INVARIANT RADON MEASURES FOR HOROCYCLE FLOWS ON ABELIAN COVERS

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Dedicated to the memory of M. Babillot

Abstract. We classify the ergodic invariant Radon measures for horocycle flows on Z^d–covers of compact Riemannian surfaces of negative curvature, thus proving a conjecture of M. Babillot and F. Ledrappier. An important tool is a result in the ergodic theory of equivalence relations concerning the reduction of the range of a cocycle by the addition of a coboundary.

1. Introduction

This paper treats the problem of classifying the invariant measures of horocycle flows on unit tangent bundles of connected hyperbolic surfaces. The first result of this kind is due to Furstenberg [**F**], who showed that in the compact case, there is just one invariant probability measure (proportional to the Liouville measure of the geodesic flow). Dani [**D**] extended this to finite volume surfaces, and showed that every ergodic invariant measure is either proportional to the Liouville measure or is supported on a periodic orbit. Ratner's theory [**R**] implies that there are no finite invariant measures in the infinite volume case, other than those supported on periodic orbits. There are however invariant *Radon* measures. Burger [**Bu**] classified these for a large class of geometrically finite hyperbolic surfaces.

The geometrically infinite situation is still mainly open. The only result I am aware of is of Babillot & Ledrappier [**BL**], who considered regular connected $\mathbb{Z}^d$–covers of compact hyperbolic surfaces. These are surfaces $\widetilde{M}$ for which there exists a compact hyperbolic surface M and a regular covering map $p : \widetilde{M} \to M$ such that the group of deck transformations $\mathbb{G}$ is isomorphic to $\mathbb{Z}^d$ (see §5.2 for details). Let $\widetilde{h}$ and $\widetilde{g}$ be the horocycle and geodesic flows on the unit tangent bundle of $\widetilde{M}$. Babillot & Ledrappier showed that for every continuous homomorphism $\alpha : \mathbb{G} \to \mathbb{R}$, there exists an $\widetilde{h}$–ergodic invariant Radon measure (e.i.r.m.) m_α such that

$$m_\alpha \circ D_* = e^{\alpha(D)} m_\alpha \quad \text{for all } D \in \mathbb{G}. \tag{1}$$

They also showed that this measure is unique up to a multiplicative factor, and that it is $\widetilde{g}$–quasi-invariant. This gives a d–parameter family of $\widetilde{h}$–e.i.r.m.'s, and it is natural to ask whether there are others. It was proved in [**Ba**] and [**ASS**] that this family contains (up to a constant) all e.i.r.m.'s which are $\widetilde{g}$–quasi-invariant. Babillot and Ledrappier [**BL**] conjectured that all $\widetilde{h}$–e.i.r.m's are $\widetilde{g}$–quasi-invariant, or equivalently that every $\widetilde{h}$–e.i.r.m. is proportional to m_α for some α.

Our aim here is to prove this, and thus obtain the first classification of invariant Radon measures for the horocycle flow on a geometrically infinite surface.

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In fact, our result is valid in a more general setting, which we now explain. Let $\widetilde{M}$ be a regular $\mathbb{Z}^d$–cover of a compact connected orientable C^∞ Riemannian surface M whose sectional curvatures are negative, and let $S\widetilde{M}$ be its unit tangent bundle. Let $\widetilde{g}^s$ be the geodesic flow on $S\widetilde{M}$. Using the work of Margulis [**M**], Marcus [**M1**] constructed a continuous flow $\widetilde{h}^t : S\widetilde{M} \to S\widetilde{M}$ for which (see §5.1):

(a) The $\widetilde{h}$–orbit of x is equal to $W^{ss}(x) := \{y \in S\widetilde{M} : d(\widetilde{g}^s x, \widetilde{g}^s y) \xrightarrow[s\to\infty]{} 0\}$;

(b) There exists μ such that $\forall s, t,\ \widetilde{g}^{-s} \circ \widetilde{h}^t \circ \widetilde{g}^s = \widetilde{h}^{\mu^s t}$.

We regard this flow to be a generalization of the horocycle flow to the variable negative curvature case. We prove:

Theorem 1. *For every homomorphism $\alpha : \mathbb{G} \to \mathbb{R}$ there exists an $\widetilde{h}$–invariant ergodic Radon measure m_α which satisfies* (1). *This measure is unique up to a multiplicative factor. Every $\widetilde{h}$–ergodic invariant Radon measure is of this form.*

The proof of Theorem 1 is based on some technical results in the ergodic theory of equivalence relations. These are presented in the next section.

Remark. The assumption that $\mathbb{G}$ is Abelian cannot be removed: Theorem 1 and the ergodic decomposition imply that every i.r.m. which satisfies (1) is ergodic. But this is false for general finitely generated groups $\mathbb{G}$ even for $\alpha \equiv 0$ (Kaimanovich [**Ka**]). The ergodicity of m_α in the Abelian case was first proved by Babillot & Ledrappier [**BL**]. There are other proofs by Pollicott [**Po1**], Coudene [**C**], and Solomyak [**So**]. Babillot [**Ba**], Kaimanovich [**Ka**], Coudene [**C**], and Pollicott [**Po2**] give ergodicity results for certain classes of non-Abelian $\mathbb{G}$'s. It would be interesting to know if theorem 1 extends to these cases.

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2. Cocycle Reduction

Let $(X, \mathcal{F})$ be a polish space together with its Borel σ–algebra. A *countable Borel equivalence relation* on X is an equivalence relation $\mathfrak{G} \in \mathcal{F} \otimes \mathcal{F}$ whose equivalence classes are countable. A $\mathfrak{G}$*–holonomy* is a bi-measurable bijection $\kappa : A \to B$ with $A, B \in \mathcal{F}$ s.t. for all $\omega \in X$, $\big(\omega, \kappa(\omega)\big) \in \mathfrak{G}$. A measure m on $(X, \mathcal{F})$ is called $\mathfrak{G}$*–invariant*, if every $\mathfrak{G}$–holonomy satisfies $m \circ \kappa|_{dom(\kappa)} = m|_{dom(\kappa)}$. A measure m is called $(\mathfrak{G}, \Psi)$*–conformal* for some $\Psi : \mathfrak{G} \to \mathbb{R}$, if for every $\mathfrak{G}$–holonomy, $m \circ \kappa|_{dom(\kappa)} \sim m|_{dom(\kappa)}$ and $\frac{dm\circ\kappa}{dm}(x) = \exp \Psi(x, \kappa x)$. A measure m on $(X, \mathcal{F})$ is called $\mathfrak{G}$*–ergodic* if every $f \in L^1$ which is invariant under all $\mathfrak{G}$–holonomies is constant. We say that a Borel property $P(x, y)$ holds a.e. in $\mathfrak{G}$ if the set $\{x : P\big(x, y\big) \text{ holds for all } y \text{ equivalent to } x\}$ has full measure. Such sets are always measurable for countable Borel equivalence relations [**FM**].

Let $\langle \mathbb{G}, +\rangle$ be a polish Abelian group with a norm $\|\cdot\|$ such that every bounded subset of $\mathbb{G}$ is precompact (e.g. $\mathbb{R}^d$ or $\mathbb{Z}^d$ with the euclidean norm). A $\mathbb{G}$*–valued* $\mathfrak{G}$*–cocycle* is a function $\Phi : \mathfrak{G} \to \mathbb{G}$ for which

$$(\omega_1, \omega_2), (\omega_2, \omega_3) \in \mathfrak{G} \Longrightarrow \Phi(\omega_1, \omega_2) + \Phi(\omega_2, \omega_3) = \Phi(\omega_1, \omega_3).$$

Every $\mathbb{G}$–valued $\mathfrak{G}$–cocycle determines an equivalence relation $\mathfrak{G}_\Phi$ on $X \times \mathbb{G}$ via

$$\mathfrak{G}_\Phi := \{\big((x,\xi),(y,\eta)\big) \in (X\times\mathbb{G})^2 : (x,y)\in\mathfrak{G} \text{ and } \eta-\xi = \Phi(x,y)\}.$$

It turns out that the orbit equivalence relation of a suitable Poincaré section of $\widetilde{h}$ is of this form, and this leads us to consider the general problem of describing $\mathfrak{G}_\Phi$–invariant measures in terms of $\mathfrak{G}$ and Φ.

Let m be a $\mathfrak{G}_\Phi$–ergodic invariant measure on $X\times\mathbb{G}$, and assume m is *locally finite*: $m(X\times K)<\infty$ for all compact $K\subseteq\mathbb{G}$. Define

$$\mathbb{H}_m := \{a\in\mathbb{G} : m\circ Q_a \sim m\} \text{ where } Q_a(x,\xi) = (x,\xi+a).$$

It follows from [**ANSS**] that this is a closed subgroup of $\mathbb{G}$, and that

Lemma 1. *If $\mathbb{H}_m = \mathbb{G}$, then there exists a continuous homomorphism $\alpha:\mathbb{G}\to\mathbb{R}$ such that $dm = e^{-\alpha(\xi)}d\nu_\alpha(x)dm_{\mathbb{G}}(\xi)$ where ν_α is a finite $(\mathfrak{G},\alpha\circ\Phi)$–conformal measure on X, and $m_{\mathbb{G}}$ is a Haar measure of $\mathbb{G}$.*

This gives a classification of locally finite $\mathfrak{G}_\Phi$–ergodic invariant measures, when $\mathbb{H}_m = \mathbb{G}$. The problem is, of course, that $\mathbb{H}_m$ can be much smaller than $\mathbb{G}$. The following observation is very useful in this respect (for a proof see §3.1):

Lemma 2. *Let $\mathbb{H}$ be a closed subgroup of $\mathbb{G}$ and suppose $\exists u : X\to\mathbb{G}$ Borel s.t. $\Phi_u(x,y) := \Phi(x,y)+u(x)-u(y)\in\mathbb{H}$ a.e. in $\mathfrak{G}_\Phi$. Set $\vartheta_c(x,\xi) := \big(x,\xi-u(x)-c\big)$. Then $\exists c\in\mathbb{G}$ s.t. $m\circ\vartheta_c^{-1}$ is a $\mathfrak{G}_{\Phi_u}$–ergodic invariant measure on $X\times\mathbb{H}$, and $\mathbb{H}_{m\circ\vartheta_c^{-1}} = \mathbb{H}_m$. If $\operatorname{ess\,sup}\|u\|<\infty$, then $m\circ\vartheta_c^{-1}$ is locally finite.*

Cocycles of the form $u(x)-u(y)$ are called *coboundries.* Two cocycles which differ by a couboundry a.e. in $\mathfrak{G}$ are said to be *cohomologous mod m.* The function $u(x)$ is called the *transfer function.* The process of reducing the range of a cocycle by adding a coboundry is called *cocycle reduction.* The lemma suggests using cocycle reduction in cases when $\mathbb{H}_m\neq\mathbb{G}$. The crux of this paper is that this is possible:

Theorem 2 (Cocycle Reduction Theorem)**.** *Let $\mathfrak{G}$ be a countable Borel equivalence relation on a polish space X, and let $\Phi:\mathfrak{G}\to\mathbb{R}$ be a Borel cocycle. For every $\mathfrak{G}_\Phi$–ergodic invariant locally finite measure m on $X\times\mathbb{R}$, there exists a Borel function $u:X\to\mathbb{R}$ such that $\Phi(x,y)+u(x)-u(y)\in\mathbb{H}_m$ m–a.e. in $\mathfrak{G}_\Phi$.*

We show later that the range of the cocycle cannot be reduced any further (§3.1, Lemma 8). It is not true in general that $\operatorname{ess\,sup}\|u\|<\infty$ or that the cohomology holds at every point (see §6). This is however the case for the cocycles used in the proof of Theorem 1. We describe these cocycles.

Let (Σ^+,T) be a one–sided subshift of finite type with finite set of states S and transition matrix $A=(t_{ij})_{S\times S}$, $t_{ij}\in\{0,1\}$, i.e.,

$$\Sigma^+ = \{(x_0,x_1,\ldots)\in S^{\mathbb{N}\cup\{0\}} : \forall i,\ t_{x_ix_{i+1}} = 1\}$$

with the metric $d(x,y) := \sum_{i\ge0}\frac{1}{2^i}(1-\delta_{x_iy_i})$, and the map $T:\Sigma^+\to\Sigma^+$ given by $T(x_0,x_1,\ldots) = (x_1,x_2,\ldots)$. Let $\phi:\Sigma^+\to\mathbb{R}^d$ be a function with *summable variations*: $\sum var_n\phi<\infty$ where $var_n\phi = \sup\{\|\phi(x)-\phi(y)\|_2 : x_0^{n-1} = y_0^{n-1}\}$.

The *tail relation* and *grand tail relation* of (Σ^+,T) are defined, respectively, by:

$$\begin{aligned}\mathfrak{T} &:= \{(x,y)\in\Sigma^+\times\Sigma^+ : \exists n \text{ s.t } T^nx = T^ny\}\\ \widetilde{\mathfrak{T}} &:= \{(x,y)\in\Sigma^+\times\Sigma^+ : \exists p,q \text{ s.t. } T^px = T^qy\}.\end{aligned}$$

Let Per be a collection of periodic points in Σ^+ which intersects every periodic orbit of T at exactly one point. If x is preperiodic, set $n(x) := \inf\{k : T^k x \in Per\}$ and $u_{Per}(x) := \sum_{k=0}^{n(x)-1} \phi(T^k x)$. Set $\phi_n := \phi + \phi\circ T + \ldots + \phi\circ T^{n-1}$, and define $\Phi : \mathfrak{T} \to \mathbb{R}^d$ and $\widetilde{\Phi} : \widetilde{\mathfrak{T}} \to \mathbb{R}^d$ by

$$\begin{aligned}\Phi(x,y) &:= \sum_{n\geq 0}[\phi(T^n y) - \phi(T^n x)]\\ \widetilde{\Phi}(x,y) &:= \begin{cases}\phi_q(y) - \phi_p(x) & x,y \text{ not preperiodic and } T^p x = T^q y,\\ u_{Per}(y) - u_{Per}(x) & x,y \text{ are preperiodic.}\end{cases}\end{aligned} \tag{2}$$

Two sided versions of these cocycles are considered in [**KS**].

We show in §5.3 that there is a correspondence between $\widetilde{h}$–ergodic invariant Radon measures and $\widetilde{\mathfrak{T}}_{\widetilde{\Phi}}$–ergodic invariant locally finite measures, for a suitable choice of Σ^+ and ϕ. This reduces the proof of theorem 1 to the classification of $\widetilde{\mathfrak{T}}_{\widetilde{\Phi}}$–ergodic invariant locally finite measures. Set $Fix(T^n) := \{x : T^n x = x\}$ and

$$\begin{aligned}\mathbb{H}_\phi &:= \overline{\langle \phi_n(x) - \phi_n(y) : x,y \in Fix(T^n), n\in\mathbb{N}\rangle},\\ \widetilde{\mathbb{H}}_\phi &:= \overline{\langle \phi_n(x) : x \in Fix(T^n), n\in\mathbb{N}\rangle}.\end{aligned}$$

The following statements are proved in §4:

Proposition 1. *$\exists u_\phi : \Sigma^+ \to \mathbb{R}^d$ with summable variations and $c_\phi \in \mathbb{R}^d$ such that $\phi + u_\phi - u_\phi\circ T + c_\phi \in \mathbb{H}_\phi$, $\phi + u_\phi - u_\phi \circ T \in \widetilde{\mathbb{H}}_\phi$, and $\widetilde{\mathbb{H}}_\phi = \overline{\mathbb{H}_\phi + c_\phi\mathbb{Z}}$. $\mathbb{H}_\phi$ (resp. $\widetilde{\mathbb{H}}_\phi$) is contained in any closed group $\mathbb{H}$ for which $\exists u$ continuous such that $\phi + u - u\circ T + c \in \mathbb{H}$ for some c (resp. $\phi + u - u\circ T \in \mathbb{H}$).*

Proposition 1 generalizes Proposition 5.1 in [**PS**]. In what follows we use the abbreviation *e.i.l.f.* for 'ergodic, invariant, and locally finite'.

Theorem 3. *Let m be a measure on $\Sigma^+ \times \mathbb{R}^d$.*

(1) *If m is $\mathfrak{T}_\Phi$–e.i.l.f., then $\Phi(x,y) + u_\phi(x) - u_\phi(y) \in \mathbb{H}_m$ everywhere in $\mathfrak{T}$, and $\mathbb{H}_m = \mathbb{H}_\phi$.*
(2) *If m is $\widetilde{\mathfrak{T}}_{\widetilde{\Phi}}$–e.i.l.f., then $\widetilde{\Phi}(x,y) + u_\phi(x) - u_\phi(y) \in \mathbb{H}_m$ everywhere in $\widetilde{\mathfrak{T}}$, and $\mathbb{H}_m = \widetilde{\mathbb{H}}_\phi$.*

The description of $\mathbb{H}_m$ in terms of weights of periodic orbits generalizes Bowen's description of the ratio set of the Anosov foliations w.r.t. the volume measure [**B2**] (see also Series [**S**]). Theorem 3 and Lemma 2 lead to

Theorem 4. *Define for $c \in \mathbb{R}^d$,$\vartheta_c(x,\xi) = (x, \xi - u_\phi(x) - c)$, and let $m_{\mathbb{H}_\phi}$, $m_{\widetilde{\mathbb{H}}_\phi}$ be the Haar measures on $\mathbb{H}_\phi, \widetilde{\mathbb{H}}_\phi$. Let m be a measure on $\Sigma^+ \times \mathbb{R}^d$.*

(1) *If m is $\mathfrak{T}_\Phi$–e.i.l.f., then $dm\circ\vartheta_c^{-1}(x,\xi) = e^{-\langle a,\xi\rangle} d\nu_a(x) dm_{\mathbb{H}_\phi}(\xi)$ for some $c, a \in \mathbb{R}^d$ and ν_a finite s.t. $\frac{d\nu_a}{d\nu_a\circ T} \propto \exp[\langle a, \phi + u_\phi - u_\phi\circ T\rangle]$. Such a measure exists for all $a \in \mathbb{R}^d$, and is unique up to a constant factor.*
(2) *If m is $\widetilde{\mathfrak{T}}_{\widetilde{\Phi}}$–e.i.l.f., then $dm\circ\vartheta_c^{-1}(x,\xi) = e^{-\langle a,\xi\rangle} d\nu_a(x) dm_{\widetilde{\mathbb{H}}_\phi}(\xi)$ for some $c, a \in \mathbb{R}^d$ and ν_a finite s.t. $\frac{d\nu_a}{d\nu_a\circ T} = \exp[\langle a, \phi + u_\phi - u_\phi\circ T\rangle]$. Such a measure exists iff $P_{top}(\langle \alpha, \phi\rangle) = 0$, and when it exists it is unique up to a constant factor.*

Part (1) generalizes Theorem 2.2 in [**ANSS**], by removing the assumptions of finite memory and aperiodicity. Part (2) generalizes Theorem 3.1 in [**ASS**], by removing the assumptions that m is quasi–invariant under $\sigma^s(x,t,\xi) = (x, t+s, \xi)$

and that ϕ is non-arithmetic. Theorem 1 follows from Theorem 4, Part (2) and a symbolic dynamics argument. This is explained in detail in §5.

