Lecture 1: Dynamical Systems and Their Orbits

The Basic Problem of Dynamical Systems: Given a map $T: X \to X$ and a point $x \in X$ ("initial condition"), describe the long term behavior of the forward orbit of x

$$(x, T(x), T^2(x), T^3(x), \dots),$$

$T^0 = id$

$T^n = \underbrace{T \circ \cdots \circ T}_{n}$.

Motivating Example: Equation of Motion ($F = ma$)

$$\begin{cases} m \dfrac{d^2\vec{x}}{dt^2} = \vec{F}(\vec{x}) \\ \vec{x}(0) = \vec{x}_0, \quad \vec{x}'(0) = \vec{y}_0 \end{cases}$$

Bring to 1^{st}-order form by introducing $y = \dot{x}$:

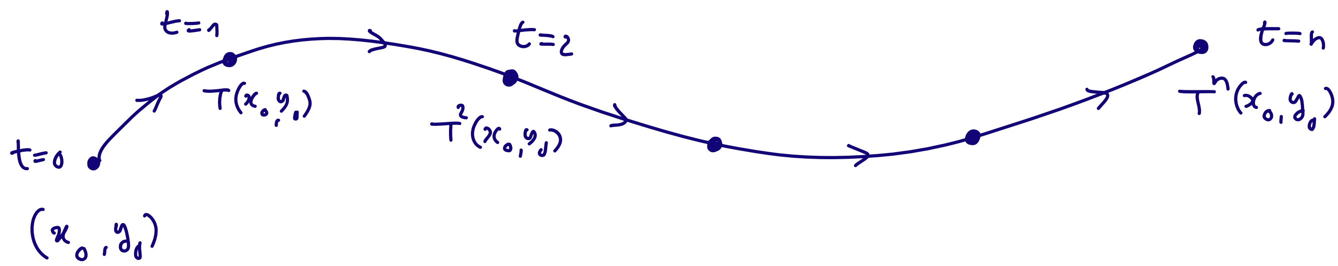
$$(*) \quad \begin{cases} \dfrac{d\vec{x}}{dt} = y \\ \dfrac{d\vec{y}}{dt} = \vec{F}(\vec{x})/m \end{cases} ; \quad \begin{pmatrix} \vec{x}(0) \\ \vec{y}(0) \end{pmatrix} = \begin{pmatrix} \vec{x}_0 \\ \vec{y}_0 \end{pmatrix}$$

Existence & Uniqueness Thm: Suppose $\vec{F}(\cdot)$ is continuously differentiable and $\|\vec{F}(\vec{x})\| \leq$ const, then $(*)$ has a unique solution defined for all $t \in \mathbb{R}$.

Let

- X = set of all possible pairs (x_0, y_0)
- $T: X \to X$ the map

$$T((x_0, y_0)) = \left(\begin{matrix} (x(1), y(1)) \text{ for the unique solution } (x(t), y(t)) \\ \text{of } (*) \text{ with initial condition } (x_0, y_0) \end{matrix} \right)$$



Then $T^n(x_0,y_0) = \left(\begin{array}{l}\text{state of the} \\ \text{system at time } n\end{array}\right)$.

We'd like to know what happens as $n \to \infty$.

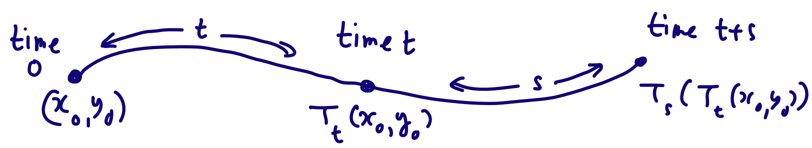
<u>Variations on the Basic Problem</u>: A <u>semi-group action</u> is a collection of maps $T_g : X \to X$ $(g \in G)$, where G is a semi-group, s.t. $T_g \circ T_h = T_{gh}$.

The orbit of $x \in X$ is $\langle T_g(x) : g \in G \rangle$.

