Lecture 13: Proof of the Diagram Fixing Theorem

In this lecture we finish the proof of the structural stability theorem, and Anosov's shadowing and closing lemmas by proving:

"Diagram Fixing Thm"*: Suppose M is compact and $T: M \to M$ is a C^1 diffeomorphism with a hyperbolic set Λ. There exist

- $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$
- an open set $\mathcal{O} \supseteq \Lambda$, as follows.

Suppose S is a diffeo s.t. $\mathrm{dist}_{C^1}(T, S) < \varepsilon_1$,
$h: X \to X$ is a homeomorphism of a metric space X
and $\phi: X \to \mathcal{O}$ is a continuous map s.t. $\overline{\phi(X)} \subseteq \mathcal{O}$ and
$\mathrm{dist}_{C^0}(\phi \circ h, S \circ \phi) < \varepsilon_2$, i.e.

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "\phi"'] & X \arrow[d, "\phi"] \\ \mathcal{O} \arrow[r, "S"'] & M \end{tikzcd}$$

"ε_2-commutes"

Then: $\exists!$ continuous $\psi: X \to \mathcal{O}$ s.t. $\mathrm{dist}_{C^0}(\psi, \phi) < \varepsilon_3$ and $\psi \circ h = S \circ \psi$, i.e.

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "\psi"'] & X \arrow[d, "\psi"] \\ \mathcal{O} \arrow[r, "S"'] & M \end{tikzcd}$$

exactly commutes

Moreover: $\forall \varepsilon \, \exists \delta$ s.t.

$$\mathrm{dist}_{C^0}(\phi \circ h, S \circ \phi) < \delta \Rightarrow \mathrm{dist}_{C^0}(\psi, \phi) < \varepsilon .$$

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* this is not a standard name.

Banach's Fixed Point Theorem

The basic tool is the following result:

Banach's Fixed-Point Theorem: Suppose (X,d) is a complete metric space and $F: X \to X$ is a contraction:

$$\exists\, 0<\lambda<1 \text{ s.t. } \forall x,y \quad d(F(x),F(y)) \leq \lambda\, d(x,y).$$

Then: There exists a unique $x \in X$ s.t. $F(x)=x$.

Proof. Fix some arbitrary $x_0 \in X$. The sequence $F^n(x_0)$ is a Cauchy sequence:

$$d(F^{n+m}(x_0), F^n(x_0)) \leq \sum_{k=0}^{m-1} d(F^{n+k}x_0, F^{n+k+1}x_0)$$

$$\leq \sum_{k=0}^{m-1} \lambda^{n+k} d(x_0, F(x_0)) \leq \frac{\lambda^n}{1-\lambda} d(x_0, F(x_0)) \xrightarrow[n,m\to\infty]{} 0$$

The limit $x := \lim_{n\to\infty} F^n(x_0)$ exists by completeness, and since F is continuous, $F(x)=x$.

This fixed point is unique, because if $F(y)=y$ for some other y, then

$$d(x,y) = d(F^n(x), F^n(y)) \leq \lambda^n d(x,y) \xrightarrow[n\to\infty]{} 0, \text{ so } x=y. \square$$

Exercise: Show that in the setup of the previous theorem, $\forall \varepsilon\ \exists \delta$ s.t. if $G: X \to X$ is another map s.t. $\forall x,y\ d(G(x),G(y)) \leq (\lambda+\delta)\, d(x,y)$; $\forall x\ d(G(x), F(x)) \leq \delta$ then the unique fixed point of G is ε-close to the unique fixed point of F.