3. Proof of the Cocycle Reduction Theorem

3.1. **Preliminaries on Equivalence Relations.** Throughout this section, $\mathfrak{G}$ is a countable Borel equivalence relation on a polish space (X, d), $\mathbb{G}$ is a locally compact Abelian polish group with norm $\|\cdot\|$ such that bounded sets are precompact, and $\Phi : \mathfrak{G} \to \mathbb{G}$ is a Borel cocycle. We also fix some $\mathfrak{G}_\Phi$–ergodic invariant locally finite measure m on $X \times \mathbb{G}$. The lemmas in this section are standard.

Lemma 3. *If κ is a $\mathfrak{G}$–holonomy, then $\kappa_\Phi(x, \xi) = (\kappa x, \xi + \Phi[x, \kappa x])$ is a $\mathfrak{G}_\Phi$–holonomy, and these holonomies generate $\mathfrak{G}_\Phi$: $(x, \xi) \sim (y, \eta)$ iff $\exists$ a $\mathfrak{G}$–holonomy κ for which $(y, \eta) = \kappa_\Phi(x, \xi)$.*

In what follows, κ_Φ and κ are always assumed to be related in this way.

Lemma 4. *If m is $\mathfrak{G}_\Phi$–ergodic, and $A, B \subseteq X \times \mathbb{G}$ are measurable with positive measure, then there exists a $\mathfrak{G}_\Phi$–holonomy κ_Φ such that $m[A \cap \kappa_\Phi(B)] > 0$. In particular, almost every $x \in A$ has a holonomy κ_Φ such that $\kappa_\Phi(x) \in B$.*

Proof. The equivalence classes of $\mathfrak{G}_\Phi$ are countable. By a theorem of Feldman and Moore [**FM**], there is a countable group of transformations G acting on $X \times \mathbb{G}$ s.t.

$$\big((x, \xi), (y, \eta)\big) \in \mathfrak{G}_\Phi \text{ iff } \exists S \in G \text{ s.t. } y = S(x).$$

The set $E := \bigcup_{S \in G} S(B)$ is measurable, because G is countable. It is $\mathfrak{G}_\Phi$–invariant, by choice of G. Since m is ergodic, $E = X \times \mathbb{G} \bmod m$, whence $m(E \cap A) > 0$. The countability of G implies that $\exists S \in G$ s.t. $m[A \cap S(B)] > 0$, and this S is a holonomy by definition.

The second statement of the Lemma follows from this, because if $E := \{x \in A : S(x) \notin B \text{ for all } S \in G\}$, then $m(E) = 0$, otherwise $\exists S \in G, x \in E$ s.t. $S(x) \in B$ in contradiction to the definition of E. □

Lemma 5. *For every $B \in \mathcal{F}$ of positive measure, set $\mathfrak{G}[B] := \mathfrak{G} \cap (B \times B)$. If m is $\mathfrak{G}$–ergodic, then $m|_B$ is $\mathfrak{G}[B]$–ergodic.*

Proof. Suppose $f : B \to \mathbb{R}$ is $\mathfrak{G}[B]$–invariant. Define $F : X \to \mathbb{R}$ by $F(x) = f(y)$ where $y \in B$ satisfies $(x, y) \in \mathfrak{G}$. It is easy to see that this is a proper definition, and the previous lemma says that F is defined almost everywhere. This function is measurable, because if $G = \{S_i\}_{i \geq 1}$ is as in the previous proof, then

$$[F < t] = \bigcup_{i=1}^{\infty} S_i^{-1}[f < t].$$

It is obvious that F is $\mathfrak{G}$–invariant, and therefore F is equal a.e. to a constant. But $F|_B = f$, so f is also equal a.e. to constant. □

Lemma 6. *Suppose $\mathbb{G} = \mathbb{R}^d$, $\mathbb{H} \leq \mathbb{G}$ is closed, and μ is $\mathfrak{G}$–ergodic. If $\exists u_0(x)$ measurable s.t. $\Phi(x, y) + u_0(x) - u_0(y) \in \mathbb{H}$ a.e. in $\mathfrak{G}[B]$ for B with positive measure, then $\exists u(x)$ measurable s.t. $\Phi(x, y) + u(x) - u(y) \in \mathbb{H}$ a.e. in $\mathfrak{G}$.*

Proof. Define $U : X \to \mathbb{G}/\mathbb{H}$ by $U(x) = u_0(y) + \Phi(y, x) + \mathbb{H}$ whenever $y \in B, (x, y) \in \mathfrak{G}$. Such a y exists for a.e. x, because of the ergodicity of $\mathfrak{G}$. This is a proper definition, because for a.e. x, if $y, y' \in B$ and $(x, y), (x, y') \in \mathfrak{G}$ then

$$[u_0(y) + \Phi(y, x)] - [u_0(y') + \Phi(y', x)] = \Phi(y, y') + u_0(y) - u_0(y') \in \mathbb{H}.$$

For a.e. $x \in X$, $\exists y \in B$ s.t. $(x,y) \in \mathfrak{G}$. For this y and all x' s.t. $(x,x') \in \mathfrak{G}$,

$$U(x) - U(x') + \Phi(x,x') = u_0(y) + \Phi(y,x) - u_0(y) - \Phi(y,x') + \Phi(x,x') + \mathbb{H} = \mathbb{H}.$$

Since $\mathbb{H} \le \mathbb{G} = \mathbb{R}^d$ is closed, $\exists c : \mathbb{R}^d \to \mathbb{R}^d$ measurable s.t. $\forall \xi \in \mathbb{R}^d$, $c(\xi)+\mathbb{H} = \xi+\mathbb{H}$ and $\xi - \xi' \in \mathbb{H} \Rightarrow c(\xi) = c(\xi')$. The function $C(\xi + \mathbb{H}) := c(\xi) + \mathbb{H}$ is well–defined and Borel measurable. A standard calculation demonstrates that $u(x) := C(U(x))$ satisfies our requirements. □

Lemma 7. *Let m be a σ–finite $\mathfrak{G}$–invariant Borel measure on X. Then there exists a standard probability space $(Y, \mathcal{D}, \pi)$ and a family $\{\mu_y : y \in Y\}$ of σ–finite $\mathfrak{G}$–ergodic invariant Borel measures on X such that:*

1. *$\mu_{y_1} \perp \mu_{y_2}$ whenever $y_1 \neq y_2$;*
2. *$y \mapsto \mu_y(E)$ is $\mathcal{D}$–measurable for every $E \subseteq X$ Borel;*
3. *$m(E) = \int_Y \mu_y(E)d\pi(y)$ for all $E \subseteq X$ Borel.*

Proof. Since $\mathfrak{G}$ is a Borel equivalence relation with countable equivalence classes, there exists a countable group of bi-measurable invertible Borel maps G such that $(x,y) \in \mathfrak{G} \Leftrightarrow \exists S \in G$ s.t. $y = Sx$ ([**FM**], Theorem 1). The Lemma now follows from the ergodic decomposition for countable group actions ([**Sch**], Cor. 6.9). □

Proof of Lemma 2. Suppose $\Phi_u(x,y) := \Phi(x,y) + u(x) - u(y) \in \mathbb{H}$ a.e. in $\mathfrak{G}$ and set $\vartheta_c(x,\xi) := \big(x, \xi - u(x) - c\big)$. One checks that $\kappa_\Phi \circ \vartheta_c^{-1} = \vartheta_c^{-1} \circ \kappa_{\Phi_u}$. Consequently, if m is $\mathfrak{G}_\Phi$–ergodic invariant, then $m \circ \vartheta_c^{-1}$ is $\mathfrak{G}_{\Phi_u}$–ergodic invariant.

It is easy to see that ϑ_c^{-1} commutes with $Q_a(x,\xi) = (x, \xi + a)$ for every $a \in \mathbb{G}$. It follows that $\mathbb{H}_{m\circ\vartheta_c^{-1}} = \mathbb{H}_m$.

We claim that $\exists c \in \mathbb{G}$ for which m is carried by $X \times \mathbb{H}$. Consider $F : X \times \mathbb{G} \to \mathbb{G}/\mathbb{H}$ given by $F(x,\xi) := \xi - u(x) + \mathbb{H}$. This function is $\mathfrak{G}_\Phi$–invariant, whence constant. Therefore, $\exists c$ s.t. $F(x,\xi) = c + \mathbb{H}$ m–a.e. Therefore, m is carried by $\{(x,\xi) : \xi - u(x) \in c + \mathbb{H}\} = \vartheta_c^{-1}(X \times \mathbb{H})$.

Now suppose that m is locally finite and $\|u\| \in L^\infty(m)$. For every $K \subseteq \mathbb{G}$ compact, $\vartheta_c^{-1}(X \times K) \subseteq X \times \{\xi \in \mathbb{G} : dist(\xi, K) \le \|c\| + \operatorname{ess\,sup} \|u\|\} \bmod m$. By assumption bounded subsets of $\mathbb{G}$ are precompact and m is locally finite, so $m \circ \vartheta_c^{-1}(X \times K) < \infty$. □

Lemma 8. *Suppose m is $\mathfrak{G}_\Phi$–ergodic and invariant. Every closed group $\mathbb{H} \le \mathbb{G}$ such that Φ is cohomologous mod m to an $\mathbb{H}$–valued cocycle must contain $\mathbb{H}_m$.*

Proof. Let ϑ_c be as in Lemma 2. Then $m \circ \vartheta_c^{-1}$ is supported in $X \times \mathbb{H}$, and therefore $m \circ \vartheta_c^{-1} \circ Q_a \perp m \circ \vartheta_c^{-1}$ for all $a \notin \mathbb{H}$. It follows that $\mathbb{H} \supseteq \mathbb{H}_{m\circ\vartheta_c^{-1}} = \mathbb{H}_m$. □

3.2. **Square holes and the proof of Theorem 2.** Let $\mathfrak{G}$ be a countable Borel equivalence relation on a polish space (X,d), and suppose $\mathbb{G}$ is a normed locally compact polish Abelian group.

Definition 1. *A Borel measure m on $X \times \mathbb{G}$ is said to have* square holes *if there exists $B \subseteq X$ Borel, $\varnothing \neq U \subseteq \mathbb{G}$ open s.t. $m(B \times \mathbb{G}) > 0$ and $m(B \times U) = 0$.*

Proposition 2. *Let $\Phi : \mathfrak{G} \to \mathbb{G}$ be a Borel cocycle, and suppose m is a $\mathfrak{G}_\Phi$–e.i.l.f. measure on $X \times \mathbb{G}$. If m has no square holes, then $\mathbb{H}_m = \mathbb{G}$.*

Proof. Assume by way of contradiction that $m \not\sim m \circ Q_a$ for some $a \in \mathbb{G}$. Then necessarily $m \not\sim m \circ Q_{-a}$. By Lemma 3, Q_a commutes with all $\mathfrak{G}_\Phi$–holonomies, so $m \circ Q_{\pm a}$ are $\mathfrak{G}_\Phi$–ergodic invariant. Ergodic measures are equivalent or mutually singular, so $m \perp \overline{m} := m \circ Q_a + m \circ Q_{-a}$.

Choose $f : X \times \mathbb{G} \to \mathbb{R}_+$ uniformly continuous with the following properties: $\int f dm = 1$, $\int f d\overline{m} < \frac{1}{4}$, and $m[f > 0] < \infty$. Since f is uniformly continuous,

$$\exists \delta > 0 \text{ s.t. } \left.\begin{array}{c} d(x,y) < \delta \\ \|\xi - \eta\| < \delta \end{array}\right\} \Longrightarrow |f(x,\xi) - f(y,\eta)| < \frac{1}{4m[f \neq 0]}.$$

We use the no square holes condition to construct a $\mathfrak{G}$–holonomy κ such that $dom(\kappa) \times \mathbb{G} = X \times \mathbb{G} \bmod m$ for which $\forall x \in dom(\kappa)$,

(H1) $d(x, \kappa x) < \delta$;
(H2) $\min\big\{\|\Phi(x,\kappa x) - a\|, \|\Phi(x,\kappa x) + a\|\big\} < \delta$.

Before constructing κ, we show how it can be used to complete the proof. The inverse of κ satisfies (H1),(H2) for all $x \in im(\kappa)$, so

$$\min\{|f \circ \kappa_\Phi^{-1} - f \circ Q_a|, |f \circ \kappa_\Phi^{-1} - f \circ Q_{-a}|\} < \frac{1}{4m[f \neq 0]}.$$

Since $dom(\kappa_\Phi) = X \times \mathbb{G} \bmod m$, we have

$$\begin{aligned} 1 = \int f dm = \int f dm \circ \kappa_\Phi = \int f \circ \kappa_\Phi^{-1} dm = \int_{\kappa_\Phi[f \neq 0]} f \circ \kappa_\Phi^{-1} dm \\ \leq \int_{\kappa_\Phi[f \neq 0]} \left(f \circ Q_a + f \circ Q_{-a} + \frac{1}{4m[f \neq 0]} \right) dm \leq \int f d\overline{m} + \frac{1}{4} < \frac{1}{2}. \end{aligned}$$

This contradiction proves that $m \sim m \circ Q_a$ for all a, whence $\mathbb{H}_m = \mathbb{G}$.

Construction of κ: Let α be a countable partition of X into pairwise disjoint Borel sets with diameter less than $\frac{\delta}{2}$, and choose some Borel probability measure $P \sim m$ (such a measure exists because m is σ–finite[1]).

Fix $A \in \alpha$, and let $\mathcal{H}[A]$ be the collection of $\mathfrak{G}$–holonomies κ' with (H2) s.t. $dom(\kappa'), im(\kappa') \subseteq A$. This collection is not empty, because it contains the empty function. There is a natural partial order on $\mathcal{H}[A]$:

$$\kappa_1 \preccurlyeq \kappa_2 \text{ iff } dom(\kappa_1) \subseteq dom(\kappa_2) \text{ and } \kappa_2|_{dom(\kappa_1)} = \kappa_1.$$

Choose some $\epsilon_n \downarrow 0$, and define by induction $\mathcal{H}_n[A] \subseteq \mathcal{H}[A]$, $s_n \geq 0$, $\kappa_n \in \mathcal{H}_n[A]$ as follows: If $n = 0$, $\mathcal{H}_0[A] := \mathcal{H}[A]$, $\kappa_0 \in \mathcal{H}_0[A]$ is arbitrary, and $s_0 := \sup\{P[dom(\kappa') \times \mathbb{G}] : \kappa' \in \mathcal{H}_0[A]\}$. If $n > 1$, then

$$\begin{aligned} \mathcal{H}_n[A] &:= \{\kappa' \in \mathcal{H}_{n-1}[A] : \kappa' \succcurlyeq \kappa_{n-1}\} \\ s_n &:= \sup\{P[dom(\kappa') \times \mathbb{G}] : \kappa' \in \mathcal{H}_n[A]\} \end{aligned}$$

and choose some $\kappa_n \in \mathcal{H}_n[A]$ for which $P[dom(\kappa_n) \times \mathbb{G}] > s_n - \epsilon_n$.

By construction, $\kappa_n \preccurlyeq \kappa_{n+1}$. Let $\kappa : dom(\kappa) \to im(\kappa)$ be their common extension. This extension obviously satisfies (H2). We claim that κ is maximal in the sense that $\overline{\kappa} \in \mathcal{H}[A], \overline{\kappa} \succcurlyeq \kappa \Rightarrow P[dom(\overline{\kappa}) \times \mathbb{G}] = P[dom(\kappa) \times \mathbb{G}]$. This is because $\overline{\kappa}, \kappa \in \mathcal{H}_n[A]$ for all n, so

$$s_n - \epsilon_n \leq P[dom(\kappa_n) \times \mathbb{G}] \leq P[dom(\kappa) \times \mathbb{G}] \leq P[dom(\overline{\kappa}) \times \mathbb{G}] \leq s_n$$

[1]G is σ–compact, because the identity has a countable system of compact neighborhoods and G is separable. Since m is locally finite, it must be σ-finite.

whence $0 \leq P[dom(\overline{\kappa}) \times \mathbb{G}] - P[dom(\kappa) \times \mathbb{G}] \leq \epsilon_n \xrightarrow[n\to\infty]{} 0$.

