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TAIL-INVARIANT MEASURES FOR SOME SUSPENSION SEMIFLOWS

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Abstract. We consider suspension semiflows over abelian extensions of one-sided mixing subshifts of finite type. Although these are not uniquely ergodic, we identify (in the "ergodic" case) all tail-invariant, locally finite measures which are quasi-invariant for the semiflow.

1. Introduction.

1.1. The Tail Relations.

We start with some background on equivalence relations, (see [F-M] for more detail). Let $(X, \mathcal{B})$ be a standard Borel space, and let $R \subset X \times X$ be an equivalence relation. Assume that $R \in \mathcal{B} \otimes \mathcal{B}$, and that each equivalence class $R(x) := \{y : (x, y) \in R\}$ is countable. Then for any $A \in \mathcal{B}$, *the saturation* $R(A) = \cup\{R(x) : x \in A)\}$ is again a Borel set. A σ-finite measure μ on X is called *non-singular* for R if $\mu(R(A)) = 0$ whenever $\mu(A) = 0$, and is, in addition, called *ergodic* if any saturated set $A = R(A)$ has either zero or full measure.

A Borel isomorphism ϕ defined on some $A \in \mathcal{B}$ with image $B \in \mathcal{B}$ is a *holonomy* if $(x, \phi(x)) \in R$ for any $x \in A$. A measure μ is *invariant* for R, if it is invariant under all the holonomies of R.

Let S be a finite set, and let Σ be a subshift of finite type over S:

$$\Sigma := \{x \in S^{\mathbb{N}} : \forall k \geq 1, A_{x_k, x_{k+1}} = 1\}$$

where $A = (t_{ij})_{S\times S}$ with $t_{ij} \in \{0, 1\}$. We endow Σ with the topology generated by cylinders $[a_1, \ldots, a_n] := \{x \in \Sigma : x_1^n = a_1^n\}$, where $x_i^j := (x_i, \ldots, x_j)$. Note that the collection of cylinders of length n is exactly α_0^{n-1} where $\alpha := \{[a] : a \in S\}$.

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Define the left shift $T : \Sigma \to \Sigma$ by $(Tx)_i = x_{i+1}$. Let $\mathcal{P}(\Sigma)$ denote the collection of Borel probability measures on Σ.

Henceforth we assume that (Σ, T) is topologically mixing. It is well-known that this is equivalent to the existence of N_0 such that all the entries of A^{N_0} are positive (see [Bo]).

Let $h : \Sigma \to \mathbb{R}_+$, $f : \Sigma \to \mathbb{Z}^d$ be Hölder continuous. Set

$$\Sigma^h := \{(x, s) : x \in \Sigma,\, 0 \le s < h(x)\},$$

and define the semiflows $g_t : \Sigma^h \to \Sigma^h$ and $G_t : \Sigma^h \times \mathbb{Z}^d \to \Sigma^h \times \mathbb{Z}^d$ by

$$\left.\begin{array}{lcl} g_t(x,s) & := & \big(T^n x, s + t - h_n(x)\big) \\ G_t(x,s,\nu) & := & \big(T^n x, s+t-h_n(x), \nu + f_n(x)\big) \end{array}\right\} \text{ where } s+t \in [h_n(x), h_{n+1}(x)).$$

Define the *tail equivalence relations* $\mathfrak{T}(g)$ on Σ^h, and $\mathfrak{T}(G)$ on $\Sigma^h \times \mathbb{Z}^d$ as follows:

$$\begin{array}{lcl} \mathfrak{T}(g) & := & \big\{ \big((x,s),(x',s')\big) \,\big|\, g_t(x,s) = g_t(x',s') \text{ for some } t > 0 \big\} \\ \mathfrak{T}(G) & := & \big\{ \big((x,s,\nu),(x',s',\nu')\big) \,\big|\, G_t(x,s,\nu) = G_t(x',s',\nu') \text{ for some } t > 0 \big\}. \end{array}$$

It is not difficult to verify that

$$\big((x,s),(x',s')\big) \in \mathfrak{T}(g) \Leftrightarrow \exists n, m > 0 \text{ s.t. } \left\{ \begin{array}{l} T^n(x) = T^m(x') \\ s - h_n(x) = s' - h_m(x') \end{array} \right.$$

and that

$$\big((x,s,\nu),(x',s',\nu')\big) \in \mathfrak{T}(G) \Leftrightarrow \exists n, m > 0 \text{ s.t. } \left\{ \begin{array}{l} T^n(x) = T^m(x') \\ s - h_n(x) = s' - h_m(x') \\ \nu + f_n(x) = \nu' + f_m(x') \end{array} \right.$$

As shown in[B-M], the relation $\mathfrak{T}(g)$ is a symbolic model for the strong stable foliation of a topologically mixing basic set Ω_k of an Axiom A flow, in the sense that, given such a flow, there exists Σ , h as above, and a one-to one correspondence between invariant measures for the strong stable foliation of Ω_k and locally-finite invariant measures for $\mathfrak{T}(g)$. The reader is referred to [B-M] for the definition of the these geometric objects.

In the same sense, $\mathfrak{T}(G)$ is a symbolic model for the strong stable foliation of a $\mathbb{Z}^d$-extension of an Axiom A flow, see [B-L],[Po], [C].

