\phi := \lim_{N\to\infty} \sum_{k=-\|\varphi\|_\infty N-1}^{\|\varphi\|_\infty N+1} \frac{k}{N} 1_{A_{k,N}}$.

(The previous formula shows that the limit exists everywhere) $\square$

Entropy Formula: Our aim is to show

Thm. Suppose m_ϕ is the equilibrium measure of a (bounded) Hölder continuous potential on a Markov shift with finite top entropy. Then:

$$\boxed{h_{m_\phi}(\sigma) = -\int \log g_{m_\phi}\, dm_\phi}$$

(Here $g_{m_\phi} = dm_\phi / dm_\phi \circ \sigma$ is the Jacobian of m_ϕ.)

Review of Entropy Theory: Let m be a σ-inv prob measure.

- Natural Partition: $\alpha = \{[a] : a \in S\}$
- Entropy of α: $H_m(\alpha) = -\sum_{A \in \alpha} m(A) \log m(A)$
- Information Function: $I_m(\alpha|\alpha_1^\infty) = -\sum_{a \in S} 1_{[a]} \log \mathbb{E}_m(1_{[a]}|\alpha_1^\infty)$
- Conditional Entropy: $H_m(\alpha|\alpha_1^\infty) = \int I_m(\alpha|\alpha_1^\infty)\, dm$
- Sinai-Rokhlin Thms: $\boxed{\text{If } H_m(\alpha) < \infty,}$ then

$$h_m(\sigma) \overset{!}{=} h_m(\sigma, \alpha) \overset{!}{=} H_m(\alpha|\alpha_1^\infty) \equiv \int I_m(\alpha|\alpha_1^\infty)\, dm$$

(the marked identities could be false when $H_m(\alpha) = \infty$.)

Ledrappier Identity: For any inv Borel prob measure on a Markov shift,

$$\boxed{I_m(\alpha|\alpha_1^\infty) = -\log g_m}$$

Exercise: Use the transfer operator to prove Ledrappier's identity.

Solution: $I_m(\alpha|\alpha_1^\infty)(x) = -\sum_{a\in S} 1_{[a]}(x)\log \mathbb{E}_m(1_{[a]}|\alpha_1^\infty)(x)$

$$= -\sum_{a\in S} 1_{[a]}(x)\log(\hat{T}_m 1_{[a]})(\sigma x) \qquad (\because\ \mathbb{E}_m(f|\alpha_1^\infty) = (\hat{T}_m f)\circ\sigma)$$

$$= -\sum_{a\in S} 1_{[a]}(x)\log \sum_{\sigma y=\sigma x} g_m(y)1_{[a]}(y)$$

$$= -\log \sum_{\sigma y=\sigma x} g_m(y)\, 1_{[x_0]}(y) \qquad (\because\ a\neq x_0 \Rightarrow 1_{[a]}(x)=0)$$

$$= -\log g_m(x), \text{ because } \left.\begin{matrix}\sigma y = x_0\\ y\in[x_0]\end{matrix}\right\} \Longrightarrow y = x. \quad \square$$

Proof of Entropy Formula: $\boxed{\text{If } H_{m_\phi}(\alpha)<\infty}$, then the Sinai-Rokhlin theory tells us that

$$h_{m_\phi}(\sigma) \underset{\substack{\uparrow\\ \text{Sinai-Rokhlin}}}{=} \int I_{m_\phi}(\alpha|\alpha_1^\infty)\, dm_\phi \underset{\substack{\uparrow\\ \text{Ledrappier}}}{=} -\int \log g_{m_\phi}\, dm_\phi.$$

The rest of the proof treats the case $H_{m_\phi}(\alpha)=\infty$.