Solution: Fix $\delta > 0$ so small that $\mu := \lambda + \delta < \frac{1+\lambda}{2} < 1$. and suppose $G : X \to X$ is a map s.t.

$$\sup_{x,y} d(G(x), G(y)) \leq \mu d(x,y) ; \quad \sup_x d(G(x), F(x)) < \delta .$$

Let x be the fixed point of F, and y be the fixed point of G. Then

$$d(G^n(x), y) = d(G^n(x), G^n(y)) \leq \mu^n d(x,y) \longrightarrow 0 \text{ exponentially fast.}$$

Thus

$$\begin{aligned}
d(x,y) &\leq d(x, G^n(x)) + d(G^n(x), y) \\
&\leq \sum_{k=0}^{n-1} d(G^k(x), G^{k+1}(x)) + d(G^n(x), y) \\
&\leq \sum_{k=0}^{n-1} \mu^k d(x, G(x)) + o(1) \\
&\xrightarrow[n \to \infty]{} \left(\sum_{k=0}^{\infty} \mu^k \right) d(x, G(x)) \\
&= \frac{d(x, G(x))}{1-\mu} \underset{\substack{F(x)=x \\ \mu < (1+\lambda)/2}}{\uparrow =} \frac{d(F(x), G(x))}{1 - \frac{1+\lambda}{2}} \underset{d_{C^0}(F,G) < \delta}{\uparrow <} \frac{2\delta}{1-\lambda} . \quad \square
\end{aligned}$$

Proof of the Diagram Fixing Theorem

Step 1: Setting-Up a Fixed Point Theorem

We're given a diagram s.t. $\phi \circ h \approx S \circ \phi$. We want to find ψ s.t. $\psi \circ h = S \circ \psi$. We'll try to set this up as a <u>fixed point problem</u>.

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "\phi"'] & X \arrow[d, "\phi"] \\ O \arrow[r, "S"'] & M \end{tikzcd}$$

<u>Attempt 1</u>: Embedd $M \subseteq \mathbb{R}^N$. Then $\psi, \phi : X \to M$ could be viewed as function $\psi, \phi : X \to \mathbb{R}^N$. Write

$$\psi(x) = \phi(x) + v(x), \quad v(x) = \text{unknown}.$$

Then $\psi \circ h = S \circ \psi$

$$\iff \phi(h(x)) + v(h(x)) = S(\phi(x) + v(x))$$

$$\iff v(h(x)) = S(\phi(x) + v(x)) - \phi(h(x))$$

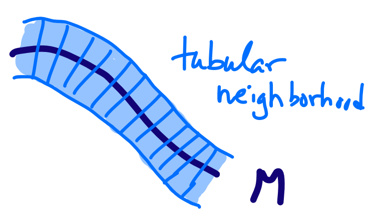
$$\iff \boxed{v(x) = S(\phi(h^{-1}(x)) + v(h^{-1}(x))) - \phi(x)}$$

<u>Problem</u>: For general $v : X \to \mathbb{R}^N$, it's possible that

$$\phi(h^{-1}(x)) + v(h^{-1}(x)) \notin M.$$

If this is so, $S(\phi(h^{-1}(x)) + v(h^{-1}(x)))$ is not defined.

<u>Attempt 2</u>: Insert a "projection" $\pi : \text{Neigh}(M) \to M$ s.t. $\pi|_M = \text{id}$. This is done using a <u>tubular neighborhood</u> of M inside $\mathbb{R}^N$:



Formally, the tubular neighborhood theorem in differential topology guarantees the existence of a neigh N of M in $\mathbb{R}^N$, and a differentiable map $\pi : N \to M$ s.t.

$$\pi|_M = id, \qquad d\pi|_{T_xM} = id \text{ for all } x \in M.$$

Fix some α s.t. the α-neigh of M in $\mathbb{R}^N$ is inside the tubular neigh N of M. Let

$$\Gamma_\alpha := \{ v : X \to \mathbb{R}^N : \text{continuous; } \|v(x)\| \leq \alpha \text{ for all } x \in X \}$$

Consider the fixed point problem

$$\boxed{v(x) = (S \circ \pi)\big(\phi(h^{-1}x) + v(h^{-1}x)\big) - \phi(x)}$$

Now everything's well-defined, and if v is a solution, then

$$\psi(x) := \phi(x) + v(x) = (S \circ \pi)(\psi(h^{-1}(x))) \underset{S(M)=M}{\in} M,$$

$$\Rightarrow \psi(x) = S(\psi(h^{-1}(x))) \quad (\because \psi(h^{-1}(x)) \in M, \pi|_M = id)$$

$$\Rightarrow \psi(h(x)) = S(\psi(x))$$

and we got our semi-conjugacy.