We use this to show that either $dom(\kappa) \times \mathbb{G} = A \times \mathbb{G} \bmod m$, or $im(\kappa) \times \mathbb{G} = A \times \mathbb{G} \bmod m$. Define $A_1 := A \setminus dom(\kappa)$, $A_2 := A \setminus im(\kappa)$, and suppose by way of contradiction that $P[A_1 \times \mathbb{G}] > 0$ and $P[A_2 \times \mathbb{G}] > 0$. Let $U_1 := \{\xi \in \mathbb{G} : \|\xi\| < \frac{\delta}{2}\}$, $U_2 := \{\xi \in \mathbb{G} : \|\xi - a\| < \frac{\delta}{2}\}$. Since m has no square holes,

$$m(A_1 \times U_1) > 0 \text{ and } m(A_2 \times U_2) > 0.$$

Ergodicity implies that $\exists E_i \subseteq A_i \times U_i$ of positive measure ($i = 1, 2$) with a $\mathfrak{G}_\Phi$ holonomy $E_1 \to E_2$ (Lemma 4). W.l.o.g. this holonomy is of the form

$$(x, \xi) \mapsto \big(\kappa' x, \xi + \Phi(x, \kappa' x)\big), \text{ where } \kappa' \text{ is a } \mathfrak{G}\text{-holonomy.}$$

(If not, use Lemma 3.)

Let μ be the Borel measure on X given by $\mu(A) := P(A \times \mathbb{G})$. The set $proj(E_1) := \{x \in X : \exists \xi \in \mathbb{G} \text{ s.t. } (x, \xi) \in E_1\}$ is analytic. The universal measurability theorem (see e.g. [**Ar**], Theorem 3.2.4) says that $\exists A_1', A_1'' \subseteq X$ Borel such that $A_1' \subseteq proj(E_1) \subseteq A_1''$ and $\mu(A_1'' \setminus A_1') = 0$. Clearly $\mu(A_1') = P(A_1'' \times \mathbb{G}) \geq P(E_1) > 0$, and if $\kappa'' : A_1' \to \kappa''(A_1')$ is the restriction of κ' to A_1', then

(1) $dom(\kappa'') \subseteq A_1 = A \setminus dom(\kappa)$ and $im(\kappa'') \subseteq A_2 = A \setminus im(\kappa)$,
(2) $\forall x \in dom(\kappa'')$, $\|\Phi(x, \kappa'' x) - a\| < \delta$, since $\exists \xi$ such that $(x, \xi) \in A_1 \times U_1$ and $\big(\kappa'' x, \xi + \Phi(x, \kappa'' x)\big) \in A_2 \times U_2$.

Note that κ, κ'' have disjoint domains and images. It follows that the following map is well-defined, and belongs to $\mathcal{H}[A]$:

$$\overline{\kappa}(x) := \begin{cases} \kappa(x) & x \in dom(\kappa) \\ \kappa''(x) & x \in dom(\kappa''). \end{cases}$$

But this contradicts the maximality of κ, because $\overline{\kappa} \succneqq \kappa$ and

$$P[dom(\overline{\kappa}) \times \mathbb{G}] = P[dom(\kappa) \times \mathbb{G}] + P[A_1' \times \mathbb{G}] > P[dom(\kappa) \times \mathbb{G}].$$

This contradiction proves that

$$dom(\kappa) \times \mathbb{G} = A \times \mathbb{G} \bmod m \text{ or } im(\kappa) \times \mathbb{G} = A \times \mathbb{G} \bmod m.$$

In the first case, define $\kappa_A := \kappa$ and in the second define $\kappa_A := \kappa^{-1}$. This gives us $\kappa_A \in \mathcal{H}[A]$ which is defined a.e. in A. Now define $\overline{\kappa} : \bigcup_{A \in \alpha} A \to \bigcup_{A \in \alpha} A$ by $\overline{\kappa}(x) = \kappa_A(x)$ whenever $x \in A$. This is a well-defined bijection, because the domains and images of κ_A are pairwise disjoint. This bijection is a holonomy, and its domain is equal to $X \bmod m$. It satisfies (H1) because for every $x \in A$, $d(x, \overline{\kappa}(x)) \leq diam(A) < \delta$, and it satisfies (H2) because $\overline{\kappa}|_A = \kappa_A$ do for all A. ☐

Proposition 3. *Suppose $\Phi : \mathfrak{G} \to \mathbb{R}$ is a Borel cocycle, and m is a $\mathfrak{G}_\Phi$–e.i.l.f. measure on $X \times \mathbb{R}$. If m has a square hole, then $\exists u : X \to \mathbb{R}$ Borel and Φ–measurable such that $\Phi(x, y) + u(x) - u(y) \in \mathbb{H}_m$ m–a.e. in $\mathfrak{G}_\Phi$.*

Proof. Fix $p : X \times \mathbb{R} \to (0, \infty)$ Borel s.t. $\int_{X \times \mathbb{R}} p dm < \infty$, and define $dP := p dm$. Let μ be the measure on X given by $\mu(E) := P(E \times \mathbb{R})$. There exists a measurable map $x \mapsto P_x \in \mathcal{P}(\{x\} \times \mathbb{R})$ s.t.

$$\forall f \in L^1(P), \ \int f dP = \int_X \left(\int_{\mathbb{R}} f dP_x \right) d\mu(x).$$

Claim 1. Almost every $x \in X$ has the following property: every $\mathfrak{G}_\Phi$ holonomy κ_Φ s.t. $x \in dom(\kappa)$ satisfies $P_{\kappa(x)} \circ \kappa_\Phi \sim P_x$.

Proof. A calculation shows that for every $\mathfrak{G}_\Phi$–holonomy κ_Φ, $P_{\kappa(x)} \circ \kappa_\Phi \sim P_x$ for almost every $x \in dom(\kappa)$. Since $\mathfrak{G}_\Phi$ is a countable equivalence relation, it is generated by a countable group of holonomies [**FM**], and the claim follows.

Claim 2. $\exists a' < b' < \alpha < \beta < a < b$, $|\alpha - \beta| < \frac{1}{2}\min\{|a'-b'|, |b'-\alpha|, |\beta - a|, |a-b|\}$ *and* $\exists B \subset X$ *measurable s.t.* $\mu(B) > 0$ *and for all* $x \in B$,

(i) $P_x(\{x\} \times [a', b']) = 0, P_x(\{x\} \times [\alpha, \beta]) > 0, P_x(\{x\} \times [a, b]) = 0$;
(ii) *if* κ_Φ *is a* $\mathfrak{G}_\Phi$*–holonomy and* $x \in dom(\kappa)$*, then* $P_x \sim P_{\kappa(x)} \circ \kappa_\Phi$.

Proof. By assumption, $\exists a_0 < b_0$ and $B_0 \in X$ such that $m(B_0 \times \mathbb{R}) > 0$ and $m(B_0 \times [a_0, b_0]) = 0$. Fix $\epsilon < \frac{b_0 - a_0}{6}$. Since $m[B_0 \times \mathbb{R}] > 0$, $\exists s \in \mathbb{R} \setminus [a_0, b_0]$ such that $m[B_0 \times B_\epsilon(s)] > 0$. Suppose w.l.o.g that $s < a_0$ (the case $s > b_0$ is symmetric).

Let $B_1 \subseteq B_0$ be measurable s.t. $m[B_1 \times \mathbb{R}] > 0$, $m(B_1 \times [a_0, b_0]) = 0$ and for which $\forall x \in B_1$:

(1) $P_x \sim P_{\kappa(x)} \circ \kappa_\Phi$ for every $\mathfrak{G}_\Phi$–holonomy κ_Φ;
(2) $P_x\big(\{x\} \times [a_0, b_0]\big) = 0$;
(3) $P_x\big(\{x\} \times B_\epsilon(s)\big) > 0$.

Such a set exists, because the last two properties cannot fail on a set of full μ–measure, and the first is guaranteed by claim 1.

Consider $t := s - (\frac{a_0+b_0}{2} - s) = 2s - \frac{a_0+b_0}{2}$. We claim that $m[B_1 \times B_\epsilon(t)] = 0$. Otherwise, by Lemma 4, P–almost every $(x, \xi) \in B_1 \times B_\epsilon(t)$ has a $\mathfrak{G}_\Phi$–holonomy κ_Φ such that $\kappa_\Phi(x, \xi) \in B_1 \times B_\epsilon(s)$. This κ_Φ satisfies $\kappa(x) \in B_1$,$\Phi(x, \kappa x) \in B_{2\epsilon}(s - t)$ for all $x \in B_1$. Therefore,

$$P_x|_{\{x\}\times B_\epsilon(s)} \sim P_{\kappa x} \circ \kappa_\Phi|_{\{x\}\times B_\epsilon(s)} \ll P_{\kappa x}|_{\{\kappa x\}\times B_{3\epsilon}(2s-t)} \circ \kappa_\Phi \equiv 0,$$

(because $B_{3\epsilon}(2s - t) = B_{3\epsilon}(\frac{a_0+b_0}{2}) \subset [a_0, b_0]$). But this contradicts property (3).

Choose $B \subseteq B_1$ of full μ–measure such that $\forall x \in B_1$, $P_x\big(\{x\} \times B_\epsilon(t)\big) = 0$. Evidently, $m[B \times B_\epsilon(t)] = 0$, $m[B \times B_\epsilon(s)] > 0$, $m(B \times [a_0, b_0]) = 0$, and by construction $t + \epsilon < s - \epsilon < a_0 < b_0$.

Finally, set $a' = t-\epsilon, b' = t+\epsilon, a = a_0, b = b_0$ and choose $\alpha, \beta \in B_\epsilon(s)$ sufficiently close so that $m(B \times [\alpha, \beta]) > 0$, $|\alpha - \beta| < \frac{1}{2}\min\{|a' - b'|, |b' - \alpha|, |\beta - a|, |a - b|\}$ and $b' < \alpha < \beta < a_0$.

Claim 3. $\exists u_0 : B \to \mathbb{R}$ *measurable s.t.* $\forall\big((x, \xi), (x', \eta)\big) \in \mathfrak{G}_\Phi\big[B\times[\alpha, \beta]\big]$, $\Phi(x, x') = u_0(x') - u_0(x)$ *and* $\xi = u_0(x)$ *for a.e.* $(x, \xi) \in B \times [\alpha, \beta]$.

Proof. Set

$$\begin{aligned} U(x) &:= \inf\{b' \le t \le b : P_x\big(\{x\} \times (t, b]\big) = 0\}, \\ L(x) &:= \sup\{a' \le t \le a : P_x\big(\{x\} \times [a', t)\big) = 0\}. \end{aligned}$$

If $x \in B, x' = \kappa(x) \in B$, and $|\Phi(x, x')| < |\alpha - \beta|$, then

$$\begin{aligned} U(x') &= U\big(\kappa(x)\big) = \inf\{b' \le t \le b : P_{\kappa(x)}\big(\{\kappa(x)\} \times (t, b]\big) = 0\} \\ &= \inf\{\alpha \le t \le a : P_{\kappa(x)}\big(\{\kappa(x)\} \times (t, \tfrac{a+b}{2}]\big) = 0\} \\ &= \inf\{\alpha \le t \le a : P_{\kappa(x)} \circ \kappa_\Phi\big(\{x\} \times (t - \Phi(x, x'), \tfrac{a+b}{2} - \Phi(x, x')]\big) = 0\} \\ &= \inf\{\alpha \le t \le a : P_x\big(\{x\} \times (t - \Phi(x, x'), a]\big) = 0\} \\ &= \inf\{\alpha - \Phi(x, x') \le s \le a - \Phi(x, x') : P_x\big(\{x\} \times (s, a]\big) = 0\} + \Phi(x, x') \\ &\ge \inf\{b' \le s \le b : P_x\big(\{x\} \times (s, a]\big) = 0\} + \Phi(x, x') \\ &= \inf\{b' \le s \le b : P_x\big(\{x\} \times (s, b]\big) = 0\} + \Phi(x, x') \equiv U(x) + \Phi(x, x') \end{aligned}$$

In the same way one shows that if $x \in B, x' = \kappa(x) \in B$, and $|\Phi(x,x')| < |\alpha - \beta|$, then $L(x') \leq L(x) + \Phi(x,x')$.

It follows that if $\big((x,\xi),(x',\eta)\big) \in \mathfrak{G}_\Phi \cap \big(B \times [\alpha,\beta]\big)^2$, then

$$L(x') - L(x) \leq \Phi(x,x') \leq U(x') - U(x),$$

whence $(L-U)(x') \leq (L-U)(x)$. By symmetry, the inequality is an equality and thus $\big((x,\xi),(x',\eta)\big) \in \mathfrak{G}_\Phi \cap \big(B \times [\alpha,\beta]\big)^2 \Rightarrow (L-U)(x') = (L-U)(x)$.

Since $m(B\times[\alpha,\beta]) > 0$ and $\mathfrak{G}_\Phi$ is ergodic, $\mathfrak{G}_\Phi \cap \big(B\times[\alpha,\beta]\big)^2$ is ergodic (Lemma 5), whence $L - U = const$. This proves that

$$\big((x,\xi),(x',\eta)\big) \in \mathfrak{G}_\Phi \cap \big(B \times [\alpha,\beta]\big)^2 \Rightarrow \Phi(x,x') = U(x') - U(x).$$

It follows that $F(x,\xi) = \xi - U(x)$ is $\mathfrak{G}_\Phi \cap \big(B \times [\alpha,\beta]\big)^2$–invariant, and therefore $\xi - U(x) = c_0$ a.e. in $B \times [\alpha,\beta]$. Setting $u_0 := U + c_0$ we see that

$$B \times [\alpha,\beta] \subseteq (B \times \mathbb{R}) \cap [\xi = u_0(x)] \mod m,$$

and $\Phi(x,x') = U(x') - U(x) = u_0(x') - u_0(x)$.

Claim 4. Define $u_1(x) := \sup\{t \geq u_0(x) : P_x\big(\{x\} \times (u_0(x),t)\big) = 0\} \leq \infty$. *There exists* $c_1 \in (0,\infty]$ *such that* $u_1 = u_0 + c_1$ μ*–a.e. in* B.

Proof. Suppose $\big((x,\xi),(x',\eta)\big) \in \mathfrak{G}_\Phi\big[B \times [\alpha,\beta]\big]$ and let κ_Φ be some holonomy such that $(x',\eta) = \kappa_\Phi(x,\xi)$. If $u_1(x') < \infty$, then

$$\begin{aligned} P_{x'}\big[\{x'\} \times \big(u_0(x'), u_1(x')\big)\big] &= 0,\\ P_{x'}\big[\{x'\} \times [u_1(x'), u_1(x') + \epsilon)\big] &> 0 \text{ for all } \epsilon > 0. \end{aligned}$$

We know that $P_{x'} = P_{\kappa x} \sim P_x \circ \kappa_\Phi^{-1}$, so

$$\begin{aligned} P_x\big(\{x\} \times (u_0(x') - \Phi(x,x'), u_1(x') - \Phi(x,x'))\big) &= 0,\\ P_x\big(\{x\} \times [u_1(x') - \Phi(x,x'), u_1(x') - \Phi(x,x') + \epsilon)\big) &> 0 \text{ for all } \epsilon > 0. \end{aligned}$$

But $u_0(x') - \Phi(x,x') = u_0(x)$, so $u_1(x) < \infty$ and

$$u_1(x) = u_1(x') - \Phi(x,x') = u_1(x') - u_0(x') + u_0(x).$$

This shows that $u_1 - u_0$ is $\mathfrak{G}_\Phi\big[B \times [\alpha,\beta]\big]$–invariant. By Lemma 5, this relation is ergodic, so either $u_1 = \infty$ almost everywhere, or $u_1 = u_0 + const$. We write $u_1 = u_0 + c_1$ with $c_1 \in [0,\infty]$.

We claim that $c_1 > 0$. It is enough to consider the case when c_1 is finite. Observe that $u_1(x) \geq \beta \geq u_0(x)$ a.e. in B, because $P_x|_{\{x\}\times[\alpha,\beta]} \ll \delta_{(x,u(x_0))}$ for a.e. $x \in B$. It is enough to show that one of these inequalities is strict on a set of positive measure, because this implies that $c_1 = u_1 - u_0 > 0$.

Assume by way of contradiction that $u_1(x) = \beta = u_0(x)$ a.e. in B. The sets $B \times \{\beta\}$ and $B \times (\beta, \beta+\delta)$ must have positive P–measure for all $\delta > 0$. By Lemma 4, $\exists B' \subseteq B$ and $\exists$ a holonomy κ defined on B' s.t. $P(B' \times \{\beta\}) > 0$, $\kappa_\Phi(B' \times \{\beta\}) \subseteq B \times (\beta, \beta+\delta)$. It follows that $0 < \Phi(x,\kappa x) < \delta$ for μ–a.e. $x \in B'$. Fixing $0 < \delta < \beta - \alpha$, we see that $P \circ \kappa_\Phi\big(B' \times (\alpha,\beta)\big) \geq P\big(\kappa(B') \times \{\beta\}\big) = P\big(\kappa(B') \times [\alpha,\beta]\big) = \int_{\kappa(B')} P_x(\{x\} \times [\alpha,\beta]) d\mu(x) > 0$. Since κ_Φ is P–nonsingular,

$$P\big(B' \times (\alpha,\beta)\big) > 0.$$

Since $B' \subseteq B$, $P\big(B \times (\alpha,\beta)\big) > 0$. It follows that $\{(x,\xi) \in B \times [\alpha,\beta] : \xi < u_0(x)\}$ has positive measure, a contradiction to Claim 3.

Claim 5: Set $c := c_1$ if c_1 is finite, and $c := 0$ if c_1 is infinite. Then $m \circ Q_c \sim m$, and for m a.e. $(x, \xi) \in B \times \mathbb{R}$, $\xi - u_0(x) \in c\mathbb{Z}$.