1.2. **The Babillot–Ledrappier Measures.** The relation $\mathfrak{T}(g)$ is uniquely ergodic [B-M], but $\mathfrak{T}(G)$ is not: [B-L] provides a d-parameter family of pairwise disjoint $\mathfrak{T}(G)$-invariant measures, called here *Babillot-Ledrappier (B-L) measures.* These are given as follows. Fix $\alpha \in \mathbb{R}^d$. By [Bo], [Ru] there exists a unique $\tau_\alpha \in \mathbb{R}$ and a unique Borel probability measure μ_α on Σ which is $(e^{-\tau_\alpha h + \langle \alpha, f\rangle}, T)$-*conformal* in the sense that $\mu_\alpha \circ T \sim \mu_\alpha$ and

$$\frac{d\mu_\alpha \circ T}{d\mu_\alpha} = e^{-\tau_\alpha h + \langle \alpha, f \rangle}.$$

The B-L measure indexed by $\alpha \in \mathbb{R}^d$ is the measure on $X = \Sigma^h \times \mathbb{Z}^d$ given by

$$m_\alpha(A \times B \times \{\nu\}) := e^{-\langle \alpha, \nu\rangle} \mu_\alpha(A) \int_B e^{\tau_\alpha r} dr.$$

These are $\mathfrak{T}(G)$-invariant measures. They are infinite, but *locally finite*: compact subsets of $\Sigma^h \times \mathbb{Z}^d$ have finite measure.

1.3. **Main Results.** It is known that ([C] and [Po])

Proposition 1.1. *m_α is $\mathfrak{T}(G)$–ergodic iff $T_{(-h,f)} : \Sigma \times \mathbb{R} \times \mathbb{Z}^d \to \Sigma \times \mathbb{R} \times \mathbb{Z}^d$ given by $T_{(-h,f)}(x, s, \nu) = \big(Tx, s - h(x), \nu + f(x)\big)$ is ergodic with respect to $\mu_\alpha \times m_{\mathbb{R}\times\mathbb{Z}^d}$, where $m_{\mathbb{R}\times\mathbb{Z}^d}$ denotes Haar measure.*

The purpose of this note is

1. To characterize this situation of ergodicity in terms of a cocycle condition for $(-h, f) : \Sigma \to \mathbb{R} \times \mathbb{Z}^d$ by showing that if one of the B-L measures is ergodic, then $(-h, f) : \Sigma \to \mathbb{R} \times \mathbb{Z}^d$ is non-arithmetic (as defined below) and that this implies that all the B-L measures are ergodic (see [C], and theorem 2.1 and corollary 2.4 below, which imply proposition 1.1).
2. To identify the locally finite $\mathfrak{T}(G)$-invariant measures by showing that in the case when the B-L measures are ergodic, that every locally finite, $\mathfrak{T}(G)$-invariant, ergodic measure which is G-quasi-invariant must be proportional to a B-L measure (Theorem 3.1 below). Theorem 2.2 in [A-N-S-S] can be viewed as a (more complete) discrete time version of this result.

As shown in [B-L], horocycle foliations of $\mathbb{Z}^d$-covers of compact manifolds of constant negative curvature are ergodic with respect to the B-L measures. This is implied (via theorem 2.1 below) by ergodicity with respect to Lebesgue measure which was established earlier in [L-S] (see also [K] and [Po]).

It follows from our results that a locally finite measure which is ergodic and invariant for the strong stable foliation of a basic set Ω_k of an Axiom A flow, and which is quasi-invariant under the flow must be proportional to a B-L measure. (In the case of a surface of constant negative curvature this can also be shown via a geometric argument, [Ba].)

2. **Ergodicity and non-arithmeticity of $\mathbb{G}$-extensions.** Let $\mathbb{G}$ be a locally compact, second countable, Abelian topological group; let $(X, \mathcal{B}, m, T)$ be a probability preserving transformation and let $\phi : X \to \mathbb{G}$ be measurable. Consider the skew product $T_\phi : X \times \mathbb{G} \to X \times \mathbb{G}$ defined by $T_\phi(x, y) := (Tx, y + \phi(x))$ with respect to the (invariant) product measure $m \times m_{\mathbb{G}}$ where $m_{\mathbb{G}}$ denotes Haar measure.

Following [G], we say that ϕ is *non-arithmetic* if

$$\gamma(\phi) = \overline{g} \cdot g \circ T$$

has no nontrivial solution in $\gamma \in \hat{\mathbb{G}}$ and $g : X \to \mathbb{S}^1$ measurable; and that ϕ is *aperiodic* if

$$\gamma(\phi) = z\overline{g} \cdot g \circ T$$

has no nontrivial solution in $\gamma \in \hat{\mathbb{G}},\ z \in \mathbb{S}^1$ and $g : X \to \mathbb{S}^1$ measurable. It is not hard to show that if T_ϕ is ergodic, and T is weakly mixing, then ϕ is non-arithmetic, and in this case T_ϕ is weakly mixing iff ϕ is aperiodic (see e.g. [K-N]).

Since $\mathbb{G}$ is a locally compact Abelian polish group topological group, there are norms $\|\cdot\|$ generating the topology of $\mathbb{G}$ which are Lipschitz in the sense that each character $\gamma : \mathbb{G} \to \mathbb{S}^1$ is $\|\cdot\|$-Lipschitz. Indeed, if Y is a metric space, and $f : Y \to \mathbb{G}$ is such that $\gamma \circ f : Y \to \mathbb{S}^1$ is Lipschitz $\forall$ characters γ, then $\exists$ a Lipschitz norm $\|\cdot\|$ such that $f : Y \to \mathbb{G}$ is $\|\cdot\|$-Lipschitz.