Of course to get a solution, we need to guarantee that

$$\Phi(v) := (S \circ \pi)(\phi \circ h^{-1} + v \circ h^{-1}) - \phi$$

is a contraction on $\Gamma_\alpha := \{ v : X \to \mathbb{R}^N : \|v\| := \sup_x \|v(x)\| < \alpha \}$, for some α.

"Sanity Check": Let's calculate the linearization $(d\Phi)_0$ of Φ at $v=0$, and see if it has norm <1.

[Recall: A non-linear map $F: B \to B$ on a Banach space B is differentiable at $v_0 \in B$, if $\exists$ linear map $L: B \to B$ s.t. $\| F(v_0+w) - (F(v_0) + Lw) \| = o(\|w\|)$.
L is called the derivative of F at v_0, and is denoted by $(dF)_{v_0} : B \to B$.

Exercise:

(1) If F is linear, then $(dF)_{v_0} = F$

(2) If F, G are differentiable, then $(d(FG))_{v_0} = dF_{G(v_0)} \, dG_{v_0}$.]

Exercise: $[(d\Phi)_0 v](x) = dS_{\phi(h^{-1}x)} \, d\pi_{\phi(h^{-1}x)} \, v(h^{-1}x)$

(Here we use: $\pi(\phi(h^{-1}x)) = \phi(h^{-1}x)$, because $\phi(x) \in M$.)

Problem: $dS_{\phi(h^{-1}x)}$ expands some vectors (because $dT_{\phi(h^{-1}x)}$ expands on $E^u(\phi(h^{-1}x))$ and $S \approx T$ in C^1)

$\Longrightarrow$ no reason why Φ should contract.

Attempt 3: "Annihilate" $d\Phi_0$ from the fixed point problem.

Write $\Phi(\sigma) = \Phi(0) + (d\Phi)_0(\sigma) + H(\sigma)$, where

$$H(\sigma) := \Phi(\sigma) - \Phi(0) - (d\Phi)_0(\sigma).$$

Then $H(0) = 0$, $(dH)_0 = 0$. Since H is continuously differentiable, $dH_\sigma \to 0$ as $\sigma \to 0$. So $\forall \varepsilon\ \exists \alpha$ s.t.

$$\|(dH)_\sigma\| \leq \varepsilon \text{ whenever } \|\sigma\| < \alpha.$$

$\Rightarrow$ If $\|\sigma\|, \|\omega\| < \alpha$, then $\|H(\sigma) - H(\omega)\| < \varepsilon \|\sigma - \omega\|$

Exercise: Prove this by applying the Mean Value Theorem to the functions $\varphi_i(t) = \langle \underline{e}_i, H(t\sigma(x) + (1-t)\omega(x)) \rangle_{\mathbb{R}^N}$, $i = 1, \dots, N$, $x \in X$ ($\underline{e}_i \in$ standard basis of $\mathbb{R}^N$)

$\Rightarrow$ H is an ε-contraction of Γ_α.

The fixed point theorem $\Phi(\sigma) = \sigma$ translates to:

$$\Phi(0) + (d\Phi)_0 \sigma + H(\sigma) = \sigma$$

$$\iff \Phi(0) + H(\sigma) = (I - (d\Phi)_0)\sigma$$

$$\iff \boxed{\sigma = (I - (d\Phi)_0)^{-1} [\Phi(0) + H(\sigma)]}$$

provided $I - (d\Phi)_0$ can be inverted.

Suppose we could prove:

<u>Lemma</u>: $\exists$ open neigh $\mathcal{O}$ of Λ and $\exists\ \varepsilon_1^*, \varepsilon_2^*, \alpha_0, K_0$ s.t. if $\mathrm{dist}_{C^1}(T, S) < \varepsilon_1^*$ and $\mathrm{dist}_{C^0}(\phi \circ h, S \circ \phi) < \varepsilon_2^*$, then $(I - (d\Phi)_0)^{-1}$ is invertible on $\Gamma_{\alpha_0} := \{\sigma : X \to \mathbb{R}^N : \text{cts}; \|\sigma\| < \alpha_0\}$ and $\|(I - (d\Phi)_0)^{-1}\| < K_0$ on Γ_{α_0}.