Proof. Suppose $c = c_1 < \infty$. We show

$$E := (B \times \mathbb{R}) \cap [u_1 - c < \xi < u_1 + c] = (B \times \mathbb{R}) \cap [\xi = u_1(x)] \bmod m.$$

By the definition of u_1, for every δ, $P\big((B \times \mathbb{R}) \cap [u_1 \leq \xi < u_1 + \delta]\big) > 0$. Almost every (x, ξ) in this set has a holonomy κ_Φ s.t. $\kappa_\Phi(x, \xi) \in (B \times \mathbb{R}) \cap [\xi = u_0(x)]$ (Lemma 4). For this κ_Φ, $\Phi(x, \kappa x) \in (-c - \delta, -c]$ and

$$P_x|_{\{x\}\times(u_1+\delta, u_1+c-\delta)} \sim P_{\kappa x} \circ \kappa_\Phi|_{\{x\}\times(u_1+\delta,u_1+c-\delta)} \ll \\ \ll P_{\kappa x}|_{\{\kappa x\}\times(u_0,u_1-\delta)} \circ \kappa_\Phi \equiv 0.$$

Passing to the limit $\delta \downarrow 0$, we see that $P_x|_{\{x\}\times(u_1,u_1+c)} = 0$ for μ–almost every $x \in B$, and consequently, $(B\times\mathbb{R})\cap[u_1-c < \xi < u_1+c] = (B\times\mathbb{R})\cap[\xi = u_1(x)] \bmod P$.

Set $E_0 := \{(x, \xi) \in (B\times\mathbb{R})\cap[\xi = u_0(x)] : P_x\{(x, u_0(x))\}, P_x\{(x, u_0(x)+c)\} > 0\}$. This set has positive P measure, because:

(1) The measure of the subset of $B \times [\alpha, \beta]$ where $P_x(x, u_0(x)) = 0$ is

$$\int_B \int_\alpha^\beta 1_{[P_x(x,u_0(x))=0]} dP_x(\xi) d\mu(x) = \int_B 1_{[P_x(x,u_0(x))=0]} P_x(x, u_0(x)) d\mu(x) = 0.$$

(2) If $P_x\{(x, u_1(x))\} = 0$, then $P_x\big(\{x\} \times (u_1(x), u_1(x) + c)\big) > 0$ (otherwise $u_1(x)$ would not be maximal). Therefore

$$\mu\{x \in B : P_x\{(x, u_1(x))\} = 0\} \leq \mu\{x \in B : P_x\big(\{x\} \times (u_1(x), u_1(x) + c)\big) > 0\}.$$

The last measure is zero, because $P([B \times \mathbb{R}] \cap [u_1 < \xi < u_1 + c]) = 0$ as we saw in the beginning of the proof of the claim.

We use this to show that $m \circ Q_c \sim m$. Suppose by way of contradiction that $m \not\sim m \circ Q_c$. Since both measures are ergodic, $P \sim m \perp m \circ Q_c \sim P \circ Q_c$, and it follows that $\exists E \subseteq E_0$ such that $P(E) > 0$ and $P(Q_c E) = 0$. The function $f(x, \xi) := P_x\{(x, \xi + c)\}/P_x\{(x, \xi)\}$ is well-defined and positive a.e. in E_0, so

$$\begin{aligned} 0 &= P(Q_c E) = \int_X \int_\mathbb{R} 1_E(x, \xi - c) dP_x(\xi) d\mu(x) \\ &= \int_X 1_E(x, u_1(x) - c) P_x(x, u_1(x)) d\mu(x) \\ &= \int_X 1_E(x, u_0(x)) f(x) P_x(x, u_0(x)) d\mu(x) = \int_{X\times\mathbb{R}} 1_E f dP. \end{aligned}$$

Since $f > 0$ P–a.e. on E, $P(E) = 0$. This contradiction proves that $m \circ Q_c \sim m$.

The first part of the proof says that $B \times [u_1 - c < \xi < u_1 + c] = (B \times \mathbb{R}) \cap [\xi = u_1(x)] \bmod m$. The Q_c^n–translates of these sets cover $B \times \mathbb{R}$. Combining this with $u_1 = u_0 + c$ and $m \circ Q_c \sim m$ gives $B \times \mathbb{R} = (B \times \mathbb{R}) \cap [\xi \in u_0(x) + c\mathbb{Z}] \bmod m$. This proves the claim in the case $c_1 < \infty$. The case $c_1 = \infty$ is treated similarly.

We can now prove the proposition. Let $u_0 : B \to \mathbb{R}$ be the function given by Claim 3. There is a $\mathfrak{G}_\Phi[B \times \mathbb{R}]$–invariant set of full measure in $B \times \mathbb{R}$ such that every (x, ξ) in this set satisfies $\xi \in u_0(x) + c\mathbb{Z}$. Therefore, for a.e. $(x, \xi) \in B \times \mathbb{R}$

$$\big((x, \xi), (x', \eta)\big) \in \mathfrak{G}_\Phi[B \times \mathbb{R}] \Rightarrow \Phi(x, x') \equiv \eta - \xi \in u_0(x') - u_0(x) + c\mathbb{Z}.$$

This implies that $\Phi(x,x')+u_0(x)-u_0(x')\in c\mathbb{Z}$ μ–a.e. in $\mathfrak{G}[B]$. The $\mathfrak{G}_\Phi$–ergodicity of m implies the $\mathfrak{G}$–ergodicity of μ. Therefore by Lemma 6, $\exists u(x)$ Borel for which $\Phi(x,x')+u(x)-u(x')\in c\mathbb{Z}$ m–a.e. in $\mathfrak{G}_\Phi$.

It remains to show that $\mathbb{H}_m=c\mathbb{Z}$. We saw that Φ is cohomologous $\bmod m$ to a $c\mathbb{Z}$–valued cocycle, so $\mathbb{H}_m\subseteq c\mathbb{Z}$ (Lemma 8). By Claim 5 $c\in\mathbb{H}_m$, so $\mathbb{H}_m=c\mathbb{Z}$. □

Proof of Theorem 2. If $\mathbb{H}_m=\mathbb{G}$ there is nothing to prove. If $\mathbb{H}_m\neq\mathbb{G}$, then m has a square hole (Proposition 2), and the theorem reduces to Proposition 3. □

4. Tail Cocycles: Proofs of Proposition 1, Theorem 3, and Theorem 4

Throughout this section (Σ^+,T) is a topologically mixing subshift of finite type with (finite) alphabet S and transition matrix $A=(t_{ij})_{S\times S}$, and $\phi=(\phi^{(1)},\dots,\phi^{(d)}):\Sigma^+\to\mathbb{R}^d$ has summable variations (see §2). Cylinder sets are denoted by

$$[a_0,\dots,a_{n-1}]:=\{x\in\Sigma^+: x_0=a_0,\dots,x_{n-1}=a_{n-1}\}\quad (a_i\in S).$$

A word $(a_0,\dots,a_{n-1})\in S^n$ is called *admissible* if $[a_0,\dots,a_{n-1}]\neq\varnothing$. The partition $\alpha:=\{[a]:a\in S\}$ is called the *natural partition*, and its iterates are defined by $\alpha_N:=\bigvee_{i=0}^{N-1}T^{-i}\alpha$. The cocycles Φ, $\widetilde{\Phi}$ are given by (2). Two continuous functions $\phi,\psi:\Sigma^+\to\mathbb{R}^d$ are called *cohomologous* if $\phi-\psi=h-h\circ T$ for some continuous h (called the *transfer function*).

4.1. Preliminaries and the proof of Proposition 1.

Lemma 9. *Let $g:\Sigma^+\to S^1$ be some continuous function, and $\gamma\in\widehat{\mathbb{R}}$ a non–trivial character. There exists $u:\Sigma^+\to\mathbb{R}$ s.t. $g=\gamma\circ u$ and $var_n u=O(var_n g)$.*

Lemma 10. *Let ν be a $\mathfrak{T}$–ergodic and T–nonsingular measure on Σ^+, and suppose $\mathbb{H}\subsetneqq\mathbb{R}$ is closed. If $\Phi(x,y)+u(x)-u(y)\in\mathbb{H}$ ν–a.e., then $\exists\lambda\in S^1$ and $g:\Sigma^+\to S^1$ measurable s.t. $\gamma\circ\phi=\lambda g/g\circ T$ ν–a.e. where $\gamma\in\widehat{\mathbb{R}}$ and $\mathbb{H}=\ker\gamma$.*

Proof. Set $g:=\gamma\circ u$, and $F(x):=\gamma[\phi(x)]g(Tx)/g(x)$. This is well defined ν–a.e. because $\nu\circ T^{-1}\ll\nu$, and is $\mathfrak{T}$–invariant, whence constant. □

Proof of Proposition 1. Suppose first that $\mathbb{H}_\phi=\mathbb{Z}^r\times\mathbb{R}^{r'}\times\{0\}^{r''}$ $(r+r'+r''=d)$. Set

$$\gamma_i(t_1,\dots,t_d):=\begin{cases}\exp[2\pi i t_i] & i=1,\dots,r\\ 1 & i=r+1,\dots,r+r'\\ \exp[t_i] & i=r+r'+1,\dots,d\end{cases}$$

Observe that $\mathbb{H}_\phi=\bigcap_{i=1}^d\ker\gamma_i$, where $\ker\gamma_i:=\{x\in\mathbb{R}^d:\gamma_i(x)=1\}$. Fix some $z\in\Sigma^+$ s.t. $T^Nz=z$, set $c=(c_1,\dots,c_d):=\frac{1}{N}\phi_N(z)$, and define $\psi:=\phi-c$.

For every k, $\forall x\in Fix(T^{kN})$, $(\gamma_i\circ\psi_{kN})(x)=\gamma_i[\phi_{kN}(x)-\phi_{kN}(z)]=1$. By Livsic's theorem ([**PP**], proposition 5.2), $\exists g_i:\Sigma^+\to S^1$ with summable variations s.t. $\gamma_i\circ\psi_N=g_i/g_i\circ T^N$. If $(x,y)\in\mathfrak{T}$, then $\exists k$ s.t. $T^{kN}x=T^{kN}y$. Thus,

$$\gamma_i[\Phi(x,y)]=\gamma_i\left(\sum_{j=0}^{k-1}[\psi_N(T^{jN}y)-\psi_N(T^{jN}x)]\right)=$$

$$=\frac{\prod_{j=0}^{k-1}g_i(T^{jN}y)/g_i(T^{(j+1)N}y)}{\prod_{j=0}^{k-1}g_i(T^{jN}x)/g_i(T^{(j+1)N}x)}=\frac{g_i(y)}{g_i(x)}=\gamma_i(0,\dots,u_i(y)-u_i(x),\dots,0),$$

where u_i has summable variations and $g_i = \gamma_i(0,\dots,u_i,\dots,0)$.

It follows that $u_\phi := (u_1,\dots,u_d)$ is a function with summable variations for which $\Phi(x,y)+u_\phi(x)-u_\phi(y)\in\bigcap_{i=1}^d \ker\gamma_i \equiv \mathbb{H}_\phi$.

Now consider $F:\Sigma^+\to\mathbb{R}^d/\mathbb{H}_\phi$, $F(x) := \phi+u_\phi-u_\phi\circ T+\mathbb{H}_\phi$. Observe that if $(x,y)\in\mathfrak{T}$, then $(Tx,Ty)\in\mathfrak{T}$ and

$$F(y)-F(x) = [\Phi(x,y)+u_\phi(y)-u_\phi(x)]-[\Phi(Tx,Ty)+u_\phi(Ty)-u_\phi(Tx)]+\mathbb{H}_\phi = \mathbb{H}_\phi.$$

Therefore $F(x)$ is $\mathfrak{T}$–invariant. Since F is continuous and $\mathfrak{T}$–equivalence classes are dense in Σ^+, F is constant, whence $\exists c_\phi$ s.t. $\phi+u_\phi-u_\phi\circ T+c_\phi\in\mathbb{H}_\phi$.

We now show that $\mathbb{H}_\phi$ is minimal with this property. Suppose $\phi+u-u\circ T+c\in\mathbb{H}$ for some closed group $\mathbb{H}$, constant vector c, and continuous function u. Then $T^n x = x, T^n y = y \Rightarrow \phi_n(y)-\phi_n(x)\in\mathbb{H}$, and this implies that $\mathbb{H}\supseteq\mathbb{H}_\phi$.

This proves the first half of Proposition 1, when $\mathbb{H}_\phi = \mathbb{Z}^r\times\mathbb{R}^{r'}\times\{0\}^{r''}$. If this is not the case, find an isomorphism $\pi:\mathbb{R}^d\to\mathbb{R}^d$ for which $\pi[\mathbb{H}_\phi]$ is of this form (see e.g. [**HR**], Theorem 9.11), and work with $\pi\circ\phi$.

We turn to $\widetilde{\mathbb{H}}_\phi$. It is obvious that $\mathbb{H}_\phi\subseteq\widetilde{\mathbb{H}}_\phi\subseteq\overline{\mathbb{H}_\phi+c_\phi\mathbb{Z}}$. We claim that $c_\phi\in\widetilde{\mathbb{H}}_\phi$. By topological mixing $\exists N,x,y$ such that $T^N x = x, T^{N+1}y = y$. We have $\phi_N(y),\phi_{N+1}(x)\in\widetilde{\mathbb{H}}_\phi$ and $\phi_{N+1}(x)-\phi_N(y)+c_\phi\in\mathbb{H}_\phi\subseteq\widetilde{\mathbb{H}}_\phi$. It follows that $c_\phi\in\widetilde{\mathbb{H}}_\phi$, whence $\widetilde{\mathbb{H}}_\phi = \overline{\mathbb{H}_\phi+c_\phi\mathbb{Z}}$, and $\phi+u_\phi-u_\phi\circ T\in\widetilde{\mathbb{H}}_\phi$ everywhere. The proof that $\widetilde{\mathbb{H}}_\phi$ is minimal is the same as the proof the $\mathbb{H}_\phi$ is minimal. □

4.2. **Cohomology via bounded transfer functions.** The change of coordinates $\vartheta(x,\xi) = (x,\xi-u(x)-c)$ does not, in general, preserve local finiteness. If $\operatorname{ess\,sup}\|u\|<\infty$, then it does (Lemma 2). Fortunately,

Proposition 4. *Let m be a $\mathfrak{T}_\Phi$–e.i.l.f. measure on $\Sigma^+\times\mathbb{R}^d$. If $\exists\mathbb{H}\le\mathbb{R}$ closed, $u^{(i)}(x)$ Borel s.t. $\Phi^{(i)}(x,y)+u^{(i)}(x)-u^{(i)}(y)\in\mathbb{H}$ m–a.e., then $\exists v^{(i)}(x)$ Borel and essentially bounded s.t. $\Phi^{(i)}(x,y)+v^{(i)}(x)-v^{(i)}(y)\in\mathbb{H}$ m–a.e.*

Remark: Had we known a priori that $m\circ\vartheta^{-1}\sim\mu\times m_{\mathbb{H}_\phi}$ with μ an equilibrium measure, then the proposition would have followed from proposition 2.6 in [**S**]. We will see eventually that this is true for $\mathfrak{T}_\Phi$–e.i.l.f. measures (Theorem 4), but we cannot assume this in advance.

Lemma 11. *If $\phi^{(i)}$ is not cohomologous to a constant via a transfer function with summable variations, then there are M_0^*, M_0 arbitrarily large and $n_0\in\mathbb{N}$ with the following property: $\forall[\underline{b}]\in\alpha_{n_0}\exists[\underline{c}]\in\alpha_{n_0}$ such that*

$$b_0 = c_0, b_{n_0-1} = c_{n_0-1},\ \textit{and}\ \forall x\in T[b_{n_0-1}]\ \left(\left|\Phi^{(i)}(\underline{b}x,\underline{c}x)\right|\in[M_0,M_0^*]\right).$$

Proof. We claim that $\Phi^{(i)}:\mathfrak{T}\to\mathbb{R}$ is not bounded. Suppose it were. Fix some x, and let $\underline{p} = (p_0,\dots,p_{s_0-1})$ be some word such that $z := (\underline{p},\underline{p},\dots)$ is an admissible periodic orbit. By Ruelle's Perron–Frobenius theorem

$$P_{top}(t\phi^{(i)}) = \lim_{k\to\infty}\frac{1}{ks_0}\log\sum_{T^{ks_0}y=z} e^{t[\Phi^{(i)}(z,y)+\phi^{(i)}_{ks_0}(z)]} = h_{top}(T)+\frac{\phi^{(i)}_{s_0}(z)}{s_0}t,$$

and therefore $\left.\frac{d^2}{dt^2}\right|_{t=0}P_{top}(t\phi^{(i)}) = 0$. This can only happen if $\phi^{(i)}$ is cohomologous to a constant via a transfer function with summable variations ([**PP**], Proposition 4.12). Since we are assuming that this is not the case, $\Phi^{(i)}$ cannot be bounded.

Fix m_0 so large that every two states a, b can be connected by an admissible path of length m_0. Set $B := \sum_{n\geq 1} var_n(\phi^{(i)})$ and fix $M > 2m_0\|\phi^{(i)}\|_\infty + 2B$.

The unboundedness of $\Phi^{(i)}$ implies the existence of $\xi, \eta \in \Sigma^+$, $n_1 \in \mathbb{N}$ such that $\xi_{n_1+1}^\infty = \eta_{n_1+1}^\infty$ and $\Phi^{(i)}(\xi, \eta) > 2M$. We may assume without loss of generality that

$$\phi_{n_1}^{(i)}(\xi) \leq -M \text{ and } \phi_{n_1}^{(i)}(\eta) \geq M.$$

Otherwise set $\psi^{(i)} := \phi^{(i)} - \frac{1}{n_1}[M + \phi_{n_1}^{(i)}(\xi)]$. This function generates the same $\mathfrak{T}$ cocycle as $\phi^{(i)}$, and satisfies $\psi_{n_1}^{(i)}(\xi) = -M$, $\psi^{(i)}(\eta) = \Phi^{(i)}(\xi, \eta) - M \geq M$.