Livsic's theorem (see [L]) states that if $(\Sigma, \mathcal{B}, m, T)$ is a mixing subshift of finite type equipped with a Gibbs measure, $\phi : X \to \mathbb{G}$ is Hölder continuous (w.r.t some Lipschitz norm), and $\gamma \in \hat{\mathbb{G}}$ and $g : X \to \mathbb{S}^1$ measurable with $\gamma(\phi) = \overline{g} \cdot g \circ T$ a.e., then $g : X \to \mathbb{S}^1$ is also Hölder continuous (w.r.t the same Lipschitz norm). Thus if a Hölder continuous $\phi : X \to \mathbb{G}$ is non-arithmetic with respect to some Gibbs measure, then it is non-arithmetic with respect to all Gibbs measures.

Recall that a non-singular subshift of finite type $(\Sigma, \mathcal{B}, m, T)$ has the *Rényi property* if there is a constant $C > 0$ such that for every cylinder of positive measure $a = [a_1, \dots, a_n]$

$$\frac{v_a'(x)}{v_a'(y)} \le C \quad \text{for } m \times m \text{ a.e. } (x,y) \in a \times a,$$

where $v_a := \left(T^n|_a\right)^{-1}$ and $v_a' := \frac{dm\circ v_a}{dm}$. The following is a generalization of a theorem in [C].

Theorem 2.1. *Suppose that $(\Sigma, \mathcal{B}, m, T)$ is a mixing subshift of finite type with the Rényi property and that ϕ is Hölder continuous and non-arithmetic; then T_ϕ is ergodic.*

Lemma 2.2. *Assume $u : \Sigma \to \mathbb{S}^1$ is Hölder continuous. At least one of the following statements is true:*

1. *$u = \overline{g} \cdot g \circ T$ for some Hölder continuous $g : \Sigma \to \mathbb{S}^1$.*
2. *Let $\epsilon \in (0,1)$ and $N \in \mathbb{N}$ be arbitrary constants. There exists $n \ge N$ such that for every $z \in \Sigma$ there are $x \in \Sigma$ and $k \le n$ such that*

$$x_1^N = z_1^N,\ T^k x = T^n z \text{ and } |u_n(z) - u_k(x)| \ge \epsilon.$$

Proof. Let μ be the Parry measure (i.e. measure of maximal entropy on Σ), then $d\mu = \psi d\nu$ where $\nu \in \mathcal{P}(\Sigma)$ is $(1,T)$-conformal and $\psi > 0$ is Hölder continuous. Let $P : L^1(\nu) \to L^1(\nu)$ be the transfer operator, then

$$Pf(x) = \sum_{Ty=x} e^{-h_{\text{top}(T)} f(y)}$$

and $P^n f \to \psi \int_X f d\nu$ uniformly $\forall\ f \in C(X)$. Define $P_u : C(\Sigma) \to C(\Sigma)$ by $P_u(f) := P(uf)$, then $P_u^n f = P^n(u_n f)$ where $u_n := \prod_{i=0}^{n-1} u \circ T^i$. By [G-H] either $\exists\ \varphi : \Sigma \to \mathbb{S}^1$ Hölder continuous such that $P_u(\varphi) = \varphi$ (which implies (1) with $g := \varphi/\psi$), or $\left\| \frac{1}{n} \sum_{k=0}^{n-1} P_u^k f \right\|_\infty \to 0\ \forall\ f \in C(\Sigma)$. If (2) fails, then $\exists\ \epsilon \in (0,1),\ N \ge 1$ such that $\forall\ n \ge N,\ \exists\ z = z^{(n)}$ satisfying

$$k \le n,\ x \in T^{-k}\{T^n z\},\ x_1^N = z_1^N\ \Rightarrow\ |u_k(x) - u_n(z)| < \epsilon.$$

There are only finitely many possibilities for the N-prefix of $z^{(n)}$. We may therefore assume without loss of generality that $\exists a = [a_1, \dots, a_N]$ such that $z^{(n)} \in a$

for all n.

$$\begin{aligned}\left\|\frac{1}{n}\sum_{k=0}^{n-1}P_u^k 1_a\right\|_\infty &\geq \left|\frac{1}{n}\sum_{k=0}^{n-1}P_u^k 1_a(T^n z^{(n)})\right| \\ &= \left|\frac{1}{n}\sum_{k=0}^{n-1}e^{-kh_{\mathrm{top}(T)}\sum_{y\in T^{-k}\{T^n z^{(n)}\}}u_k(y)1_a(y)}\right| \\ &= \left|\frac{1}{n}\sum_{k=0}^{n-1}e^{-kh_{\mathrm{top}(T)}\sum_{y\in T^{-k}\{T^n z^{(n)}\}}1_a(y)\left(u_n(z^{(n)})-\left[u_n(z^{(n)})-u_k(y)\right]\right)}\right| \\ &\geq \frac{1}{n}\sum_{k=0}^{n-1}e^{-kh_{\mathrm{top}(T)}\sum_{y\in T^{-k}\{T^n z^{(n)}\}}1_a(y)\left(1-|u_n(z^{(n)})-u_k(y)|\right)} \\ &\geq (1-\epsilon)\frac{1}{n}\sum_{k=0}^{n-1}e^{-kh_{\mathrm{top}(T)}\sum_{y\in T^{-k}\{T^n z^{(n)}\}}1_a(y)} \\ &= (1-\epsilon)\frac{1}{n}\sum_{k=0}^{n-1}P^k 1_a(T^n z^{(n)}).\end{aligned}$$

Now $\frac{1}{n}\sum_{k=0}^{n-1}P^k 1_a \to \nu(a)\psi$ uniformly, whence

$$\liminf_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}P^k 1_a(T^n z^{(n)}) \geq \nu(a)\inf\,\psi > 0.$$

Let W_n denote the collection of admissible words of length n in Σ, that is $W_n := \{(\epsilon_1,\dots,\epsilon_n)\in S^n :\ A_{\epsilon_j,\epsilon_{j+1}} = 1\ \forall\ 1\leq j\leq n-1\}$. We denote the concatenation of $a\in W_n$ and $b\in W_m$ with $A_{a_n,b_1}=1$, by $a\cdot b$, and the concatenation of $a\in W_n$ and $x\in\Sigma$ with $A_{a_n,x_1}=1$ by (a,x).