Then we can set

$$F(\sigma) := \left(I - (d\Phi)_0\right)^{-1}\left[\Phi(0) + H(\sigma)\right]$$

and observe:

<u>Claim</u>: For all α, ε_2 sufficiently small, if $\mathrm{dist}_{C^1}(\phi\circ h, S\circ\phi) < \varepsilon_2$ then $F(\Gamma_\alpha) \subseteq \Gamma_\alpha$ and $F : \Gamma_\alpha \to \Gamma_\alpha$ is a contraction

<u>Proof</u>.

(1) $\forall\, \sigma, w \in \Gamma_\alpha$, $\|F(\sigma) - F(w)\| \leq K_0 \|H(\sigma) - H(w)\|$
$$\leq \varepsilon K_0 \|\sigma - w\|$$

where $\varepsilon := \sup_{\|\sigma\| \leq \alpha} \|(dH)_\sigma\| \xrightarrow[\alpha\to 0]{} 0$. So for α sufficiently small, F is a contraction, say by factor $1/2$.

(2) $F(\Gamma_\alpha) \subseteq \Gamma_\alpha$: If $\|\sigma\| \leq \alpha$, then

$$\|F(\sigma)\| \leq \|F(0)\| + \|F(\sigma) - F(0)\| \leq \|F(0)\| + \tfrac{1}{2}\|\sigma\|_\infty$$
$$\leq K_0 \|\Phi(0)\| + \tfrac{1}{2}\alpha$$
$$= K_0 \| S(\phi\circ h^{-1}) - \phi\|_\infty + \tfrac{1}{2}\alpha$$

$\Phi(\sigma) = (S\circ\pi)(\phi\circ h^{-1} + \sigma\circ h^{-1}) - \phi$

$\Phi(0) = S(\phi\circ h^{-1}) - \phi$

$$= K_0 \|S\circ\phi - \phi\circ h\|_\infty + \tfrac{1}{2}\alpha$$
$$\leq K_0\, \mathrm{dist}_{C^0}(S\circ\phi, \phi\circ h) + \tfrac{1}{2}\alpha$$
$$\leq K_0\, \varepsilon_2 + \tfrac{1}{2}\alpha$$

So if $\varepsilon_2 \leq \min\{\varepsilon_2^*, \alpha/3K_0\}$, $F(\Gamma_\alpha) \subseteq \Gamma_\alpha$.

By the claim, F has a fixed point v, and we saw above that

$$F(v) = v \iff \Phi(v) = v \iff \psi = \phi + v \text{ satisfies } \psi \circ h = S \circ \psi.$$

Moreover, ψ is the unique fixed point s.t.

$$\|\psi - \phi\| = \|v\| < \alpha$$

In particular:

- ψ is locally unique
- if ϕ is close to the identity, ψ is close to the identity.

It remains to prove the lemma.

<u>Step 2</u>: <u>Proof of the Lemma</u>

<u>Recall</u>: $\Phi(v) = (S \circ \pi)(\phi \circ h^{-1} + v \circ h^{-1}) - \phi$

$$[(d\Phi)_0 v](x) = d(S \circ \pi)_{\phi(h^{-1}(x))} v(h^{-1}(x))$$

We need to find

- an open neigh $\mathcal{O}$ of Λ
- bounds $\varepsilon_1, \varepsilon_2$ for $\mathrm{dist}_{C^1}(T, S)$, $\mathrm{dist}_{C^0}(\phi \circ h, S \circ \phi)$
- bound α_0 for α

so that we can invert $I - d\Phi_0$ on

$$\Gamma_\alpha = \{ v : X \to \mathbb{R}^N : \|v\|_\Delta < \alpha \}$$

Recall: $\mathcal{O}$ is an open neigh of a hyperbolic set Λ for T.
Extend $E^s(x)$, $E^u(x)$ and the Lyapunov metric $\|\cdot\|'$ continuously to $\mathcal{O}$ s.t. $\|dT|_{E^s(x)}\| < \lambda < 1$, $\|dT^{-1}|_{E^u}\| < \lambda < 1$ on $\mathcal{O}$.
(this is possible if $\mathcal{O}$ is small enough).