The main idea is to consider the <u>induced map</u> on some partition set $[a]$, and show that it's a Markov shift, and that its natural partition has finite entropy for the "induced" measure $\overline{m}_\phi = m_\phi(\cdot\,|[a])$. Now

$$\begin{pmatrix}\text{Rokhlin's formula}\\ \text{for the induced map}\end{pmatrix} + \begin{pmatrix}\text{Abramov's}\\ \text{formula}\end{pmatrix} + \begin{pmatrix}\text{Kac}\\ \text{formula}\end{pmatrix} \Longrightarrow \begin{pmatrix}\text{Rokhlin's}\\ \text{formula for}\\ m_\phi\end{pmatrix}$$

For details, see:

Buzzi, Sarig: Uniqueness of equilibrium measures for countable Markov shifts and multidimensional piecewise expanding maps. Ergodic Theory & Dynamical Syst. <u>23</u>, 1383–1400 (2003). $\square$

Proof of the Structure Theorem (1):

Suppose $\phi : \Sigma^+ \to \mathbb{R}$ is a (bounded) Hölder function on a top transitive Markov shift Σ^+ with finite top entropy. Necessarily,

$$P_\sigma(\phi) = \sup\{h_m(\sigma) + \int \phi\, dm\} < \infty.$$

Let $\lambda = \exp P_\sigma(\phi)$. Recall that $\exists\, h \geq 0$ continuous s.t. $L_\phi h \leq \lambda h$. Let

$$\boxed{g(x) := \frac{e^{\phi(x)} h(x)}{\lambda h(\sigma x)}}$$

<u>Exercise</u>: For each x

(1) $\sum_{\sigma y = x} g(y) \leq 1$,

(2) $0 < g(x) \leq 1$,

(3)* $\log g \in L^1(m_\phi)$ for any equilibrium measure m_ϕ,

(4) $\log h - \log h \circ \sigma \in L^1(m_\phi)$, therefore $\int (\log h - \log h\circ\sigma)\, dm_\phi = 0$.

(<u>Sol$^{\underline{n}}$ of (3)</u>: See [Buzzi - Sarig, ETDS (2003)], p. 1391).

<u>Exercise</u>: Suppose m is an inv prob measure with finite entropy. Let $g_m = \log \frac{dm}{dm\circ\sigma}$. Then:

(1) $\sum_{\sigma y = x} g_m(y) = 1$ $\quad m$-a.e. (Hint: Prove that $\hat{T}_m 1 = 1$)

(2) $0 \leq g_m(y) \leq 1$ $\quad m$-a.e.

(3) If m satisfies Rokhlin's Entropy formula, then $g_m \in L^1(m)$

Now suppose m_ϕ is an equilibrium measure of ϕ. Then

$$0 = h_{m_\phi}(\sigma) + \int \phi \, dm_\phi - P_\sigma(\phi)$$

$$\overset{!}{=} h_{m_\phi}(\sigma) + \int [\phi + (\log h - \log h\circ\sigma) - P_\sigma(\phi)] \, dm_\phi$$

prev. exercise

$$= h_{m_\phi}(\sigma) + \int \log g \, dm_\phi$$

$$= -\int \log g_{m_\phi} \, dm_\phi + \int \log g \, dm_\phi = \int \log\left(\frac{g}{g_{m_\phi}}\right) dm_\phi$$

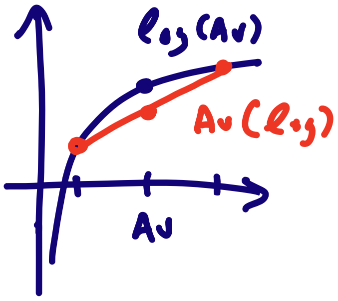
Rokhlin formula

$$= \int \hat{T}_{m_\phi}\left[\log \frac{g}{g_{m_\phi}}\right] dm_\phi \qquad \left(\because \int \hat{T}_m \varphi \, dm = \int 1\circ\sigma \cdot \varphi \, dm = \int \varphi \, dm\right)$$

$$= \int \underbrace{\sum_{\sigma y = x} g_{m_\phi}(y) \log \frac{g(y)}{g_{m_\phi}(y)}}_{\text{weighted average since } \sum_{\sigma y = x} g_{m_\phi}(y) = 1} dm_\phi$$

$$\leq \int \log\left(\sum_{\sigma y = x} g_{m_\phi}(y) \frac{g(y)}{g_{m_\phi}(y)}\right) dm_\phi = \int \log \underbrace{\sum_{\sigma y = x} g(y)}_{\leq 1} dm_\phi \leq 0.$$

Jensen's Inequality for the concave $\log x$



Necessarily the red ineq. is an equality.