Now define $U := \phi_{n_1}^{(i)}(\eta)$, $D := \phi_{n_1}^{(i)}(\xi)$, $\underline{u} = (\eta_0, \ldots, \eta_{n_1-1})$, and $\underline{d} = (\xi_0, \ldots, \xi_{n_1-1})$. Observe that

$$\begin{aligned} U - B \leq \phi_{n_1}^{(i)} \leq U + B \text{ in } [\underline{u}] \quad &\text{and} \quad U \geq M, \\ D - B \leq \phi_{n_1}^{(i)} \leq D + B \text{ in } [\underline{d}] \quad &\text{and} \quad D \leq -M. \end{aligned}$$

Finally, set $n_0 := n_1 + 2m_0$.

We construct for every $[\underline{b}] \in \alpha_{n_0}$ a cylinder $[\underline{c}]$ as required. Define $\underline{\epsilon} \in \alpha_{n_1}$ by:

$$\underline{\epsilon} := \begin{cases} \underline{u} & \text{if } \sup\{\phi_{n_0}^{(i)}(x) : x \in [\underline{b}]\} < 0 \\ \underline{d} & \text{if } \inf\{\phi_{n_0}^{(i)}(x) : x \in [\underline{b}]\} > 0 \\ \underline{u} & \text{otherwise.} \end{cases}$$

Find $[\underline{\alpha}], [\underline{\beta}] \in \alpha_{m_0}$ s.t. $\alpha_0 = b_0$, $\beta_{m_0-1} = b_{n_0-1}$, and $[\underline{\alpha}, \underline{\epsilon}, \underline{\beta}] \neq \varnothing$. Define $[\underline{c}] := [\underline{\alpha}, \underline{\epsilon}, \underline{\beta}]$. Fix some $x \in T[b_{m_0-1}]$ and set $\Phi^{(i)} := \Phi^{(i)}(\underline{b}x, \underline{c}x)$. We write this as

$$\Phi^{(i)} = [\phi_{n_1}^{(i)}(\underline{\epsilon\beta}x) - \phi_{n_0}^{(i)}(\underline{b}x)] + \phi_{m_0}^{(i)}(\underline{c}x) + \phi_{m_0}^{(i)}(\underline{\beta}x)$$

and estimate it case by case (in what follows $a = b \pm c$ means $b - c \leq a \leq b + c$):

(1) If $\sup\{\phi_{n_0}^{(i)}(x) : x \in [\underline{b}]\} < 0$, then $\phi_{n_0}^{(i)}(\underline{b}x) \in [-n_0\|\phi^{(i)}\|_\infty, 0]$ and $\phi_{n_1}^{(i)}(\underline{\epsilon\beta}x) = U \pm B$, whence $\Phi^{(i)} \in [U - B - 2m_0\|\phi^{(i)}\|_\infty, U + B + (2m_0 + n_0)\|\phi^{(i)}\|_\infty]$.

(2) If $\inf\{\phi_{n_0}^{(i)}(x) : x \in [\underline{b}]\} > 0$, then $\phi_{n_0}^{(i)}(\underline{b}x) \in [0, n_0\|\phi^{(i)}\|_\infty]$ and $\phi_{n_1}^{(i)}(\underline{\epsilon\beta}x) = D \pm B$, whence $\Phi^{(i)} \in [D - B - (2m_0 + n_0)\|\phi^{(i)}\|_\infty, D + B + 2m_0\|\phi^{(i)}\|_\infty]$.

(3) In the remaining case we must have $\phi_{n_0}^{(i)}(\underline{b}x) = \pm B$ and $\phi_{n_1}^{(i)}(\underline{\epsilon\beta}x) = U \pm B$, so $\Phi^{(i)} \in [U - 2B - 2m_0\|\phi^{(i)}\|_\infty, U + 2B + 2m_0\|\phi^{(i)}\|_\infty]$.

In all cases $|\Phi^{(i)}| \in [M_0, M_0^*]$ where

$$\begin{aligned} M_0 &:= \min\{U, |D|\} - 2B - 2m_0\|\phi^{(i)}\|_\infty, \\ M_0^* &:= \max\{U, |D|\} + 2B + (2m_0 + n_0)\|\phi^{(i)}\|_\infty. \end{aligned}$$

This shows that $[\underline{c}]$ is as required. Finally, note that n_0, M_0 and M_0^* can be made arbitrarily large (by taking M, m_0 arbitrarily large). $\square$

Proof of Proposition 4. If $\mathbb{H} = \mathbb{R}$, take $u^{(i)} \equiv 0$. If $0 \subsetneq \mathbb{H} \subsetneq \mathbb{R}$, then $\exists c \neq 0$ such that $\mathbb{H} = c\mathbb{Z}$. In this case define $u^{(i)}(x)$ to be the unique $0 \leq u^{(i)}(x) < c$ for which $v(x) \in u^{(i)}(x) + c\mathbb{Z}$. It remains to treat the case $\mathbb{H} = \{0\}$. We do this by proving that if $\Phi^{(i)}(x, y) + u^{(i)}(x) - u^{(i)}(y) = 0$ m–a.e., then $\phi^{(i)} = h^{(i)} - h^{(i)} \circ T + c$ for some $c \in \mathbb{R}$ and $h^{(i)}$ with summable variations. This proves the lemma, because in this case $\Phi^{(i)}(x, y) = h^{(i)}(x) - h^{(i)}(y)$ with $h^{(i)}$ bounded (being continuous).

Assume by way of contradiction that $\phi^{(i)}$ is not cohomologous to a constant via a transfer function with summable variations.

The function $F(x,\xi) := \xi_i - u^{(i)}(x)$ is $\mathfrak{T}_\Phi$–invariant, and therefore constant m-a.e. Without loss of generality $F(x,\xi) = 0$ a.e. (otherwise modify $u^{(i)}$ by a constant), and this implies that m is supported inside $\{(x,\xi) : \xi_i = u^{(i)}(x)\}$.

If $\operatorname{ess\,sup}|u^{(i)}| < \infty$, then there is nothing to prove, so assume $\operatorname{ess\,sup}|u^{(i)}| = \infty$. Let $B := \sum_{n\geq 1} var_n\phi$, and fix A_0 large enough that $m(\Sigma^+ \times I) > 0$, where

$$I := [-A_0, A_0]^d.$$

Choose $M_0^*, M_0 > 2A_0 + B, n_0 \in \mathbb{N}$ as in lemma 11. We claim that it is possible to choose M_0^* and A_j large enough that $m(\Sigma^+ \times J) > 0$, where

$$J := \{\xi \in \mathbb{R}^d : M_0 - A_0 - B < |\xi_i| < M_0^* + A_0 + B \text{ and } |\xi_j| < A_j \text{ for } j \neq i\}.$$

This is because M_0^* can always be increased without spoiling Lemma 11, and $m\{(x,\xi) : |\xi_i| > M_0 - A_0 - B\} > 0$ (else $\operatorname{ess\,sup}|\xi_i| = \operatorname{ess\,sup}|u^{(i)}| < \infty$). We further increase A_j to ensure $A_j > A_0 + B + 2n_0\|\phi^{(j)}\|_\infty$ for $j \neq i$.

Define two measures μ_I, μ_J on Σ^+ by $\mu_I(E) := m(E \times I)$, $\mu_J(E) := m(E \times J)$. These are finite positive measures, because m is locally finite. They are mutually singular because μ_I is supported inside $\{x : u^{(i)}(x) \in [-A_0, A_0]\}$, and μ_J is supported inside $\{x : |u^{(i)}(x)| > A_0\}$.

There must therefore exist some N, $[\underline{a}] \in \alpha_N$ such that

$$0 < \mu_J[\underline{a}] < \frac{1}{2|\alpha_{n_0}|}\mu_I[\underline{a}].$$

Choose $[\underline{b}] \in \alpha_{n_0}$ which maximizes $\mu_I[\underline{a},\underline{b}]$. Clearly

$$\mu_I[\underline{a},\underline{b}] \geq \frac{1}{|\alpha_{n_0}|}\mu_I[\underline{a}].$$

Choose $[\underline{c}] \in \alpha_{n_0}$ s.t. $c_0 = b_0, c_{n_0-1} = b_{n_0-1}$ and $|\Phi^{(i)}(\underline{b}x, \underline{c}x)| \in [M_0, M_0^*]$ for all $x \in T[b_{n_0-1}]$ (this is how we chose M_0, M_0^*). Define $\kappa : [\underline{a},\underline{b}] \to [\underline{a},\underline{c}]$ by $\kappa(\underline{a}\,\underline{b}x) = (\underline{a}\,\underline{c}x)$ and set $\kappa_\Phi(x,\xi) := \big(\kappa x, \xi + \Phi(x,\kappa x)\big)$. Then $\forall x \in [\underline{a},\underline{b}]$,

$$\Phi^{(i)}(x,\kappa x) = \Phi^{(i)}(\underline{b}x,\underline{c}x) \pm B \in [M_0 - B, M_0^* + B] \cup [-M_0^* - B, -M_0 + B].$$

Since $|\Phi^{(j)}(x,\kappa x)| \leq 2n_0\|\phi^{(j)}\|_\infty + B$, $\kappa_\Phi([\underline{a},\underline{b}] \times I) \subseteq [\underline{a},\underline{c}] \times J$. Thus,

$$\mu_I[\underline{a}] \leq |\alpha_{n_0}| \cdot \mu_I[\underline{a},\underline{b}] = |\alpha_{n_0}| \cdot m([\underline{a},\underline{b}] \times I) = |\alpha_{n_0}| \cdot m[\kappa_\Phi([\underline{a},\underline{b}] \times I)] \leq$$
$$\leq |\alpha_{n_0}| \cdot m([\underline{a},\underline{c}] \times J) \leq |\alpha_{n_0}| \cdot \mu_J[\underline{a}] < \frac{1}{2}\mu_I[\underline{a}],$$

so $1 < \frac{1}{2}$. This contradiction proves the proposition. □

4.3. **Proof of Part (1) in Theorems 3 and 4.** Let m be a locally finite $\mathfrak{T}_\Phi$–ergodic invariant measure. W.l.o.g. $\mathbb{H}_m = \mathbb{R}^r \times c_{r+1}\mathbb{Z} \times \ldots \times c_d\mathbb{Z}$ with $c_i \geq 0$. We construct $u^{(i)}(x)$ essentially bounded s.t. $\big(\Phi^{(i)}(x,y) + u^{(i)}(x) - u^{(i)}(y)\big)_{i=0}^d \in \mathbb{R}^r \times \prod_{i=r+1}^d c_i\mathbb{Z}$ a.e. in $\mathfrak{T}_\Phi$. Clearly, we can take $u^{(i)} \equiv 0$ for $i = 1,\ldots,r$, so it is enough to consider $i \geq r+1$.

Choose a s.t. $\forall i\ m(X^{(i)}) > 0$, where $X^{(i)} := \Sigma^+ \times [-a,a]^{i-1} \times \mathbb{R} \times [-a,a]^{d-i-1}$. Let $m^{(i)}$ be the restriction of m to $X^{(i)}$. By lemma 5, $m^{(i)}$ is $\mathfrak{E}^{(i)} := \mathfrak{T}_\Phi[X^{(i)}]$–ergodic and invariant. Since m is locally finite as a measure on $\Sigma^+ \times \mathbb{R}^d$, $m^{(i)}$ is locally finite as a measure on $X^{(i)} \cong (\Sigma^+ \times [-a,a]^{d-1}) \times \mathbb{R}$. Observing that

$$\mathfrak{E}^{(i)} \cong \mathfrak{T}^{(i)}_{\Phi^{(i)}} \text{ with } \mathfrak{T}^{(i)} := \mathfrak{T}_{(\Phi^{(1)},\ldots,\Phi^{(i-1)},\Phi^{(i+1)},\ldots,\Phi^{(d)})}[\Sigma^+ \times [-a,a]^{d-1}],$$

we see that $m^{(i)}$ can be identified with a locally finite $\mathfrak{T}^{(i)}_{\Phi^{(i)}}$–ergodic invariant measure on $\Sigma^+ \times [-a,a]^{d-1} \times \mathbb{R}$.

It is easy to check that $\mathbb{H}_{m^{(i)}} = c_i\mathbb{Z}$. The cocycle reduction theorem provides a Borel function $w^{(i)} : \Sigma^+ \times [-a,a]^{d-1} \to \mathbb{R}$ such that

$$\Phi^{(i)}(x,y) + w^{(i)}(x,\xi^{(i)}) - w^{(i)}(y,\eta^{(i)}) \in c_i\mathbb{Z} \;\; m^{(i)}\text{–a.e. in } \mathfrak{E}^{(i)},$$

where here and throughout, $\xi = (\xi_1,\ldots,\xi_d)$ and $\xi^{(i)} = (\xi_1,\ldots,\xi_{i-1},\xi_{i+1},\ldots,\xi_d)$. Since $\mathfrak{E}^{(i)} = \mathfrak{T}_\Phi[X^{(i)}]$, the proof of lemma 6 gives $v^{(i)} : \Sigma^+ \times \mathbb{R}^{d-1} \to \mathbb{R}$ Borel s.t.

$$\Phi^{(i)}(x,y) + v^{(i)}(x,\xi^{(i)}) - v^{(i)}(y,\eta^{(i)}) \in c_i\mathbb{Z} \; m\text{–a.e. in } \mathfrak{T}_\Phi.$$

We show that $v^{(i)}(x,\xi^{(i)})$ has a version which is independent of $\xi^{(i)}$.

We begin by constructing a version which is independent of ξ_k for $k \geq r+1$. The function $\xi_i - v^{(i)} + c_i\mathbb{Z}$ is $\mathfrak{T}_\Phi$–invariant, whence constant. Add a constant to $v^{(i)}$ to ensure that $\xi_i - v^{(i)} + c_i\mathbb{Z} = c_i\mathbb{Z}$ a.e. Modify $v^{(i)}$ if necessary to ensure that $0 \leq v^{(i)} < c_i$ whenever $c_i \neq 0$. Then there is a set of full measure $\Omega' \subseteq \Sigma^+ \times \mathbb{R}^d$ such that for every $(x,\xi) \in \Omega'$ and all $i = r+1,\ldots,d$

$$\xi_i \in v^{(i)}(x,\xi^{(i)}) + c_i\mathbb{Z}, \text{ and } 0 \leq v^{(i)}(x,\xi^{(i)}) < c_i \text{ if } c_i \neq 0.$$

For every $a \in \{0\}^r \times \prod_{i=r+1}^d c_i\mathbb{Z}$, $m \circ Q_a \sim m$. Construct $\Omega \subseteq \Omega'$ of full measure s.t. for all $a \in \{0\}^r \times \prod_{i=r+1}^d c_i\mathbb{Z}$, $Q_a(\Omega) = \Omega \subseteq \Omega'$. For every $a \in \{0\}^r \times \prod_{i=r+1}^d c_i\mathbb{Z}$ and all $(x,\xi) \in \Omega$,

$$\xi_i \in v^{(i)}(x,\xi^{(i)} + a^{(i)}) + c_i\mathbb{Z}, \text{ and } 0 \leq v^{(i)}(x,\xi^{(i)} + a^{(i)}) < c_i \text{ if } c_i \neq 0.$$

In particular, $v^{(i)}(x,\xi^{(i)}) = v^{(i)}(x,\xi^{(i)} + a^{(i)}) \bmod c_i\mathbb{Z}$ in Ω. Since $0 \leq v^{(i)} < c_i$ when $c_i \neq 0$, we have (regardless of c_i)

$$v^{(i)} \circ Q_{a^{(i)}} = v^{(i)} \text{ on } \Omega \;\; \text{for all } a \in \{0\}^r \times \prod_{i=r+1}^d c_i\mathbb{Z}.$$

Fix now $k \geq r+1$ and suppose $(x,\xi),(y,\eta) \in \Omega$ and $(x,\xi^{(k)}) = (x,\eta^{(k)})$. Then $\xi_k, \eta_k = v^{(k)}(x,\xi^{(k)}) \bmod c_k\mathbb{Z}$, whence $\xi - \eta \in \{0\}^{k-1} \times c_k\mathbb{Z} \times \{0\}^{d-k-1}$ and so $v^{(i)}(x,\eta^{(i)}) = v^{(i)}Q_{(\xi-\eta)^{(i)}}(x,\eta^{(i)}) = v^{(i)}(x,\xi^{(i)})$. This shows that $v^{(i)}|_\Omega$ is independent of ξ_k. Repeating this for other coordinates, we see that $v^{(i)}|_\Omega$ does not depend on $\xi_{r+1},\ldots,\xi_d$.