Lemma 2.3. *Suppose that ϕ is Hölder continuous, $\gamma\in\widehat{\mathbb{G}}$ is non-constant, $\epsilon\in(0,1)$ and $N\in\mathbb{N}$. If ϕ is non-arithmetic, then there exists $\ell\geq 1$ arbitrarily large and infinitely many $n\geq N$ with the following property:*

$$\left.\begin{array}{c} a\in W_n \\ c\in W_\ell \\ a\cdot c\in W_{n+\ell}\end{array}\right\} \Rightarrow \begin{array}{c}\exists k\in[N,n] \\ \textit{and} \\ \exists b\in W_k\end{array} \quad s.t. \left\{\begin{array}{c} b_1^N = a_1^N \\ b_k = a_n \\ \forall x\in c, |\gamma\circ\phi_n(a,x)-\gamma\circ\phi_k(b,x)|\geq\epsilon\end{array}\right.$$

Proof. Fix $\gamma\in\widehat{\mathbb{G}}$ non-constant, $\epsilon\in(0,1)$, and $N\geq 1$. Choose $0<\delta<\frac{1-\epsilon}{2}$ and $\ell\geq 1$ such that

$$\eta_\ell := \sup\left\{|\gamma\circ\phi_n(x)-\gamma\circ\phi_n(y)| : n\geq 1,\ x,y\in\Sigma,\ x_1^{n+\ell}=y_1^{n+\ell}\right\}<\delta.$$

By lemma 2.2, $\exists\ n\geq N$ such that $\forall\ z\in\Sigma,\ \exists\ k\leq n,\ x\in T^{-k}\{T^n z\},\ x_1^N = z_1^N$ such that

$$|\gamma\circ\phi_n(z)-\gamma\circ\phi_k(x)|\geq\epsilon+2\delta.$$

Now fix $a\in W_n,\ c\in W_\ell$ with $a\cdot c\in W_{n+\ell}$, choose some $u\in\Sigma$ such that $A_{c_\ell,u_1}=1$, and set $z=(a,c,u)$. Let $k\leq n,\ x(z)\in T^{-k}\{T^n z\},\ x(z)_1^N = z_1^N$ be such that $|\gamma\circ\phi_n(z)-\gamma\circ\phi_k(x(z))|\geq\epsilon+\delta$ and let $b=x(z)_1^k$. Since $T^k x(z)=T^n z,\ x(z)=(b,c,u)$. For any $v\in\Sigma$ with $A_{c_\ell,v_1}=1$ we have that

$$|\gamma\circ\phi_n(a,c,u)-\gamma\circ\phi_n(a,c,v)|<\delta,\ |\gamma\circ\phi_k(b,c,u)-\gamma\circ\phi_k(b,c,v)|<\delta$$

whence $|\gamma\circ\phi_n(a,c,v)-\gamma\circ\phi_k(b,c,v)|\geq\epsilon$. Since this is true for all $v\in\Sigma$ with $A_{c_\ell,v_1}=1$, the lemma is proved. ∎

Proof of theorem 2.1 (c.f. §2 "Proof of theorem 1" in [AD]) For a nonsingular transformation $(Y, \mathcal{C}, \mu, Q)$, define the *Grand Tail Relation* of Q:

$$\mathfrak{G}(Q) := \{(x,y) \in Y \times Y :\ \exists\, n,k > 0,\ Q^n x = Q^k y\}.$$

This is an equivalence relation, and if $(Y, \mathcal{C}, \mu)$ is standard, then $\mathfrak{G}(Q) \in \mathcal{C} \otimes \mathcal{C}$. If Q is locally invertible, then $\mathfrak{G}(Q)$ has countable equivalence classes and is nonsingular. It is easy to check that every Q-invariant subset of Y is $\mathfrak{G}(Q)$-saturated. It follows that if $\mathfrak{G}(Q)$ is ergodic, then Q is ergodic.

It is therefore enough to prove that $\mathfrak{G}(T_\phi)$ is ergodic. Define

$$\widetilde{\phi} : \mathfrak{G}(T) \setminus \{(x,y) \in X \times X : x \text{ and } y \text{ are pre-periodic }\} \to \mathbb{G}$$

by $\widetilde{\phi}(x,y) = \phi_n(x) - \phi_k(y)$ whenever $T^n x = T^k y$. This is independent of the choice of n, k whenever x, y are not pre-periodic.