For each $y \in \mathcal{O}$, we have a continuous splitting

$$\mathbb{R}^N = T_y\mathbb{R}^N = E^s(y) \oplus E^u(y) \oplus E^\perp(y)$$

(where $E^\perp(y) = \{v \in T_y\mathbb{R}^N : v \perp T_y M\}$).

Let $P^s_y : \mathbb{R}^N \to E^s(y)$, $P^u_y : \mathbb{R}^N \to E^u(y)$, $P^\perp_y : \mathbb{R}^N \to E^\perp(y)$
be the corresponding projections. Since $E^s(y), E^u(y), E^\perp(y)$ depend continuously on y, $y \mapsto P^t_y$ $(t = u, s, \perp)$ are continuous.
We can break each $v \in \Gamma_\alpha$ into

$$v^s(x) := P^s_{\phi(x)}(v(x))$$

$$v^u(x) := P^u_{\phi(x)}(v(x))$$

$$v^\perp(x) := P^\perp_{\phi(x)}(v(x))$$

and consider the action of $d\Phi_0$ on $v^s(x), v^u(x), v^\perp(x)$.

Perpendicular Component: $\forall v \in \Gamma_\alpha$

$$d\Phi_0(v(x)) = d(S\circ\pi)_{\phi(h^{-1}x)}\, v(h^{-1}x) = dS_{\phi(h^{-1}x)}\left[(d\pi)_{\phi(h^{-1}x)}(v(h^{-1}x))\right]$$

$$\subseteq dS_{\phi(h^{-1}x)}(v^s(h^{-1}x)+v^u(h^{-1}x)), \text{ because } d\pi(v^\perp(x))=0$$

$$\subseteq dS(T_{\phi(h^{-1}x)}M) = T_{S(\phi(h^{-1}x))}M \subseteq \ker P^\perp_{S(\phi h^{-1}x)}.$$

So $P^\perp_{S(\phi h^{-1}x)}(d\Phi_0 v) = 0$. This calculation also shows that $d\Phi_0(v(x)) = d\Phi_0[v^s(x)+v^u(x)]$, so $d\Phi_0 P^\perp_{\phi(x)} = 0$

Stable Component: Since $\phi(X)\subseteq\Theta$, $\phi(h^{-1}x)\approx T^{-1}(y)$, for some $y\in\Lambda$, and the error $\longrightarrow 0$ as Θ shrinks to Λ.

$$(dS)_{\phi(h^{-1}x)}\, v^s(h^{-1}x) \equiv (dS)_{\phi(h^{-1}x)}\, P^s_{\phi(h^{-1}x)} v^s(h^{-1}x)$$

$$\approx (dS)_{T^{-1}(y)}\, P^s_{T^{-1}y}\, v^s(h^{-1}x) \qquad \left(\because \begin{array}{l}\phi(h^{-1}x)\approx T^{-1}y\\ E^s(\cdot)\text{ cts on }\Theta\end{array}\right)$$

$$\approx (dT)_{T^{-1}y}\, P^s_{T^{-1}y}\, v^s(h^{-1}x) \qquad (\because \operatorname{dist}_{C^1}(T,S)\ll 1)$$

$$\in E^s(y) \qquad (\because y\in\Lambda \text{ so } (dT)_{T^{-1}y}E^s(T^{-1}y)=E^s(y))$$

$$\approx E^s(T\phi h^{-1}(x)) \qquad (\because \phi(h^{-1}x)\approx T^{-1}y)$$

$$\approx E^s(S\phi h^{-1}(x)) \qquad (\because \operatorname{dist}_{C^1}(T,S)\ll 1)$$

$$\approx E^s(\phi h h^{-1}(x)) = E^s(\phi(x)) \qquad (\because \operatorname{dist}_{C^0}(S\phi, hS)\ll 1)$$

The symbols $\approx$ mean that the error in the approximation can be made arbitrarily small by taking Θ sufficiently close to Λ, and $\operatorname{dist}_{C^1}(T,S)$, $\operatorname{dist}_{0}(\phi h, S\phi)$ suff. small.