We show that $v^{(i)}$ can be made independent of $\xi_1,\ldots,\xi_r$. We show independence of ξ_1. Other coordinates are handled in the same way. A similar argument to that used above shows that there is $\Omega_1 \subseteq \Omega$ of full measure s.t. $\forall i > r$ and $(x,\xi) \in \Omega_1$,

$$v^{(i)}(x,\xi^{(i)} + a^{(i)}) = v^{(i)}(x,\xi^{(i)}) \text{ for all } a \in \mathbb{Q}^r \times \{0\}^{d-r}.$$

Decompose $m = \int_{X\times\mathbb{R}^{d-1}} P_{(x,\xi^{(1)})} dm(x,\xi)$ with $P_{(x,\xi^{(1)})} \in \mathcal{M}(\{x\} \times \mathbb{R} \times \{\xi^{(1)}\})$. Then for m–a.e. (x,ξ),

$$v^{(i)} \circ Q_{a^{(i)}} = v^{(i)} \;\; P_{(x,\xi^{(1)})}\text{–a.e., for all } a \in \mathbb{Q}^r \times \{0\}^{d-r}.$$

We know that $m \circ Q_{(t,0,\ldots,0)} = e^{\lambda t}m$ for some $\lambda \in \mathbb{R}$. It follows that $e^{-\lambda t}dP_{(x,\xi^{(1)})}$ is translation invariant on $\{x\} \times \mathbb{R} \times \{(\xi_2,\ldots,\xi_d)\}$ for a.e. (x,ξ). In particular, $P_{(x,\xi^{(1)})}$ is equivalent to Lebesgue's measure on the fibre. Therefore, if we write $f^{(i)}_{(x,\xi^{(1)})}(t) := v^{(i)}(x,t,\xi_2,\ldots,\xi_{i-1},\xi_{i+1},\ldots,\xi_d)$, then necessarily

$$f^{(i)}_{(x,\xi^{(1)})}(t+q) = f_{(x,\xi^{(1)})}(t) \text{ Lebesgue a.e., for all } q \in \mathbb{Q}.$$

The local finiteness of m implies that $f^{(i)}_{(x,\xi^{(1)})}$ is Lebesgue locally integrable for a.e. $(x,\xi^{(1)})$. Locally integrable functions which are invariant under all rational translations are constant[2], so $\exists c(x,\xi^{(1)})$ s.t. $f^{(i)}_{(x,\xi^{(1)})} = c^{(i)}(x,\xi^{(1)})\ P_{(x,\xi^{(1)})}$–a.e.

Since this is true for m–a.e. fibre, there is a set $\Omega_2 \subseteq \Omega_1$ of full measure such that $v^{(i)}|_{\Omega_2} = c^{(i)}|_{\Omega_2}$. Since $v^{(i)}$ is Borel, $c^{(i)}$ is measurable w.r.t the completion of m. Changing Ω_2 by a set of measure zero, we can obtain a Borel version of $c^{(i)}$ (hence of $v^{(i)}$) independent of ξ_1. Repeating this argument for other coordinates, we see that $v^{(i)}$ has a version which is independent of $\xi_1,\dots,\xi_d$.

Pass to such a version. Abusing notation we view $u^{(i)}$ as a function on Σ^+ and write $u^{(i)} = u^{(i)}(x)$. It follows that

$$\Phi^{(i)}(x,y) + u^{(i)}(x) - u^{(i)}(y) \in c_i\mathbb{Z}\ m\text{–a.e. in } \mathfrak{T}_\Phi.$$

By proposition 4, $u^{(i)}(x)$ can be assumed to be essentially bounded. Setting $u = (u^{(1)},\dots,u^{(d)})$, we have $\Phi(x,y)+u(x)-u(y) \in \mathbb{H}_m$ m–a.e. and $\operatorname{ess\,sup}\|u\| < \infty$.

By Lemma 2 $\exists c \in \mathbb{R}^d$ such that $m \circ \vartheta_c^{-1}$ is a *locally finite* $\mathfrak{T}_{\Phi_u}$–ergodic invariant on $\Sigma^+ \times \mathbb{H}_m$ and $\mathbb{H}_{m\circ\vartheta_c^{-1}} = \mathbb{H}_m$. By Lemma 1, $\exists \alpha : \mathbb{H}_m \to \mathbb{R}$ continuous homomorphism and a finite $(\mathfrak{T}, \alpha\circ\Phi_u)$–conformal measure ν on Σ^+ s.t.

$$dm\circ\vartheta_c^{-1}(x,\xi) = e^{-\alpha(\xi)}d\nu(x)dm_{\mathbb{H}_m}(\xi).$$

The measure $d\nu_0 = e^{-\alpha\circ u}d\nu$ is clearly $(\mathfrak{T},\alpha\circ\Phi)$–conformal. It is finite, because $\operatorname{ess\,sup}\|u\| < \infty$. Since ϕ has summable variations, ν_0 must be equivalent to the equilibrium measure of $\alpha\circ\phi$, which we denote by μ_α (see [**ANSS**], proposition 2.4). It follows that $\Phi^{(i)}(x,y) + u^{(i)}(x) - u^{(i)}(y) \in \mathbb{H}_i$ μ_α–a.e. in $\mathfrak{T}$, where $\mathbb{H}_i \le \mathbb{R}$, $\mathbb{H}_m = \mathbb{H}_1 \times \ldots \times \mathbb{H}_d$.

We are now in a position to apply the Livsic rigidity theory for equilibrium measures. If $\mathbb{H}_i = \mathbb{R}$ replace $u^{(i)}$ by the zero function. Otherwise pick $\gamma_i \in \widehat{\mathbb{R}}$ such that $\mathbb{H}_i = \ker\gamma_i$ and proceed as follows: Equilibrium measures are T–nonsingular so Lemma 10 applies, and $\exists\lambda \in S^1$, $g_i : \Sigma^+ \to S^1$ measurable s.t. $\gamma_i[\phi^{(i)}] = \lambda g_i/g_i\circ T$ μ_α–a.e. Livsic's Rigidity Theorem ([**PP**], Proposition 4.2) says that this equality can be made to hold everywhere with g_i with summable variations. Write $g_i = \gamma_i\circ v^{(i)}$ with $v^{(i)}$ with summable variations, and set $v = (v^{(1)},\dots,v^{(d)})$. Then $\Phi^{(i)}(x,y) + v^{(i)}(x) - v^{(i)}(y) \in \ker\gamma_i = \mathbb{H}_i$ everywhere for all $1 \le i \le d$. It follows that $\Phi(x,y) + v(x) - v(y) \in \mathbb{H}_m$ *everywhere* in $\mathfrak{T}$.[3]

We show that $\mathbb{H}_m = \mathbb{H}_\phi$ and that v can be replaced by u_ϕ. The function $F := \phi + v - v\circ T + \mathbb{H}_m$ is $\mathfrak{T}$–invariant and continuous, whence constant. Therefore, $\exists c$ s.t. $\phi + v - v\circ T + c \in \mathbb{H}_m$ everywhere. But $\mathbb{H}_\phi$ is the smallest group with such a cohomology (Proposition 1), so $\mathbb{H}_m \supseteq \mathbb{H}_\phi$. The other inclusion holds as well, because $\phi + u_\phi - u_\phi\circ T + c_\phi \in \mathbb{H}_\phi \Rightarrow \Phi(x,y) + u_\phi(x) - u_\phi(x) \in \mathbb{H}_\phi$, so $\mathbb{H}_m \subseteq \mathbb{H}_\phi$ (Lemma 8). Theorem 3, part (1) is proved.

We prove Theorem 4, part (1). We just saw that $\Phi(x,y) + u_\phi(x) - u_\phi(y) \in \mathbb{H}_m$. Clearly $\|u_\phi\| < \infty$ (u_ϕ is continuous). By Lemma 2, $\exists c \in \mathbb{R}^d$ s.t. $m\circ\vartheta_c^{-1}$ is supported on $\Sigma^+ \times \mathbb{H}_\phi$, and $\mathbb{H}_{m\circ\vartheta_c^{-1}} = \mathbb{H}_m = \mathbb{H}_\phi$. By Lemma 1, $dm\circ$

[2]If f is such function, then $d\mu = fdt$ is a Radon measure which is invariant under all rational translations. It is easy to check that $\forall\varphi \in C_c(\mathsf{R}), \forall a \in \mathsf{R}\ \int_{\mathsf{R}} \varphi(t+a)d\mu(t) = \int_{\mathsf{R}} \varphi(t)d\mu(t)$. Therefore μ is translation invariant, whence proportional to Lebesgue's measure.

[3]Note that the boundedness of $\|u\|$ was essential here: there is an abundance of (non locally finite) $(\mathsf{T},\alpha\circ\Phi)$–conformal measures which are not equilibrium measures. Livsic's theory does not apply to such measures.

$\vartheta_c^{-1}(x,\xi) = e^{-\alpha(\xi)}d\nu(x)dm_{\mathbb{H}_m}(\xi)$ with $\alpha : \mathbb{H}_m \to \mathbb{R}$ a continuous homomorphism and ν a finite $(\mathfrak{T}, \alpha\circ\Phi_{u_\phi})$–conformal measure. Proposition 2.4 in [**ANSS**] says that $\nu = \nu_\alpha$ where ν_α is the unique (up to a constant) finite measure for which $\frac{d\nu_\alpha}{d\nu_\alpha\circ T} = e^{\alpha\circ[\phi+u_\phi-u_\phi\circ T]-P_{top}(\alpha\circ\phi)}$. It follows from Ruelle's Perron–Frobenius theorem that such a measure exists and is unique up to scalar multiple. Part (1) in Theorem 4 is proved.

4.4. **Proof of Part (2) in Theorems 3 and 4.** Suppose m is a locally finite $\widetilde{\mathfrak{T}}_{\boldsymbol{\Phi}}$–invariant ergodic locally finite measure on $\Sigma^+\times\mathbb{R}^d$. If $\vartheta(x,\xi) = (x, \xi - u_\phi(x))$, then $m\circ\vartheta^{-1}$ is a locally finite $\widetilde{\mathfrak{T}}_{\boldsymbol{\Phi}_u}$–ergodic invariant measure, where $\widetilde{\Phi}_u(x,y) := \widetilde{\Phi}(x,y) + u_\phi(x) - u_\phi(y)$. (Local finiteness follows from the boundedness of u_ϕ.)

Since $\widetilde{\mathfrak{T}} \supset \mathfrak{T}$ and $\widetilde{\Phi}_u|_{\mathfrak{T}} = \Phi_u$, $m\circ\vartheta^{-1}$ is $\mathfrak{T}_{\Phi_u}$–invariant. It is not clear, however, that it is $\mathfrak{T}_{\Phi_u}$–ergodic. Let $m\circ\vartheta^{-1} = \int_Y m_y d\pi(y)$ be its $\mathfrak{T}_{\Phi_u}$–ergodic decomposition (Lemma 7). Since Σ^+ has a countable base of compact open sets, a.e. $\mathfrak{T}_{\Phi_u}$–ergodic component is a locally finite.

Part (1) of Theorem 4 shows that for a.e. $y \in Y$ $\exists c_y$ such that $m_y \sim \nu_y \times m_{c_y+\mathbb{H}_\phi}$ where $\nu_y\circ T^{-1} \sim \nu_y$ and $m_{c_y+\mathbb{H}_\phi}$ is the measure on $c_y + \mathbb{H}_\phi$ induced by the Haar measure on $\mathbb{H}_\phi$ in the natural way. It follows that every $m_y|_{T[a]\times\mathbb{R}^d}$ is quasi invariant under the following transformations (here $a \in S$ is fixed):

(1) $A(x,\xi) = (x, \xi+\phi(x)+u_\phi(x)-u_\phi(Tx)+c_\phi)$, because the second coordinate is always dilated by an element of $\mathbb{H}_\phi = \mathbb{H}_{m_y}$;
(2) $B(ax,\xi)$, where $ax := (a, x_0, x_1, \ldots)$, because $\nu_y\circ T^{-1} \sim \nu_y$.

Since $m\circ\vartheta^{-1} = \int m_y d\pi(y)$, $m\circ\vartheta^{-1}|_{T[a]\times\mathbb{R}^d}$ must also be quasi–invariant under those transformations. It is also quasi–invariant (indeed, invariant) under the map

$$\kappa(x,\xi) = \big(Tx, \xi - \phi(x) - u_\phi(x) + u_\phi(Tx)\big)\ ,\ \kappa : [a]\times\mathbb{R}^d \to T[a]\times\mathbb{R}^d$$

because this map is a $\widetilde{\mathfrak{T}}_{\boldsymbol{\Phi}_u}$–holonomy. A composition of non–singular maps is non–singular, so $Q_{c_\phi}|_{T[a]\times\mathbb{H}_\phi} = \kappa\circ A\circ B$ is nonsingular, whence $m\circ\vartheta^{-1}\circ Q_{c_\phi} \not\perp m\circ\vartheta^{-1}$. By ergodicity, $m\circ\vartheta^{-1}\circ Q_{c_\phi} \sim m\circ\vartheta^{-1}$, and this shows that $c_\phi \in \mathbb{H}_{m\circ\vartheta^{-1}} = \mathbb{H}_m$.

For every y, $m_y \sim m_y\circ Q_\xi$ $\forall\xi\in\mathbb{H}_\phi$, so $m\circ\vartheta^{-1}\circ Q_\xi \sim m\circ\vartheta^{-1}$ for all $\xi \in \mathbb{H}_\phi$, and therefore $\mathbb{H}_\phi \subseteq \mathbb{H}_m$. Since $c_\phi \in \mathbb{H}_m$ we have $\widetilde{\mathbb{H}}_\phi \equiv \overline{\mathbb{H}_\phi + c_\phi\mathbb{Z}} \subseteq \mathbb{H}_m$. Since $\phi + u_\phi - u_\phi\circ T \in \widetilde{\mathbb{H}}_\phi \Rightarrow \widetilde{\Phi}(x,y) + u_\phi(x) - u_\phi(y) \in \widetilde{\mathbb{H}}_\phi$, we also have $\mathbb{H}_m \subseteq \widetilde{\mathbb{H}}_\phi$ (Lemma 8), whence $\mathbb{H}_m = \widetilde{\mathbb{H}}_\phi$. This proves Theorem 3, part (2).

We turn to Theorem 4, part (2). We saw that $\widetilde{\Phi}(x,y)+u_\phi(x)-u_\phi(y) \in \widetilde{\mathbb{H}}_\phi = \mathbb{H}_m$ everywhere in $\widetilde{\mathfrak{T}}$. By Lemma 2, $\exists c$ such that $m\circ\vartheta_c^{-1}$ is a locally finite $\widetilde{\mathfrak{T}}_{\boldsymbol{\Phi}_{u_\phi}}$–ergodic invariant on $\Sigma^+\times\widetilde{\mathbb{H}}_\phi$. Since $\mathbb{H}_{m\circ\vartheta_c^{-1}} = \mathbb{H}_m = \widetilde{\mathbb{H}}_\phi$, $dm\circ\vartheta_c^{-1} = e^{-\alpha(\xi)}d\nu_\alpha(x)dm_{\widetilde{\mathbb{H}}_\phi}(\xi)$ where $\alpha : \widetilde{\mathbb{H}}_\phi \to \mathbb{R}$ is a continuous homomorphism and ν_α is a finite $(\widetilde{\mathfrak{T}}, \alpha\circ\widetilde{\Phi}_{u_\phi})$–conformal measure on Σ^+ (Lemma 1).

Note that $\widetilde{\mathfrak{T}}$ is the smallest equivalence relation which contains $\mathfrak{T}$ and

$$\mathfrak{Orb}(T) := \{(x,y)\in\Sigma^+\times\Sigma^+ : \exists n\geq 0 \text{ s.t. } x = T^n y \text{ or } y = T^n x\}.$$

Therefore, ν_α is $(\widetilde{\mathfrak{T}}, \alpha\circ\widetilde{\Phi}_{u_\phi})$–conformal iff ν_α is $(\mathfrak{T}, \alpha\circ\Phi_{u_\phi})$–conformal and

$$\frac{d\nu_\alpha\circ T}{d\nu_\alpha} = \exp[\alpha\circ\widetilde{\Phi}_{u_\phi}(x,Tx)] \equiv e^{-\alpha\circ[\phi+u_\phi-u_\phi\circ T]}.$$

Proposition 2.4 in [**ANSS**] says that $(\mathfrak{T}, \alpha\circ\Phi_{u_\phi})$–conformality is equivalent to the condition $\frac{d\nu_\alpha\circ T}{d\nu_\alpha}\propto e^{-\alpha\circ[\phi+u_\phi-u_\phi\circ T]}$. It follows that ν_α is $(\mathfrak{T}, \alpha\circ\widetilde{\Phi})$–conformal iff $\frac{d\nu_\alpha\circ T}{d\nu_\alpha} = e^{-\alpha\circ[\phi+u_\phi-u_\phi\circ T]}$. By Ruelle's Perron–Frobenius theorem, a finite measure like this exists iff $P_{top}(\alpha\circ\phi)=0$, and if it exists then it is unique up to a multiplicative factor. This completes the proofs of Theorems 3 and 4. □

5. Application to Horocycle Flows: Proof of Theorem 1

5.1. W^{ss}–**flows.** Let $g^s : N \to N$ be a C^2–flow on a compact connected C^∞ Riemannian manifold N. The flow g^s is called an *Anosov flow* if it has no fixed points, the tangent bundle of N is the Whitney sum of of three dg^s–invariant sub–bundles $TN = E^u\oplus E^s\oplus E^t$ where E^t is the line bundle tangent to the direction of the flow, and $\exists C>0, \theta\in(0,1)$ such that

(1) $\|dg^s(v)\|\leq C\theta^s\|v\|$ for all $v\in E^s$, $s\geq 0$;
(2) $\|dg^{-s}(v)\|\leq C\theta^s\|v\|$ for all $v\in E^u$, $s\geq 0$.

The flow is called *transitive* if it has a dense orbit, and *topologically weak mixing* if the functional equation $F\circ g^s = e^{i\lambda s}F$ $(s\in\mathbb{R})$ has no non-constant continuous solutions F. The basic example is as follows: let M be a compact connected C^∞–Riemannian surface whose sectional curvatures are all negative. The geodesic flow on the unit tangent bundle $N = SM$ is a topologically weak mixing transitive Anosov flow [**An**].

For every point $x\in N$, define

$$W^{ss}(x) \quad := \quad \{y\in N : d(g^s x, g^s y) \xrightarrow[s\to\infty]{} 0\}$$

It is known that these are submanifolds of N, and that $\mathcal{W}^{ss} := \{W^{ss}(x) : x\in N\}$ is a foliation, called the *strong stable* foliation of g^s. We will be interested in flows for which

$$W^{ss}(x) \text{ are one–dimensional, and can be oriented continuously in } x. \qquad (3)$$

B. Marcus proved in [**M1**] that if g^s is topologically weak mixing, then (3) implies the existence of a continuous flow $h^t : N\to N$, called the W^{ss}*–flow*[4] of g^s s.t.