The grand tail relation of T_ϕ is given by

$$\begin{aligned}\mathfrak{G}(T_\phi) &= \Big\{\big((x,s),(y,t)\big) \in (X \times \mathbb{G})^2 :\ \exists\, n,k > 0 \text{ such that } T^n x = T^k y, \\ &\qquad\qquad \text{and } s - t = \phi_n(y) - \phi_k(x)\Big\} \\ &= \Big\{\big((x,s),(y,t)\big) \in (X \times \mathbb{G})^2 :\ (x,y) \in \mathfrak{G}(T),\ \widetilde{\phi}(x,y) = s - t\Big\}\end{aligned}$$

We prove that $\mathfrak{G}(T_\phi)$ is ergodic by the method of Schmidt (explained in [S]), by considering the group of essential values which we now proceed to define. Set $\mathcal{B}_+ := \{B \in \mathcal{B} : m(B) > 0\}$. For every $B \in \mathcal{B}_+$, let $\mathrm{Hol}(B) = \mathrm{Hol}\big(B, \mathfrak{G}(T)\big)$ be the collection of non-singular $\mathfrak{G}(T)$-holonomies with domain B:

$$\begin{aligned}\mathrm{Hol}(B) := \big\{\tau : B \to X : \tau &\text{ is a non-singular Borel isomorphism } B \to \tau(B) \\ &\text{such that } \forall x \in B, \big(x, \tau(x)\big) \in \mathfrak{G}(T)\big\}.\end{aligned}$$

Now define

$$\begin{aligned}E\big(\mathfrak{G}(T_\phi)\big) := \Big\{t \in \mathbb{G} :\ &\forall U \text{ open neighborhood of } t \text{ and } \forall A \in \mathcal{B}_+, \\ &\exists B \in \mathcal{B}_+ \text{ and } \exists \tau \in \mathrm{Hol}(B) \text{ such that } B, \tau(B) \subseteq A \\ &\text{and } m\big(B \cap \tau^{-1} B \cap \{x \in X :\ \widetilde{\phi}(x, \tau(x)) \in U\}\big) > 0\Big\}.\end{aligned}$$

It is shown in [S] that $E\big(\mathfrak{G}(T_\phi)\big)$ is a closed subgroup of $\mathbb{G}$. To prove ergodicity, we show that $E\big(\mathfrak{G}(T_\phi)\big) = \mathbb{G}$ (see [S]).

Suppose that $E\big(\mathfrak{G}(T_\phi)\big) = H \subsetneq \mathbb{G}$, then $\exists\ \gamma \in \widehat{\mathbb{G}},\ \gamma \neq 0$ with $\gamma|_H \equiv 1$. Fix a precompact neighborhood of the identity $V \subseteq \mathbb{G}$, and let $N \in \mathbb{N}$ be so large that

$$j \geq 1, n \geq N,\ x_1^{j+n} = y_1^{j+n} \Rightarrow \phi_j(x) - \phi_j(y) \in V.$$

Fix $\epsilon \in (0,1)$ and let $\ell \geq 1$ and $n \geq N$ be as in lemma 2.3 with ℓ so large that

$$\eta_\ell := \sup\Big\{|\gamma \circ \phi_j(x) - \gamma \circ \phi_j(y)| : j \geq 1,\ x, y \in \Sigma,\ x_1^{j+\ell} = y_1^{j+\ell}\Big\} < \frac{\epsilon}{5}.$$

It follows that $\forall a \in W_n, \forall c \in W_\ell$ s.t. $a \cdot c \in W_{n+\ell}$, $\exists\ k \leq n,\ b \in W_k$ with $b_1^N = a_1^N,\ b_k = a_n$ such that $\forall\ j \geq 1, \forall u \in W_j$ s.t. $A_{u_j, a_1} = 1$,

$$\big|\gamma \circ \phi_{j+n}(u,a,c,x) - \gamma \circ \phi_{j+k}(u,b,c,x)\big| \geq \frac{4\epsilon}{5}\ \forall\ x \in Tc_\ell.$$

Let

$$K := \Big\{\phi_{j+n}(u,a,c,x) - \phi_{j+k}(u,b,c,x):\ j \ge 1, u \in W_j,\ a \in W_n,\ A_{u_j,a_1} = 1,\\ c \in W_\ell, a\cdot c \in W_{n+\ell},\ k \le n,\ b \in W_k,\ b_1^N = a_1^N,\ b_k = a_n,\\ x \in Tc_\ell,\ |\gamma\circ\phi_{n+j}(u,a,c,x) - \gamma\circ\phi_{j+k}(u,b,c,x)| \ge \frac{4\epsilon}{5}\Big\}.$$

By the choice of N and γ, $\overline{K} \subset \overline{V} \setminus E\big(\mathfrak{G}(T_\phi)\big)$ and $\overline{K}$ is compact. The methods of [S] show that $\exists\ A \in \mathcal{B}_+$ such that

$$\big(A \times A\big) \cap \mathfrak{G}(T) \cap [\widetilde{\phi} \in K] = \varnothing.$$

By the Rényi property, $\exists\ M > 1$ such that

$$M^{-1}m(u)m(v) \le m(u \cap T^{-k}v) \le Mm(u)m(v)\ \ \forall\, u \in \alpha_0^{k-1},\ v \in \alpha_0^{\ell-1},\ [v_1] \subset T[u_k].$$

Given $j \ge 1,\ u = [u_1,\dots,u_j] \subset \Sigma$ and $a \in W_n,\ b \in W_k,\ c \in W_\ell$ as above, define $\tau : [u\cdot a\cdot c] \to [u\cdot b\cdot c]$ by

$$\tau(u,a,c,y) := (u,b,c,y).$$

It follows that $\tau : [u,a,c] \to [u,b,c]$ is invertible, nonsingular and $\frac{dm\circ\tau}{dm} = M^{\pm 4}\frac{m(b)}{m(a)}$.