Thus, $dS_{\phi(h^{-1}x)} v^s(h^{-1}x)$ is approximately inside $E^s(\phi(x))$. So

$$\| P^u_{S\phi h^{-1}(x)} (dS)_{\phi(h^{-1}x)} v^s(h^{-1}x) \|' \approx \| P^u_{\phi(x)} \left(\begin{array}{c}\text{vector approx.} \\ \text{inside } E^s(\phi(x))\end{array} \right) \|$$

$$\approx 0 \cdot \|v^s(h^{-1}x)\| \text{ , and}$$

$$\| P^s_{\phi(x)} (dS_{\phi(h^{-1}x)}) v^s(h^{-1}x) \|' \approx \| dS_{\phi(h^{-1}x)} v^s(h^{-1}x) \|'$$

$$\approx \| (dT)_{T^{-1}y} P^s_{T^{-1}y} v^s(h^{-1}x) \|$$

$$\leq \lambda \| P^s_{T^{-1}y} v^s(h^{-1}x) \| , \quad \lambda := \begin{array}{l}\text{contraction coefficient} \\ \text{of } dT \text{ on } E^s \text{ on } \Lambda\end{array}$$

$$\approx \lambda \| P^s_{h^{-1}(x)} v^s(h^{-1}x) \| = \lambda \| v^s(h^{-1}x) \| .$$

It follows that by choosing θ, $\varepsilon_1, \varepsilon_2$ appropriately, we can guarantee that for some $\mu < 1$, for all $v \in \Gamma_\alpha$

$$\| P^s d\Phi_0 v^s \|' \leq \mu \| v^s \| , \quad \| P^u d\Phi_0 v^s \|' \leq \varepsilon \| v^s \| .$$

Unstable Component: Similar calculations show that by choosing θ, $\varepsilon_1, \varepsilon_2$ appropriately, we can guarantee that for some $0 < \mu < 1$ and for all $v \in \Gamma_\alpha$

$$\| P^u d\Phi_0 v^u \|' \geq \mu^{-1} \| v^s \| , \quad \| P^s d\Phi_0 v^s \|' \leq \varepsilon \| v^s \| .$$

<u>In summary</u>: Define the following operators on $\mathbb{R}^N$

$$A^{ss}(x) = P^s_{(S\phi h^{-1})(x)} \circ dS_{\phi(h^{-1}x)} \circ P^s_{\phi(h^{-1}x)} \quad \text{contracts by } \mu \approx \lambda$$

$$A^{uu}(x) = P^u_{(S\phi h^{-1})(x)} \circ dS_{\phi(h^{-1}x)} \circ P^u_{\phi(h^{-1}x)} \quad \text{expands by } \mu^{-1} \approx \lambda^{-1}$$

$$\left.\begin{aligned} A^{su}(x) &= P^s_{(S\phi h^{-1})(x)} \circ dS_{\phi(h^{-1}x)} \circ P^u_{\phi(h^{-1}x)} \\ A^{us}(x) &= P^u_{(S\phi h^{-1})(x)} \circ dS_{\phi(h^{-1}x)} \circ P^s_{\phi(h^{-1}x)} \end{aligned}\right\} \text{have arbitrarily small norms}$$

Now let's return to the inversion problem

$$\boxed{(I - d\Phi_0)v = w}$$

Decomposing $v = v^u + v^s + v^\perp$, $w = w^u + w^s + w^\perp$, we obtain that for all $x \in X$,

$$\left[I - \underbrace{\begin{pmatrix} A^{ss} & A^{su} & 0 \\ A^{us} & A^{uu} & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{A} \right] \begin{pmatrix} v^u(h^{-1}x) \\ v^s(h^{-1}x) \\ v^\perp(h^{-1}x) \end{pmatrix} = \begin{pmatrix} w^u(x) \\ w^s(x) \\ w^\perp(x) \end{pmatrix}$$

Note that A is a perturbation of the matrix $\begin{pmatrix} A^{ss} & 0 & 0 \\ 0 & A^{uu} & 0 \\ 0 & 0 & 0 \end{pmatrix}$ which has spectrum $\subseteq [0,\mu] \cup [\mu^{-1}, \infty) \subseteq \mathbb{R} \setminus \{1\}$.