(a) *Orbit property*: for all $x\in N$, $W^{ss}(x) = \{h^t(x)\}_{t\in\mathbb{R}}$;
(b) *Commutation relation*: $\exists\mu$ such that $\forall s,t\in\mathbb{R}$, $g^{-s}\circ h^t\circ g^s = h^{\mu^s t}$.

This flow is unique (up to uniform rescaling of time), and $\mu = \exp[-h_{top}(g^1)]$. In the special case of the geodesic flow on a hyperbolic surface we get the classical (stable) horocycle flow. The point of this construction is that it makes sense in the variable negative curvature case.

5.2. $\mathbb{Z}^d$–**covers.** A connected manifold $\widetilde{M}$ is called a *regular* $\mathbb{Z}^d$*–cover* of a manifold M, if there exists a continuous map $p:\widetilde{M}\to M$ for which

(1) $p:\widetilde{M}\to M$ is onto, and $\forall x\in M$ $\exists V\ni x$ open and connected for which every connected component of $p^{-1}(V)$ is mapped by p homeomorphically onto V.
(2) $Cov(\widetilde{M},p) := \{D:\widetilde{M}\to\widetilde{M} : D \text{ continuous}, p\circ D = p\}\cong\mathbb{Z}^d$. We parameterize $Cov(\widetilde{M},p) = \{D_\xi : \xi\in\mathbb{Z}^d\}$ so that $D_{\xi+\eta} = D_\xi\circ D_\eta$.
(3) for every $x\in M$, $\exists\widetilde{x}\in\widetilde{M}$ s.t. $p^{-1}(x) = \{D(\widetilde{x}) : D\in Cov(\widetilde{M},p)\}$.

[4]The term used in [**M1**, **M2**] is 'W^s–flow'.

A connected manifold with first Betti number d_0 has a regular $\mathbb{Z}^d$–cover iff $d \leq d_0$.

A $\mathbb{Z}^d$–cover $\widetilde{M} \to M$ induces a $\mathbb{Z}^d$–cover $\widetilde{N} \to N$ where $N = SM, \widetilde{N} = S\widetilde{M}$ (the unit tangent bundles). We abuse notation and use the same notation $p, D_{\underline{a}}$ for the covering map and the deck transformations of this cover.

By the unique lifting property, every continuous flow $\varphi^t : M \to M$ determines a flow $\widetilde{\varphi}^t : \widetilde{M} \to \widetilde{M}$ for which $p \circ \widetilde{\varphi}^t = \varphi^t \circ p$. Let $\widetilde{g}^s, \widetilde{h}^t : \widetilde{N} \to \widetilde{N}$ be the lifts of $g^s, h^t : N \to N$ to $\widetilde{N}$. The unique lifting property guarantees that $\widetilde{h}, \widetilde{g}$ satisfy (a) and (b), and that $\widetilde{g}, \widetilde{h}$ commute with D_ξ ($\xi \in \mathbb{Z}^d$).

Let $\omega \in M$ be a φ–periodic point with period $\lambda(\omega)$. The *Frobenius element* of ω is the unique $D_\xi \in Cov(\widetilde{M}, p)$ such that

$$\widetilde{\varphi}^\lambda(\widetilde{\omega}) = D_\xi(\widetilde{\omega}), \text{ whenever } \widetilde{\omega} \in p^{-1}\{\omega\}.$$

The flow φ is called $\mathbb{Z}^d$*–full* if $\forall \xi \in \mathbb{Z}^d$, D_ξ is a Frobenius element for some periodic point (see [**Po1**],[**Sh**]). As explained in [**Po1**], the geodesic flow on the unit tangent bundle of a C^∞ compact connected Riemannian surface is $\mathbb{Z}^d$–full.

5.3. **A Poincaré section for $\widetilde{h}$.** Throughout this section let N be a compact connected C^∞–Riemannian manifold with a regular $\mathbb{Z}^d$–cover $p : \widetilde{N} \to N$, g a topologically weak mixing C^2–Anosov flow on N with property (3), h a W^{ss}–flow of g, and $\widetilde{g}, \widetilde{h}$ the lifted flows on $\widetilde{N}$.

We construct a Poincaré section for $\widetilde{h}$ which admits a simple symbolic description. We first consider the symbolic dynamics of $\widetilde{g}$. The following lemma contains all the information we need. We omit the proof which follows from Bowen's symbolic dynamics for g and $\mathcal{W}^{ss}$ ([**B1**, **B2**], see also [**BR**]).

Lemma 12. *Fix $i : N \to \widetilde{N}$ 1-1 and continuous such that $p \circ i = id$. There exists a topologically mixing two–sided subshift of finite type (Σ, T), Hölder continuous $r : \Sigma \to \mathbb{R}$, $f : \Sigma \to \mathbb{Z}^d$ which only depend on the non-negative coordinates, and a Hölder continuous surjection $\pi : \Sigma \times \mathbb{R} \times \mathbb{Z}^d \to \widetilde{N}$ such that*

(1) *$\pi : \Sigma_r \times \{0\} \to i(N)$ is a bounded-to-one surjection, where*

$$\Sigma_r = \{(x, t) : 0 \leq t < r(x)\};$$

(2) *$\pi \circ Q_{(t,a)} = [\widetilde{g}_t \circ D_a] \circ \pi$ for all $(t, a) \in \mathbb{R} \times \mathbb{Z}^d$;*

(3) *$\pi \circ T_{(-r,f)} = \pi$ where $T_{(-r,f)}(x, t, \xi) = \big(Tx, t - r(x), \xi + f(x)\big)$;*

(4) *Suppose $\widetilde{x} = \pi(x, t, \xi), \widetilde{y} = \pi(x', t', \xi')$. Then*

$$\exists p, q \geq 0 \ s.t \begin{cases} x_p^\infty = y_q^\infty \\ t - t' = r_p(x) - r_q(y) \\ \xi - \xi' = f_q(y) - f_p(x) \end{cases} \Longrightarrow W^{ss}(\widetilde{x}) = W^{ss}(\widetilde{y}).$$

We denote the alphabet of Σ by S, the transition matrix of Σ by $(t_{ij})_{S\times S}$ (see §2), and set $\phi : \Sigma \to \mathbb{R} \times \mathbb{Z}^d$ to be $\phi := (-r, f)$. We will also need the one–sided version of Σ given by

$$\Sigma^+ := \{x^+ = (x_0, x_1, \ldots) : x_i \in S \text{ and } t_{x_i x_{i+1}} = 1 \text{ for all } i\}.$$

We will often abuse notation and view ϕ as a function on Σ^+ rather than Σ (this is possible since r, f depend only on the non-negative coordinates).

We now construct the section for $\widetilde{h}$. Fix $P : S \to S$ s.t. $\forall a \in S,\ t_{P(a),a} = 1$, and let $\rho' : \Sigma^+ \times \mathbb{R} \times \mathbb{Z}^d \to \Sigma \times \mathbb{R} \times \mathbb{Z}^d$, $\rho : \Sigma^+ \times \mathbb{R} \times \mathbb{Z}^d \to \widetilde{N}$ be

$$\begin{aligned}\rho'(x^+, t, \xi) &= (x, t, \xi), \text{ where } x_0^\infty = x^+ \text{ and } \forall k \leq -1,\ x_k = P(x_{k+1});\\ \rho(x^+, t, \xi) &= \pi[\rho'(x^+, t, \xi)].\end{aligned}$$

Set $\widetilde{K} := \rho[\Sigma^+ \times \mathbb{R} \times \mathbb{Z}^d] \subset \widetilde{N}$. We show below that this is a *global Poincaré section* for $\widetilde{h}$, i.e., that

$$\widetilde{\tau} : \widetilde{K} \to \mathbb{R}^+ \ ,\ \widetilde{\tau}(\omega) := \inf\{t > 0 : \widetilde{h}^t(\omega) \in \widetilde{K}\}$$

is well–defined, finite and strictly positive for all $\omega \in \widetilde{K}$. This will allow us to reduce the analysis of $\widetilde{h}$–ergodic invariant measures to the study of H-ergodic invariant measures, where $H : \widetilde{K} \to \widetilde{K}$ is the *section map* $H(\omega) := h^{\mathfrak{e}(\omega)}(\omega)$. Recall that a measure is ergodic invariant w.r.t to an invertible map $F : X \to X$ iff it is ergodic and invariant w.r.t the *orbit equivalence relation* of F

$$\mathfrak{Orb}(F) := \{(\omega, \omega') : \exists \ell \geq 0 \text{ s.t } \omega' = F^\ell(\omega) \text{ or } \omega = F^\ell(\omega')\}.$$

The main virtue of $\widetilde{K}$ is that $\mathfrak{Orb}(H)$ has a nice symbolic description:

Lemma 13. *$\widetilde{K}$ is a global Poincaré section for $\widetilde{h}$, and if $\omega_i = \rho(x_i^+, t_i, \xi_i)$ for $i = 1, 2$, then $\big((x_1^+, t_1, \xi_1), (x_2^+, t_2, \xi_2)\big) \in \widetilde{\mathfrak{T}}_{\mathfrak{P}} \Rightarrow (\omega_1, \omega_2) \in \mathfrak{Orb}(H)$.*

Proof. We first show that $K := p(\widetilde{K})$ is a section for $h^t : N \to N$, and then lift to $\widetilde{N}$. Write for $a \in S$, $K_a := (p \circ \rho)([a] \times \mathbb{R} \times \{0\})$. Bowen and Marcus [**BM**] showed that K_a is a transversal to the foliation $\mathcal{W}^{ss}$ for every $a \in S$. This means that $\exists k_a : K_a \times [-1, 1] \to N$ continuous and 1-1 for which

(1) $\forall \omega \in K_a,\ k_a(\{\omega\} \times [-1, 1]) \subseteq W^{ss}(\omega)$
(2) $\forall \omega \in K_a,\ k_a(\omega, 0) = \omega$
(3) $k_a(K_a \times [-1, 1])$ is compact, and has non–empty interior V.

Since h^t is a W^{ss}–flow, $\forall(\omega, s) \in K_a \times [-1, 1]\ \exists! t = t(\omega)$ s.t. $k_a(\omega, 1) = h^t(\omega)$. We claim that this t is uniformly bounded away from $\pm\infty$ and 0.

Suppose by way of contradiction that $\sup t = \infty$. Then $\exists \omega_n \in K_a$ such that $t(\omega_n) > n$. Choose $n_k \uparrow \infty$ such that $\omega_{n_k} \to \omega$ (K_a is compact). The leaves of $\mathcal{W}^{ss}$ are connected one-dimensional manifolds. The topological weak mixing of g implies that they are dense [**Pl**], so they must be homeomorphic to $\mathbb{R}$. Since $h^t(\omega), k_a(\omega, t)$ are continuous in (ω, t),

$$\{h^t(\omega_{n_k})\}_{0<t<t(\omega_{n_k})} = \{k_a(\omega_{n_k}, t)\}_{0<t<1} \subseteq K_a.$$

Fixing $t > 0$ and passing to the limit $k \to \infty$ we see that $\{h^t(\omega)\}_{t>0} \subseteq K_a$. But this is impossible because $N \setminus K_a$ is a non-empty open set, and the *forward* orbits of W^{ss}–flows of a topologically weak mixing Anosov flows are dense: W^{ss}–flows are uniquely ergodic with unique invariant measure of full support (for unique ergodicity, see Theorem 2.1 in [**M2**];[5] full support follows from [**M2**], proposition 2.2 and §4 of [**BM**]). This contradiction proves that $\sup t(\omega) \neq \infty$. A similar

[5]The reader should note the following differences in terminology: $W^s(x)$ in [**M2**] is what we call $W^{ss}(x)$, and 'topological mixing' in [**M2**] §1.4 (d) is what we call 'topological weak mixing'. The minimality condition of Theorem 2.1 in [**M2**] requires that $W^{ss}(x)$ are dense for all $x \in N$ (this is weaker than saying that the forward orbits of h are dense). Minimality follows from topological weak mixing and Theorem 1.8 in [**Pl**].

argument shows that $\inf t(\omega) \neq -\infty$. Finally, $\inf |t(\omega)| > 0$, otherwise $\exists \omega_n \to \omega$ s.t. $t(\omega_n) \to 0$. But this implies that

$$k_a(\omega, 1) = \lim_{n\to\infty} k_a(\omega_n, 1) \equiv \lim_{n\to\infty} h^{t(\omega_n)}(\omega_n) = h^0(\omega) = \omega \equiv k_a(\omega, 0)$$

in contradiction to the injectivity of k_a. It follows that $|t(\omega)|$ is bounded from zero and infinity. Fix $\epsilon_a, t_a > 0$ such that $\epsilon_a < |t(\omega)| < t_a$ for all $\omega \in K_a$.

We can now see that K_a is a Poincaré section for h: the result we just cited shows that the forward orbit of every ω visits V at arbitrarily large times t. But if $h^s(\omega) \in V$, then $\exists s' \in [s - t_a, s + t_a]$ such that $h^s(\omega) \in K_a$, and therefore every forward orbit of ω visits K_a infinitely often. Since

$$\tau_a(\omega) := \inf\{\tau > 0 : h^\tau(\omega) \in K_a\} \in [\epsilon_0, \infty) \text{ for all } \omega \in K_a,$$

we have that K_a is a Poincaré section for h.

Since S is finite, $K = \bigcup_{a\in S} K_a$ is a Poincaré section for h. It automatically follows that $\bigcup_{\xi\in\mathbb{Z}^d} D_\xi \rho(\Sigma^+ \times \mathbb{R} \times \{0\})$ is a section for $\widetilde{h}$. This set is equal to $\rho(\Sigma^+ \times \mathbb{R} \times \mathbb{Z}^d) \equiv \widetilde{K}$ because $D_\xi \circ \rho = \rho \circ Q_\xi$. We proved that $\widetilde{K}$ is a section for $\widetilde{h}$.

Now let H be the section map and suppose $\omega_i = \rho(x_i^+, t_i, \xi_i)$ $(i = 1, 2)$. If $\big((x_1^+, t_1, \xi_1), (x_2^+, t_2, \xi_2)\big) \in \widetilde{\mathfrak{T}}_{\mathfrak{P}}$, then $W^{ss}(\omega_1) = W^{ss}(\omega_2)$ (Lemma 12), so ω_1, ω_2 are in the same $\widetilde{h}$–orbit (because $\widetilde{h}$ is a W^{ss}–flow). But $\omega_1, \omega_2 \in \widetilde{K}$, so they must be in the same H–orbit, i.e., $(\omega_1, \omega_2) \in \mathfrak{Orb}(H)$. $\square$

Proposition 5. *Suppose $\widetilde{\mathbb{H}}_\phi = \mathbb{R} \times \mathbb{Z}^d$. For every $\widetilde{h}$–ergodic invariant Radon measure m, there is a locally finite $\widetilde{\mathfrak{T}}_{\mathfrak{P}}$–ergodic invariant measure m^+ on $\Sigma^+ \times \mathbb{R} \times \mathbb{Z}^d$ with the following property: If $E^+ \subseteq \Sigma^+ \times \mathbb{R} \times \mathbb{Z}^d$ is Borel and $\rho|_{E^+}$ is 1-1, then*

$$m\Big(\bigcup_{t\in I} \widetilde{h}^t \rho E^+\Big) = |I| m^+(E^+) \text{ for all intervals } I \subseteq [0, \inf_{\rho(E^+)} \widetilde{\tau}]. \tag{4}$$

Proof. The construction of m^+ is an easy modification of [**BM**], but is included for completeness. Suppose m is an $\widetilde{h}$–ergodic invariant Radon measure. Let $E \subseteq \widetilde{K}$ and $A \subseteq [0, \inf_E \widetilde{\tau}] =: I_0$ be Borel. The map $(x, t) \mapsto \widetilde{h}^t(E)$ is 1-1 and continuous on $[0, \inf_E \widetilde{\tau}] \times E$. By Souslin's theorem, it maps Borel sets to Borel sets ([**Ku**] §35 IV). We can therefore define a Borel measure on I_0 by

$$\lambda_E(A) := m\Big(\bigcup_{t\in A} \widetilde{h}^t E\Big).$$

The $\widetilde{h}$–invariance of m implies that λ_E is translation invariant on I_0, and must therefore be proportional to Lebesgue's measure on I_0, which we denote by λ. Let $m_0(E)$ be the proportionality constant: $\lambda_E(A) = \lambda(A) m_0(E)$. It is easy to see that m_0 is σ–additive, so it extends to a Borel measure on $\widetilde{K}$.

Recall the following fact ([**Ku**], §35 VII Cor. 5): *Let $f : X \to Y$ be a countable-to-one continuous function between two polish spaces. Then $X = \biguplus_{n=1}^\infty X_n$ where X_n are Borel sets s.t. $f|_{X_n}$ is 1-1 and continuous for all n. In this case $f(X_n)$ are also Borel.* Applying this to $\rho : \Sigma^+ \times \mathbb{R} \times \mathbb{Z}^d \to \widetilde{K}$ we see that $\exists!$ Borel measure m^+ on $\Sigma^+ \times \mathbb{R} \times \mathbb{Z}^d$ such that $m^+(E^+) = m_0(\rho E^+)$ whenever $\rho|_E^+$ is 1-1 and $E^+ \subseteq \Sigma^+ \times \mathbb{R} \times \mathbb{Z}^d$ is Borel. This measure is locally finite: If

$$r_n := \begin{cases} r + r \circ T \circ \ldots r \circ T^{n-1} & n > 0 \\ 0 & n = 0 \\ -r \circ T^{-1} - r \circ T^{-2} - \ldots - r \circ T^{-|n|+1} & n < 0 \end{cases}$$

then $\Sigma^+ \times \mathbb{R} \times \mathbb{Z}^d = \biguplus_{n \in \mathbb{Z}, \xi \in \mathbb{Z}^d} \{(x_0^\infty, t, \xi) : x \in \Sigma \text{ and } 0 \leq t - r_n(x) < r(T^n x)\}$, and each of the sets in this decomposition has a compact closure of finite measure. But r is bounded away from zero and infinity, so every compact subset of $\Sigma^+ \times \mathbb{R} \times \mathbb{Z}^d$ is covered by finitely many of these sets, and is therefore of finite measure.