Let $\delta > 0$ be so small that for all $k \le n, a \in W_n, b \in W_k, c \in W_\ell, k \le n$,

$$\delta < \frac{m(b)}{M^4 m(a)}\left(\frac{m([a,c])}{M} - \delta\right)$$

$\exists\ j \ge 1$ and $u = [u_1,\dots,u_j] \subset \Sigma$ such that $m(u \setminus A) < \delta m(u)$. Let $a \in W_n$ be such that $[u,a] \ne \varnothing$ and let $k \le n,\ b \in W_k,\ c \in W_\ell$ be as above. Consider the corresponding $\tau : [u,a,c] \to [u,b,c]$. Evidently $T^{j+k}\circ\tau \equiv T^{j+n}$ so $(x,\tau(x)) \in \mathfrak{G}(T)\ \forall\ x \in [u,a,c]$, and $\phi_{j+k}\circ\tau(x) - \phi_{j+n}(x) \in K\ \forall\ x \in [u,a,c]$.

To complete the proof we claim that $\exists\ B \in \mathcal{B}_+\ B \subset A \cap [u,a,c]$ such that $\tau B \subset A$. To see this we show that $m(\tau([u,a,c] \cap A)) \ge m(u \setminus A)$, because this implies $m\big(A \cap \tau([u,a,c] \cap A)\big) > 0$ since $\tau([u,a,c] \cap A) \subset u$. Now

$$\begin{aligned} m(\tau([u,a,c]\cap A)) &\ge \frac{m(b)}{M^4 m(a)} m([u,a,c]\cap A) \\ &\ge \frac{m(b)}{M^4 m(a)}\Big(m([u,a,c]) - m(u \setminus A)\Big) \\ &> \frac{m(b)}{M^4 m(a)}\left(\frac{m([a,c])}{M} - \delta\right) m(u) \\ &> \delta m(u) > m(u \setminus A). \end{aligned}$$

and this shows that $(A \times A) \cap \mathfrak{G}(T_\phi) \cap [\widetilde{\phi} \in K] \ne \varnothing$ which is a contradiction.

The following amplifies proposition 1:

Corollary 2.4. *Let m_α be a B-L measure on $\Sigma^h \times \mathbb{Z}^d$. The following are equivalent:*

1. *$(\Sigma^h \times \mathbb{Z}^d, m_\alpha, \mathfrak{T}(G))$ is ergodic;*
2. *the cocycle $(-h,f) : \Sigma \to \mathbb{R} \times \mathbb{Z}^d$ is non-arithmetic;*
3. *$T_{(-h,f)}$ is ergodic on $\Sigma \times \mathbb{R} \times \mathbb{Z}^d$ with respect to $\mu_\alpha \times m_{\mathbb{R}\times\mathbb{Z}^d}$ where $m_{\mathbb{R}\times\mathbb{Z}^d}$ denotes Haar measure and μ_α is as in §1.2.*

Proof. Set $X = \Sigma^h \times \mathbb{Z}^d$. As shown in [Po],

$$\mathfrak{G}(T_{(-h,f)}) \cap (X \times X) = \mathfrak{T}(G) \tag{2.1}$$

(1) $\Rightarrow$ (2). Suppose (1) and that $s \in \mathbb{R},\ \gamma \in \mathbb{R}^d$ and $g : \Sigma \to \mathbb{S}^1$ satisfy $e^{-ish+i\langle\gamma,f\rangle} = \frac{g}{g\circ T}$, and define $F : X \to \mathbb{C}$ by $F(x,y,z) := g(x)e^{-isy+i\langle\gamma,z\rangle}$, then

$$\begin{aligned} F\circ T_{(-h,f)}(x,y,z) &= F(Tx, y-h(x), z+f(x)) \\ &= g(Tx)e^{-isy+ish(x)+i\langle\gamma,z\rangle+i\langle\gamma,f(x)\rangle} \\ &= \tfrac{g(Tx)}{g(x)}e^{-ish(x)+i\langle\gamma,f(x)\rangle}F(x,y,z) = F(x,y,z). \end{aligned}$$

It follows that F is constant, since $F\circ T_{(-h,f)} = F$ and so every set of the form $[F \le t]$ is $\mathfrak{G}(T_{(-h,f)})$-saturated whence also $\mathfrak{T}(G)$-saturated.

Now consider $F_0 : X \to \mathbb{C}$ the restriction of F to X. It follows that for $(x,y,z) \in X,\ t \ge 0$ (choosing $n \ge 0$ such that $h_n(x) \le t < h_{n+1}(x)$):

$$\begin{aligned} F_0\circ G_t(x,y,z) &= F_0(T^n x, y+t-h_n(x), z+f_n(x)) = F\circ T^n_{(-h,f)}(x,y+t,z) \\ &= F(x,y+t,z) = e^{-ist}F_0(x,y,z) \end{aligned}$$

and F_0 is $\mathfrak{T}(G)$-invariant, whence constant. It follows that $s = 0,\ \gamma = 0$ and $g \equiv 1$, so $(-h,f) : \Sigma \to \mathbb{R}\times\mathbb{Z}^d$ is non-arithmetic. (2) $\Rightarrow$ (3) by theorem 2.1. (3) $\Rightarrow$ (1) follows from (2.1). ☐

Thus:

Corollary 2.5. *If $\mathfrak{T}(G)$ is ergodic with respect to some B-L measure, then the cocycle $(-h,f) : \Sigma \to \mathbb{R}\times\mathbb{Z}^d$ is non-arithmetic and $\mathfrak{T}(G)$ is ergodic with respect to all B-L measures.*