$\Downarrow$ strict concavity of log

$$\boxed{\text{For } m_\phi\text{-a.e. } x, \ \frac{g(y)}{g_{m_\phi}(y)} \text{ are equal for all } y \in \sigma^{-1}(x) \text{ s.t. } g_{m_\phi}(y) \neq 0}$$

Equivalently, $\exists\, c(x)$ s.t. $g(y) = c(x)\, g_{m_\phi}(y)$ $\forall y \in \sigma^{-1}(x)$ s.t. $g_{m_\phi}(y) \neq 0$.

Substituting this above, we obtain $\int \log\Big(c(x) \underbrace{\sum_{\sigma y = x} g_{m_\phi}(y)}_{=1 \text{ a.e.}}\Big) dm_\phi = 0$

So $\int \log c(x)\, dm_\phi = 0$.

Also $1 \geq \sum_{\sigma y = x} g(y) \geq c(x) \sum_{\sigma y = x} g_{m_\phi}(y) = c(x)$ m_ϕ-a.e., so $c(x) \leq 1$ a.e.

Necessarily $c(x) = 1$ a.e., and so

$$\boxed{\text{For } m_\phi\text{-a.e. } x, \quad \left.\begin{array}{l} \sigma y = x \\ g_{m_\phi}(y) > 0 \end{array}\right\} \Rightarrow g_{m_\phi}(y) = g(x)}$$

<u>Thus</u>: $(\hat{T}_{m_\phi} f)(x) = \sum_{\sigma y = x} g(y) f(y) = \lambda^{-1} h^{-1} L_\phi(hf)$

and we have identified the transfer operator of m_ϕ.

We are now nearly done: Let $\nu = h^{-1} dm_\phi$. Then:

(1) $dm_\phi = h\, d\nu$: Trivial

(2) $L_\phi^* \nu = \lambda \nu$: $\forall f \in L^1(\nu)$, $(L_\phi^* \nu)(f) = \int (L_\phi f)\, d\nu = \int h^{-1} L_\phi(h \tfrac{f}{h}) dm_\phi$

$= \lambda \int \hat{T}_{m_\phi}(f/h)\, dm_\phi = \lambda \int (f/h)\, dm_\phi = \lambda \int f d\nu = \lambda \nu(f)$.

$\Rightarrow L_\phi^* \nu = \lambda \nu$.

<u>Exercise</u>: Show that $\mathrm{Supp}(\nu) = \Sigma^+$. (Hint: Use transitivity to show that $\forall$ cylinders $[\underline{a}], [\underline{b}]$ $\exists N$ s.t. $L_\phi^N 1_{[\underline{a}]} > 0$ on $[\underline{b}]$).

(3) $L_\phi h = \lambda h$: Since m_ϕ is σ-inv, $\hat{T}_{m_\phi} 1 = 1$, whence

$\lambda^{-1} h^{-1} L_\phi(h.1) = \hat{T}_{m_\phi} 1 = 1 \Rightarrow L_\phi h = \lambda h$ m_ϕ-a.e.

$\Rightarrow L_\phi h = \lambda h$ everywhere (because both sides of the equation are continuous functions, and $\mathrm{Supp}(m_\phi) = \mathrm{Supp}(\nu) = \Sigma^+$. $\square$

Proof that m_ϕ is Bernoulli-up-to-a Period:

We do this in the special case of MMEs ($\phi \equiv 0$).
The general case can be found in §3 of:
O. Sarig: Bernoulli equilibrium states for diffeomorphisms, J. Modern Dynamics (2011)

Recall that if $\phi \equiv 0$, then h is constant on partition sets, i.e. $h = h(x_0)$. Therefore,

$$m_\phi[a_0,\dots,a_n] = m_\phi\left(1_{[a_0,\dots,a_n]}\right) = m_\phi\left(\hat{T}^n_{m_\phi} 1_{[a_0,\dots,a_n]}\right)$$

$$= m_\phi\left(\lambda^{-n} h^{-1} L^n_\phi \left(h 1_{[a_0\cdots a_n]}\right)\right) \qquad \text{(check!)}$$

$$= m_\phi\Big(\lambda^{-n} h^{-1} \sum_{\sigma^n y = x} e^{\phi_n(y)} h(y)\, 1_{[a_0,\dots,a_n]}(y)\Big)$$

$$= m_\phi\Big(\lambda^{-n} h^{-1}\, h(a_0) \underbrace{1_{[a_0,\dots,a_n]}(a_0,\dots,a_{n-1},x)}_{= 1_{[a_n]}(x)}\Big)$$