Since $\widetilde{K}$ is a section, m is $\widetilde{h}$–ergodic and invariant iff m_0 is ergodic and invariant with respect to the section map $H : \widetilde{K} \to \widetilde{K}$, equivalently w.r.t $\mathfrak{Orb}(H)$.

We show that m^+ is $\widetilde{\mathfrak{T}}_{\mathfrak{P}}$–invariant. Let $\kappa^+ : A^+ \to B^+$ be a $\widetilde{\mathfrak{T}}_{\mathfrak{P}}$–holonomy between Borel sets s.t. $\rho|_{A^+}, \rho|_{B^+}$ are 1-1, and define

$$\kappa := \rho \circ \kappa^+ \circ (\rho|_{A^+})^{-1} : \rho(A^+) \to \rho(B^+).$$

Lemma 13 says that κ is a $\mathfrak{Orb}(H)$–holonomy, and therefore, $m_0|_{\rho(B^+)} \circ \kappa = m_0|_{\rho(A^+)}$, whence $m^+|_{B^+} \circ \kappa^+ = m^+|_{A^+}$. It follows that m^+ is $\widetilde{\mathfrak{T}}_{\mathfrak{P}}$–invariant.

We show that m^+ is $\widetilde{\mathfrak{T}}_{\mathfrak{P}}$–ergodic. Let $m^+ = \int_Y \mu_y^+ d\pi(y)$ be the ergodic decomposition of m^+ into $\widetilde{\mathfrak{T}}_{\mathfrak{P}}$–ergodic invariant measures (Lemma 7). Fix some E^+ of positive finite m^+–measure. We may assume w.l.o.g. that $m^+(E^+) = 1$, $\pi(Y) = 1$, and that for all y, $\mu_y^+(E^+) = 1$. (Recall that the ergodic components here are infinite measures, so we are at liberty to normalize them any way we wish.)

By theorem 4, and since $\widetilde{\mathbb{H}}_\phi = \mathbb{R} \times \mathbb{Z}^d$, every locally finite $\widetilde{\mathfrak{T}}_{\mathfrak{P}}$–ergodic invariant measure on $\Sigma^+ \times \mathbb{R} \times \mathbb{Z}^d$ is of the form $e^{-\alpha(\xi)} d\nu_\alpha(x) dm_{\mathbb{R} \times \mathbb{Z}^d}(\xi)$ for some continuous homomorphism $\alpha : \mathbb{R} \times \mathbb{Z}^d \to \mathbb{R}$, and this homomorphism determines the measure up to a constant. (The assumption on $\widetilde{\mathbb{H}}_\phi$ allows us to set $u_\phi = 0, c = 0$.) Therefore, for every y, there exists a homomorphism $\alpha_y : \mathbb{R} \times \mathbb{Z}^d \to \mathbb{R}$ s.t. $\mu_y^+ \circ Q_{(s,\xi)} = e^{\alpha_y(s,\xi)} \mu_y^+$ for all (s, ξ), and $\alpha_{y_1} = \alpha_{y_2} \Rightarrow y_1 = y_2$.

It follows that $m^+ \circ Q_{(s,\xi)} \sim m^+$ for all $(s, y) \in \mathbb{R} \times \mathbb{Z}^d$. Since $\rho \circ Q_{(s,\xi)} = \widetilde{g}^s \circ D_\xi \circ \rho$, $m \circ \widetilde{g}^s \circ D_\xi \sim m$ for all $(s, \xi) \in \mathbb{R} \times \mathbb{Z}^d$. It follows that $m \circ \widetilde{g}^s \circ D_\xi$ is $\widetilde{h}$–ergodic. It is easy to check, using the commutation relations between $\widetilde{h}^t, \widetilde{g}^s$ and D_ξ, that $m \circ \widetilde{g}^s \circ D_\xi$ is $\widetilde{h}$–invariant. Therefore, $m \circ \widetilde{g}^s \circ D_\xi$ is proportional to m. There must therefore exist a homomorphism $\beta : \widetilde{\mathbb{H}}_\phi \to \mathbb{R}$ s.t. $m \circ \widetilde{g}^s \circ D_\xi = e^{\beta(s,\xi)} m$ for all $(s, \xi) \in \widetilde{\mathbb{H}}_\phi$. One checks that $m^+ \circ Q_{(s,\xi)} = e^{\alpha(s,\xi)} m^+$, where $\alpha(s, \xi) = \beta(s, \xi) + s \log \mu$.

The identity

$$1 = e^{-n\alpha(s,\xi)} m^+[Q_{(ns,n\xi)} E^+] = \int_Y e^{n[\alpha_y(s,\xi) - \alpha(s,\xi)]} \mu_y^+(E^+) d\pi(y)$$

shows that $\forall \epsilon > 0$, $\pi\{y : \alpha_y(s, \xi) - \alpha(s, \xi) \geq \epsilon\} \leq e^{-n\epsilon} \to 0$, whence $\alpha_y(s, \xi) \leq \alpha(s, \xi)$ for π–a.e. y. This means that

$$\begin{aligned} 1 &= e^{-n\alpha(s,\xi)} m^+[Q_{(ns,n\xi)} E^+] = \\ &= \int_{[\alpha_y(s,\xi) \leq \alpha(s,\xi)]} e^{n[\alpha_y(s,\xi) - \alpha(s,\xi)]} \mu_y^+(E^+) d\pi(y) \leq \int_Y \mu^+(E^+) d\pi(y) = 1. \end{aligned}$$

Thus, $\alpha_y(s, \xi) = \alpha(s, \xi)$ π–a.e. Since there is at most one ergodic component with $\alpha_y = \alpha$, m^+ has just one ergodic component, and is thus ergodic. □

5.4. **Proof of Theorem 1.** We outline the properties of the geodesic flow g on the unit tangent bundle SM of a compact connected orientable C^∞ Riemannian surface M with variable negative curvature that are used below. Firstly, it is a topologically weak mixing transitive Anosov flow [**An**]. Secondly, is satisfies (3) (see [**M1**], page 143). Thirdly, it is $\mathbb{Z}^d$–full [**Po1**]. Finally, it is time–reversible: $\forall\omega \in \widetilde{SM}$, let $\imath(\omega)$ be the unit tangent vector with the same base point and opposite direction, then $\widetilde{g}^s \circ \imath = \imath \circ \widetilde{g}^{-s}$.

Let Σ, π, r, f be as in Lemma 12 for $N = SM$, and recall that $\phi = (-r, f)$. Let $\Lambda_r := \Sigma \times \mathbb{R}/\sim$ where $\sim$ is the equivalence relation generated by $(x,t) \sim \big(Tx, t - r(x)\big)$. We denote by $\langle\cdot,\cdot\rangle$ the equivalence classes of the relation. Let $\sigma^s : \Lambda_r \to \Lambda_r$ be $\sigma^s\langle x,t\rangle = \langle x, t+s\rangle$. By Lemma 12 $\pi\langle x,t\rangle := \langle \pi(x,t,\underline{0})\rangle$ is a well–defined bounded-to-one continuous surjection $\pi : \Lambda_r \to SM$ such that $\pi\circ\sigma^s = g^s \circ \pi$, and π is 1-1 on a residual set. Since g^s is topologically weak mixing, σ^s is topologically weak mixing ([**PP**], Lemma 9.1)

The topological mixing of σ implies that $\widetilde{\mathbb{H}}_r = \mathbb{R}$. Otherwise $\widetilde{\mathbb{H}}_r = c\mathbb{Z}$ for some c, and by Livsic's theorem $\exists u(x)$ continuous and $\theta \neq 0$ such that $e^{-i\theta r} = u/u\circ T$. But in this case $F\langle x,t\rangle := e^{i\theta t}u(x)$ is a well–defined continuous eigenfunction of $\sigma^s : \Sigma_r \to \Sigma_r$, in contradiction to topological weak mixing.

We claim that $\widetilde{\mathbb{H}}_\phi = \mathbb{R}\times\mathbb{Z}^d$. This is implicit in the work of Sharp ([**Sh**], proof of Theorem 1, see also [**So**]), but there is a simpler argument in the reversible case due to Y. Coudene [**C**], which we now describe.

If $\omega \in SM$ is g–periodic with period λ and Frobenius element D_ξ, then $\exists\langle x,t\rangle \in \Lambda_r$ s.t. $\omega = (p\circ\pi)\langle x,t\rangle$ and $\exists n$ s.t. $T^n x = x$, $r_n(x) = \lambda$, $f_n(x) = \xi$. Since g is time reversible, $\omega' := \imath(\omega)$ is periodic with period λ and Frobenius element $D_{-\xi}$. Again, $\exists\langle x',t'\rangle \in \Lambda_r$ s.t. $\omega' = (p\circ\pi)\langle x',t'\rangle$ and $\exists n'$ s.t. $T^{n'}x' = x'$, $r_{n'}(x') = \lambda$, $f_{n'}(x') = -\xi$. It follows that $(\lambda, \pm\xi) \in \widetilde{\mathbb{H}}_\phi$, whence

$$\widetilde{\mathbb{H}}_\phi \supseteq \{(2\lambda, 0) : \lambda \text{ is the length of a periodic geodesic in } SM\} = \\ = 2\widetilde{\mathbb{H}}_r \times \{0\} = \mathbb{R}\times\{0\}.$$

The geodesic flow is $\mathbb{Z}^d$–full. Therefore, $\forall\xi \in \mathbb{Z}^d$, $\exists\lambda \in \mathbb{R}$ s.t. $(\lambda,\xi) \in \widetilde{\mathbb{H}}_\phi$. But $\forall\lambda \in \mathbb{R}, (\lambda, 0) \in \widetilde{\mathbb{H}}_\phi$. Therefore, $\widetilde{\mathbb{H}}_\phi \supseteq \{0\}\times\mathbb{Z}^d$, whence $\widetilde{\mathbb{H}}_\phi = \mathbb{R}\times\mathbb{Z}^d$.

Suppose m is an $\widetilde{h}$–ergodic invariant Radon measure. Proposition 5 shows that m determines a $\widetilde{\mathfrak{T}}_{\mathfrak{P}}$–ergodic invariant locally finite measure m^+ on $\Sigma^+\times\mathbb{R}\times\mathbb{Z}^d$ with (4). Since $\widetilde{\mathbb{H}}_\phi = \mathbb{R}\times\mathbb{Z}^d$, $\exists$ a homomorphism $\beta : \mathbb{R}\times\mathbb{Z}^d \to \mathbb{R}$ s.t.

$$m^+ \circ Q_{(t,\xi)} = e^{\beta(t,\xi)}m^+ \quad \text{for all } (t,\xi) \in \mathbb{R}\times\mathbb{Z}^d.$$

Write $\beta(s,\xi) = \lambda s + \langle\underline{a},\xi\rangle$ for some $\lambda \in \mathbb{R}, \underline{a} \in \mathbb{Z}^d$. Since $\rho\circ Q_{(0,\xi)} = D_\xi\circ\rho$, we see that $m\circ D_\xi = e^{\langle\underline{a},\xi\rangle}m$ for all $\xi \in \mathbb{R}^d$, and this proves (1). As a by–product of the proof we see that $m\circ g^s = e^{\lambda s}m$ where $\lambda = \lambda(\underline{a})$ is the unique root of the equation $P_{top}(-\lambda r + \langle\underline{a}, f\rangle) = 0$. It also follows from Theorem 4 that $\underline{a}$ determines m up to a constant factor.

Now suppose $\underline{a} \in \mathbb{Z}^d$ and set $\beta(s,\xi) := \lambda s + \langle\underline{a},\xi\rangle$ where $\lambda = \lambda(\underline{a})$ satisfies $P_{top}(-\lambda r + \langle\underline{a}, f\rangle) = 0$. Such λ always exists, because $\varphi(\lambda) := P_{top}(-\lambda r + \langle\underline{a}, f\rangle)$ is continuous (in fact convex) and $\varphi(\lambda) \xrightarrow[\lambda\to\pm\infty]{} \mp\infty$. Theorem 4 gives a $\widetilde{\mathfrak{T}}_{\mathfrak{P}}$–invariant locally finite measure m^+ s.t. $m^+\circ Q_{(s,\xi)} = e^{\beta(s,\xi)}m^+$. The structure of m^+ implies that $\rho : \Sigma^+ \times \mathbb{R} \times \mathbb{Z}^d \to \widetilde{K}$ is 1-1 on a set of full measure (see [**PP**], page 236). It follows that the $(\Rightarrow)$ in Lemma 13 is $(\Leftrightarrow)$ on a set of full measure. Therefore, the measure m on $\widetilde{N} = S\widetilde{M}$ defined by (4) is $\widetilde{h}$–invariant. This is a Radon measure, and $m \circ D_\xi = e^{\langle \underline{a}, \xi \rangle} m$. Almost every ergodic component μ_y of m is locally finite, and $\mu_y \circ D_\xi = e^{\langle \underline{a}, \xi \rangle} \mu_y$ for all $\xi \in \mathbb{Z}^d$ (see the proof of Proposition 5). The latter condition determines μ_y up to a constant, so almost all the ergodic components of m are proportional. Therefore, m is ergodic. We found an $\widetilde{h}$–ergodic invariant Radon measure which realizes $\underline{a}$. □

6. A counterexample

Our argument depended in a crucial way on the fact that for the cocycles Φ, $\widetilde{\Phi}$ the cohomology produced by Theorem 2 can be made to hold everywhere (not just a.e.) and that the transfer function can be chosen to be essentially bounded. We give an example that shows that this is not true for general cocycles, even in the case of the tail relation of the two–shift.

Let $\Sigma_2^+ := \{0, 1\}^{\mathbb{N} \cup \{0\}}$. Recall that the *adding machine* is the transformation $\tau : \Sigma_2^+ \to \Sigma_2^+$ given by $\tau(x) := \min\{y \in \Sigma_2^+ : y \succnsim x\}$, where $\preccurlyeq$ is the partial order

$$x \preccurlyeq y \Leftrightarrow \exists p \text{ s.t. } x_p \le y_p \text{ and } x_{p+1}^\infty = y_{p+1}^\infty.$$

Observe that the $\mathfrak{T}$–equivalence class of x is the (full) τ–orbit of x. We can therefore define a $\mathfrak{T}$–cocycle as follows: $\Psi(x, y) := n$ whenever $y = \tau^n(x)$. Next, fix some $z \in \Sigma_2^+$ and define a measure on $\Sigma_2^+ \times \mathbb{R}$ by $m := \sum_{n \in \mathbb{Z}} \delta_{(\tau^n z, n)}$ (δ=Dirac).

Proposition 6. *m is $\mathfrak{T}_\Psi$–e.i.l.f., and $\mathbb{H}_m = \{0\}$. But:*

(1) *$\nexists$ Borel u s.t.* $\operatorname{ess\,sup}|u| < \infty$ *and* $\Psi(x, y) + u(x) - u(y) = 0$ *a.e. in* $\mathfrak{T}_\Psi$*;*
(2) *$\nexists$ Borel u s.t.* $\Psi(x, y) + u(x) - u(y) = 0$ everywhere *in* $\mathfrak{T}_\Psi$*.*

Proof. Suppose that $\exists u(x)$ Borel s.t. $\Psi(x, y) + u(x) - u(y) = 0$ a.e. in $\mathfrak{T}$. The function $F(x, \xi) = \xi - u(x)$ is $\mathfrak{T}_\Psi$–invariant, and therefore constant a.e. It follows that $\exists c$ s.t. m is supported on $\{(x, \xi) : \xi = u(x) + c\}$. This can only happen if $u(\tau^n z) = n - c$, and it follows that $\operatorname{ess\,sup}|u| = \infty$. This proves (1).

To prove (2) assume by way of contradiction that $\Psi(x, y) + u(x) - u(y) = 0$ for all $(x, y) \in \mathfrak{T}$ with $u(x)$ Borel. We can assume w.l.o.g. that $u(x) \in \mathbb{Z}$. Otherwise, $\Psi(x, y) + \lfloor u(x) \rfloor = \lfloor \Psi(x, y) + u(x) \rfloor = \lfloor u(y) \rfloor$, so we can work with $\lfloor u \rfloor$. Now define $\varphi(x) := \tau^{-u(x)}(x)$. The set $\Omega := \varphi(X)$ is Borel, because $\Omega = \bigcup_{k \in \mathbb{Z}} \tau^{-k}[u(x) = k]$ and u is Borel. We claim that

$$X = \biguplus_{k \in \mathbb{Z}} \tau^k(\Omega).$$

Equality is clear. For disjointness, we check $\tau^k \Omega \cap \Omega \neq \varnothing \Rightarrow k = 0$: If $\tau^k \Omega \cap \Omega \neq \varnothing$, then $\exists x, y$ such that $\tau^k[\varphi(x)] = \varphi(y)$, whence $k = \Psi[\varphi(x), \varphi(y)] = \Psi[\varphi(x), x] + \Psi(x, y) + \Psi[y, \varphi(y)] = u(x) + \Psi(x, y) - u(y) = 0$.

We now obtain a contradiction as follows. Let p be the $(\frac{1}{2}, \frac{1}{2})$ Bernoulli measure on Σ_2^+. This is a τ–invariant, so $1 = p(\Sigma_2^+) = \sum_{k \in \mathbb{Z}} p[\tau^k \Omega] = \sum_{k \in \mathbb{Z}} p[\Omega] \in \{0, \infty\}$. This contradiction shows that u cannot be Borel. □

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