By choosing θ, ε_1, ε_2 appropriately, we can guarantee that $\operatorname{spec}(A) \subseteq \mathbb{R} \setminus \{1\} \Longrightarrow (I - A)$ is invertible, and $(I - A(x))^{-1}$ is uniformly bounded.*

Thus $(I - d\Phi_0)^{-1} v = (I - A(x))^{-1} \begin{pmatrix} v^s(hx) \\ v^u(hx) \\ v^\perp(hx) \end{pmatrix}$ exists, and is bounded. $\square$

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* More details on this below.

More Details on the Last Step

<u>Exercise</u>: If $A: B \to B$ is a bounded linear operator on a Banach space and $\|A\| < 1$, then $\sum_{n=0}^{\infty} A^n$ is a bounded linear operator equal to $(I-A)^{-1}$.

<u>Lemma</u>: Given $0 < \mu < 1$ there exists $\varepsilon > 0$ as follows. Suppose $A^{ss}, A^{uu}, A^{su}, A^{us}$ are square matrices s.t.

$$\|A^{su}\|, \|A^{us}\| < \varepsilon, \quad \|A^{ss}\| \leq \mu, \quad \|(A^{uu})^{-1}\| \leq \mu.$$

Then $I - \begin{pmatrix} A^{ss} & A^{su} & 0 \\ A^{us} & A^{uu} & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is invertible and the norm of the inverse is less than $2/1-\mu$.

<u>Proof</u>. First consider $D := \begin{pmatrix} A^{ss} & 0 & 0 \\ 0 & A^{uu} & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Decompose $\mathbb{R}^N = \mathbb{R}^{N_s} \oplus \mathbb{R}^{N_u} \oplus \mathbb{R}^{N_0}$ where N_s, N_u, N_0 are the number of rows and columns in A^{ss}, A^{uu}, zero matrix below them. Decompose

$$v = v^s + v^u + v^0 \quad \text{accordingly.}$$

Let's solve $(I-D)v = w$. This resolves to

$$(I-0)\, v^0 = w^0 \Rightarrow v^0 = w^0$$

$$(I - A^{ss})\, v^s = w^s \Rightarrow v^s = \sum_{k=0}^{\infty} (A^{ss})^k v^s$$

$$(I - A^{uu})\, v^s = w^u \Leftrightarrow ((A^{uu})^{-1} - I)\, v^u = (A^{uu})^{-1} w^u$$

$$\Leftrightarrow (I - (A^{uu})^{-1})\, v^u = -(A^{uu})^{-1} w^u$$

$$\Leftrightarrow v^u = -\sum_{k=0}^{\infty} (A^{uu})^{-k-1} w^u$$

Thus $(I-D)$ is invertible, and the inverse operator has norm less than

$$\sqrt{\left(\frac{1}{1-\|A^{ss}\|}\right)^2+\left(\frac{1}{1-\|(A^{uu})^{-1}\|}\right)^2+1}$$

$$\leq \sqrt{1+\frac{2}{(1-\mu)^2}} \leq \frac{\sqrt{3}}{1-\mu} < \frac{2}{1-\mu}.$$

Next, consider $(I-A)^{-1}$. Note that

$$\|A-D\| \leq \sqrt{\|A^{su}\|^2+\|A^{us}\|^2} < 2\varepsilon$$

Write: $\quad I-A=(I-D)+(D-A)=$

$$=(I-D)\left(I-(I-D)^{-1}(A-D)\right)$$

If $\|(I-D)^{-1}(A-D)\|<1/2$ (which happens as soon as $\frac{2\varepsilon}{1-\mu}<1/2$, or equiv. $\varepsilon<\frac{1-\mu}{4}$),

then we can invert:

$$(I-A)^{-1}=\left(I-(I-D)^{-1}(A-D)\right)^{-1}(I-D)^{-1}$$

$$=\sum_{n=0}^{\infty}\left((I-D)^{-1}(A-D)\right)^n(I-D)^{-1}.$$

The norm of the inverse is less than

$$\sum_{n=0}^{\infty}\frac{1}{2^n}(I-D)^{-1} \leq \frac{2}{1-\mu}.$$

$\square$