$\phi \equiv 0$

$$= m_\phi\left(\lambda^{-n} \frac{h(a_0)}{h(a_n)} 1_{[a_n]}\right)$$

$$= \frac{m_\phi[a_n]\, h(a_0)}{\lambda^n h(a_n)} = m_\phi[a_0] \cdot \frac{h(a_0)/m_\phi[a_0]}{\lambda^n h(a_n)/m_\phi[a_n]}$$

$h(x) = h(x_0)$

$$= m_\phi[a_0] \frac{1/\partial[a_0]}{1/\partial[a_n]} = m_\phi[a_0] \cdot \frac{\partial[a_n]}{\lambda^n \partial[a_0]} = m_\phi[a_0] \cdot \prod_{i=0}^{n} \frac{\partial[a_{i+1}]}{\lambda \partial[a_i]}$$

$$= P_{a_0} \cdot P_{a_0 a_1} \cdot \ldots \cdot P_{a_{n-1} a_n},$$ where

$$\boxed{P_a = m_\phi[a] \quad \text{and} \quad P_{a,b} = \begin{cases} \dfrac{\partial[b]}{\lambda \partial[a]} & a \to b \\ 0 & \text{else.} \end{cases}}$$

Exercise: (p_a) is a prob vector; $(p_{a,b})$ is a stoch. matrix, and $\forall a$ $p_a \neq 0$; If $a \to b$, then $p_{a,b} \neq 0$.

[Sol$\underline{^n}$: (p_a) is a prob. vector because m_ϕ is a prob. measure. Next, by the previous formula $p_{ab} = \dfrac{p_a p_{ab}}{p_a} = \dfrac{m_\phi[a,b]}{m_\phi(a)}$,

so $\displaystyle\sum_{b:a\to b} p_{ab} = \sum_{b:a\to b} \frac{m_\phi[a,b]}{m_\phi(a)} = \frac{m_\phi(a)}{m_\phi(a)} = 1$.]

The exercise proves:

Gurevich's Thm: The MME of a top transitive Markov shift is a globally supported Markov measure.

It's well-known that globally supported Markov measures on transitive Markov shifts are Bernoulli-up-to-a-period. $\square$

Uniqueness of the Equilibrium Measure: Suppose by way of contradiction there were two different equilibrium measures m_ϕ', m_ϕ''. Then $m_\phi := \frac{1}{2}(m_\phi' + m_\phi'')$ is a non-ergodic equilibrium measure. But we just saw that eq. measures are Bernoulli up to a period, and this implies ergodicity. $\square$

References for Lecture 2

Main Coding Theorem: We use the formulation in
Buzzi-Crovisier-Sarig: Strong Positive Recurrence and Exponential Mixing for Diffeomorphisms. ArXiv (2025).

See also:

Sarig: Symbolic dynamics for surface diffeomorphisms with positive entropy, J. of AMS (2013)

Ben Ovadia: Symbolic dynamics for non-uniformly hyperbolic diffeomorphisms of compact smooth manifolds. J. Modern Dynam (2018)

Buzzi-Crovisier-Sarig: Measures of maximal entropy for surface diffeomorphisms, Ann. Math. (2022)

Araujo-Lima-Poletti: Symbolic dynamics for non uniformly hyperbolic maps with singularities in higher dimension. Memoirs of AMS (2024)

Structure of MME for Markov Shifts:

Buzzi & Sarig: Uniqueness of equilibrium measures for countable Markov shifts and multidimensional piecewise expanding maps ETDS (2003)

Gurevich: Shift entropy and Markov measures in the space of paths of a countable graph. Dokl. Acad. Nauk. SSSR (1970)

Parry: Intrinsic Markov chains. Trans. AMS (1964)

Sarig: Thermodynamic formalism for countable Markov shifts. Lecture notes freely available from my homepage.

Bernoulli Property: Sarig: Bernoulli equilibrium states for surface diffeomorphisms. J. Modern Dynam. (2011)