3. Identification of ergodic, locally finite $\mathfrak{T}(G)$-invariant measures.

Theorem 3.1. *Let $X := \Sigma^h \times \mathbb{Z}^d$ and let $G_t\quad (t \ge 0)$ be the suspension semi-flow. Assume that $(-h,f) : \Sigma \to \mathbb{R}\times\mathbb{Z}^d$ is non-arithmetic and Hölder continuous. Suppose that m is a locally finite, $\mathfrak{T}(G)$-invariant, ergodic measure on X and that $m\circ G_t^{-1} \sim m\ \forall\ t > 0$, then m is proportional to a B-L measure.*

Proof. By assumption, $f : \Sigma \to \mathbb{Z}^d$ is Hölder continuous, and every such function is of the form $f(x) = f(x_1, \ldots, x_m)$ for some m. Recoding Σ if necessary, we assume without loss of generality that $f(x) = f(x_1, x_2)$.

For $t > 0$, define the measure $m\circ G_t$ by $m\circ G_t(A) := \sum_{a\in\alpha} m(G_t(A\cap a))$ where α is a countable partition of X such that $G_t|_a$ is 1-1 $\forall\ a \in \alpha$. Evidently $m\circ G_t \sim m$. Let $\mathfrak{M}(\Sigma\times\mathbb{Z}^d)$ denote the collection of all (possibly infinite) Borel measures on $\Sigma\times\mathbb{Z}^d$.

Claim 1: $\exists\ \tau \in \mathbb{R}$ such that $\frac{dm\circ G_t}{dm} = e^{\tau t}$, and $\exists\ \mu \in \mathfrak{M}(\Sigma\times\mathbb{Z}^d)$ locally finite, such that $\frac{d\mu\circ T_f}{d\mu} = e^{\tau h}$ and

$$m(A\times B) = \mu(A)\int_B e^{\tau r}dr \quad \big(A \in \mathcal{B}(\Sigma\times\mathbb{Z}^d),\ B \in \mathcal{B}(\mathbb{R}),\ A\times B \subset X\big). \tag{3.2}$$

Moreover $(\Sigma\times\mathbb{Z}^d, \mathcal{B}(\Sigma\times\mathbb{Z}^d), T_f, \mu)$ is ergodic.

Proof. Fix $t_0 > 0$. We prove first that $\frac{dm\circ G_{t_0}}{dm}$ is $\mathfrak{T}(G)$-invariant and hence constant. Suppose that $A \subset X$ is Borel, and that $K : A \to KA$ is a $\mathfrak{T}(G)$-holonomy. Without loss of generality, $G_{t_0}|_A,\ G_{t_0}|_{KA}$ are 1-1. It follows that

$$K_1 := G_{t_0}\circ K\circ G_{t_0}^{-1} : G_{t_0}A \to G_{t_0}KA$$

is a well-defined $\mathfrak{T}(G)$-holonomy. By the $\mathfrak{T}(G)$-invariance of m,

$$m(G_{t_0}KA) = m(K_1G_{t_0}A) = m(G_{t_0}A).$$

This shows that $\frac{dm\circ G_{t_0}}{dm}$ is indeed $\mathfrak{T}(G)$-invariant and hence constant. Disintegrating the measure m over $\Sigma\times\mathbb{Z}^d$, we see that $\exists\ \lambda\in\mathfrak{M}(\Sigma\times\mathbb{Z}^d)$ locally finite, and $m_x\in\mathfrak{M}(\mathbb{R}_+)$ such that $x\mapsto m_x$ is measurable, and such that

$$m(A\times B)=\int_A m_x(B)d\lambda(x).$$

It follows that $m_x(J+t)=e^{\tau t}m_x$ for open $J\subset(0,h(x))$ and $t\in\mathbb{R}$ small, whence $dm_x(y)=c(x)e^{\tau y}dy$ and (3.2) follows with $d\mu(x):=c(x)d\lambda(x)$. The equation $\frac{d\mu\circ T_f}{d\mu}=e^{\tau h}$ now follows from $\frac{dm\circ G_t}{dm}=e^{\tau t}$, and the ergodicity of (Σ,T_f,μ) is standard. □

Claim 2: $\exists$ a homomorphism $\alpha:\mathbb{Z}^d\to\mathbb{R}$ and $c>0$ such that $\mu(A\times\{n\})=ce^{-\alpha(n)}\nu(A)$ where $\nu\in\mathcal{P}(\Sigma)$ is $(e^{\alpha\circ f+\tau h},T)$-conformal.

Proof. We first claim it suffices to show that $H:=\{n\in\mathbb{Z}^d:\ \mu\circ Q_n\sim\mu\}=\mathbb{Z}^d$ where $Q_n(x,k):=(x,k+n)$. To see this, note that

$$\frac{d\mu\circ Q_n\circ T_f}{d\mu\circ Q_n}=\frac{d\mu\circ T_f}{d\mu}\circ Q_n=e^{\tau h}\quad\forall n\in\mathbb{Z}^d.$$

The ergodicity of (Σ,T_f,μ) ensures that $\forall n\in\mathbb{Z}^d$, either $\mu\circ Q_n\perp\mu$ or $\mu\circ Q_n=c_n\mu$ for some $c_n>0$. The condition $H=\mathbb{Z}^d$ ensures that $\mu\circ Q_n=e^{-\alpha(n)}\mu$ where $\alpha:\mathbb{Z}^d\to\mathbb{R}$ is a homomorphism. Thus, $\mu(A\times\{n\})=ce^{-\alpha(n)}\nu(A)$ where $c>0$ and $\nu\in\mathcal{P}(\Sigma)$. The $(e^{\alpha\circ f+\tau h},T)$-conformality of ν follows from the $(e^{\tau h},T_f)$-conformality of μ.

We now prove that $H=\mathbb{Z}^d$. Suppose otherwise that $H\neq\mathbb{Z}^d$, then $\exists\ \gamma\in\widehat{\mathbb{Z}^d}$ non-constant, such that $\gamma|_H\equiv 1$. Using non-arithmeticity and lemma 2.3, we fix $n\geq 1$ so that $\forall\ a\in W_n$ and $c\in S$ s.t. $a\cdot c\in W_{n+1}$, $\exists\ k=k(a)\leq n$ and $b=b(a,c)\in W_k$ such that $a_1=b_1,\ a_n=b_k$ and $\gamma\circ f_n(a,c)\neq\gamma\circ f_k(b,c)$.[1] By choice of γ, this means that $f_n(a,c)-f_k(b,c)\notin H$.

Set $J:=\{f_n(a,c)-f_k(b(a,c),c):\ a\in W_n, c\in S, a\cdot c\in W_{n+1}\}$, then $J\subset\mathbb{Z}^d\setminus H$ and J is finite. Set $\overline{\mu}:=\sum_{j\in J}\mu\circ Q_j$, then $\overline{\mu}\perp\mu$ and $\exists\ \ K\subset\Sigma$ compact and $g\in\mathbb{Z}^d$ such that $\mu(K\times\{g\})>0$, $\overline{\mu}(K\times\{g\})=0$.

Set $I:=\sup\{|h_j(x)-h_j(y)|:\ \ j\geq 1,\ x_1^j=y_1^j\}$, $L:=2\max_{k\leq n}\sup|h_k|$ and $M:=|W_{n+1}|e^{\tau(I+L)}$. Approximating K by larger open sets, we see that $\exists\ U\subset\Sigma$ open , such that $K\subset U$ and $\overline{\mu}(U\times\{g\})<\frac{\mu(K\times\{g\})}{2M}$. It follows that $\exists$ a cylinder set $d=[d_1,\ldots,d_N]$ such that $\mu(d\times\{g\})>0$ and $\overline{\mu}(d\times\{g\})<\frac{\mu(d\times\{g\})}{2M}$.

Since $d\times\{g\}=\bigcup_{a\in W_n,c\in S}[d,a,c]\times\{g\}$, $\exists\ a\in W_n,\ c\in S$ with $a\cdot c\in W_{n+1}$ such that $\mu([d,a,c]\times\{g\})\geq\frac{\mu(d\times\{g\})}{|W_{n+1}|}$. Next, $\exists\ b=(b_1,\ldots,b_k)\in W_k$ such that $a_1=b_1,\ a_n=b_k$ and $f_n(a,c)-f_k(b,c)\in J$. Define $\kappa:[d,a,c]\times\{g\}\to d\times\mathbb{Z}^d$ by $\kappa\big((d,a,x),g\big):=\big((d,b,x),g+f_k(b,c)-f_n(a,c)\big)$. Since $\frac{d\mu\circ T_f}{d\mu}=e^{\tau h}$, we have that

$$\frac{d\mu\circ\kappa}{d\mu}(x,v)=e^{\tau(h_{N+k}(d,b,x)-h_{N+n}(d,a,x))}\in[e^{-\tau(I+L)},e^{\tau(I+L)}],$$

[1] We are using here the assumption $f(x)=f(x_0,x_1)$ to note that lemma 2.3 can be used with $\ell=1$ and that f_n (resp. f_k) is constant on $(a,c)\in W_{n+1}$ (resp. $(b,c)\in W_{k+1}$) so that the notation $f_n(a,c)$, $f_k(b,c)$ makes sense.

where the last estimate follows from

$$|h_{N+k}(d,b,x) - h_{N+n}(d,a,x)| \le |h_N(d,b,x) - h_N(d,a,x)| \\ + |h_k(b,x)| + |h_n(a,x)| \le I + L.$$

Thus

$$\begin{aligned}(\mu\circ\kappa)\big([d,a,c]\times\{g\}\big) &= \int_{[d,a,c]\times\{g\}} \frac{d\mu\circ\kappa}{d\mu} d\mu \\ &\ge e^{-\tau(I+L)}\mu\big([d,a,c]\times\{g\}\big) \\ &\ge e^{-\tau(I+L)}\frac{\mu(d\times\{g\})}{|W_{n+1}|} \\ &= \frac{\mu(d\times\{g\})}{M}\end{aligned}$$

On the other hand, $\kappa([d,a,c]\times\{g\})) \subset Q_{f_k(b,a)-f_n(a,c)}(d\times\{g\})$ whence

$$\frac{\mu(d\times\{g\})}{M} \le \mu(\kappa\big([d,a,c]\times\{g\}\big)) \le \mu\big(Q_{f_k(b,c)-f_n(a,c)}(d\times\{g\})\big) \le \\ \overline{\mu}(d\times\{g\}) < \frac{\mu(d\times\{g\})}{2M}$$

and $1 < \frac{1}{2}$. This contradiction establishes claim 2. □

Since the $(e^{\alpha\circ f+\tau h}, T)$-conformal probability is unique, it follows from claim 2 that m is proportional to the corresponding B-L measure. □

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