(1) <u>"Map" $(G = \mathbb{N}_0)$</u>: $T_n = T \circ \cdots \circ T$ for some function $T : X \to X$

(2) <u>"Flow" $(G = \mathbb{R}^+, \mathbb{R})$</u>: $T_t(x)$, $T_{t+s} = T_t \circ T_s$

E.g. $T_t(x_0,y_0) = (x(t),y(t))$ where $(x(t),y(t))$ solves $(*)$



(3) <u>"$\mathbb{Z}^d$-action"</u>: $T_1, \dots, T_d : X \to X$ invertible and commuting $(T_i \circ T_j = T_j \circ T_i)$, and

$$T_{(n_1,\dots,n_d)} = T_1^{n_1} \circ T_2^{n_2} \circ \cdots \circ T_d^{n_d} \qquad (\underline{n} \in \mathbb{Z}^d)$$

etc.

The Main Difficulty in Dynamical Systems: Calculating orbits is often intractable:

(1) Often, there's no explicit formula for T (e.g. when it's a solution of an ODE)

(2) Even if a formula exists, the formula for T^n can "explode". Example:

- $T(z) = z^2+1$ (deg $= 2$)
- $T^2(z) = (z^2+1)^2 + 1$ (deg $= 4$)
- $T^3(z) = ((z^2+1)^2+1)^2+1$ (deg $= 8$)

....

- $T^n(z)$ = polynomial of degree 2^n

(3) Numerical Instabilities: Often, to predict $T^n(z)$ up to precision ε, must know z up to precision $e^{-\alpha n}\varepsilon$. This is not realistic, even for small n.

Example: $T^n(z) = z^{2^n}$ + lower order terms

Modern Approach to Dynamical Systems: Soft methods* for describing the orbits of "many" or "some" initial conditions, but without knowing the explicit coordinates of the initial conditions we talk about.

* topology, functional analysis, probability theory etc.

Behavior of Orbits

Setup: Let $T: X \to X$ be a continuous map of a metric space (X, d), possibly non-invertible.

(1) The forward orbit closure of x is $\overline{\{x, T(x), T^2(x), \ldots\}}$

(2) The ω-limit set of x is

$$\omega(x) := \{ y \in X : \exists\, n_k \to \infty \ (T^{n_k}(x) \to y) \}$$

(3) In invertible cases we also define

- The (full) orbit closure $\overline{\{ T^n(x) : n \in \mathbb{Z} \}}$
- The α-limit set

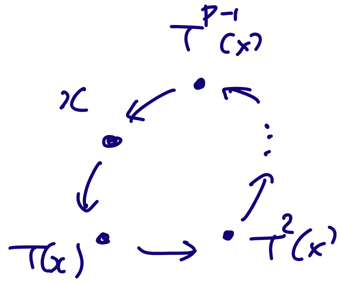
$$\alpha(x) = \{ y \in X : \exists\, n_k \to \infty \ (T^{-n_k}(x) \to y) \}$$

("I am the alpha and omega, the beginning and end")

The Basic Idea: $\begin{pmatrix} \text{large orbit closure} \\ \text{or } \omega\text{-limit set} \end{pmatrix} \longleftrightarrow \begin{pmatrix} \text{complicated} \\ \text{behavior} \end{pmatrix}$

Simple Behavior:

- fixed point: $T(x) = x$.



- periodic point: $T^p(x) = x$

- forward asymptotic to fixed/periodic point

$$d(T^n(x), T^n(y)) \xrightarrow[n\to\infty]{} 0, \quad T^p(y) = y$$

Example: $T: \mathbb{C} \to \mathbb{C}$; $T(z) = z^2$. $T^n(z) = z^{2^n}$

- Fixed Points: $z^2 = z$ $\quad z = 0, 1$

- Periodic Points: $z^{2^p} = z$ (many solutions, all at zero or on the unit circle)

- $|z| < 1$: $T^n(z) \longrightarrow 0$

  Thus $\omega(z) = \{0\}$ for all $|z| < 1$

- $|z| > 1$: $T^n(z) \longrightarrow \infty$, $\omega(z) = \emptyset$

But we'll soon see that some z on the unit circle have much more complicated behavior.

Orbits with Complicated Behavior: For our example, all such orbits must lie on $S^1 = \{z \in \mathbb{C} : |z| = 1\}$.

Symbolic Dynamics: A "change of coordinates" which simplifies the dynamics.

In our example, it's convenient to represent $z \in \mathbb{C}$, $|z|=1$ by the binary expansion $(\omega_1, \omega_2, \omega_3, \dots) \in \{0,1\}^{\mathbb{N}}$ of $\frac{1}{2\pi} \operatorname{Arg}(z)$:

$$z = \pi(\omega_1, \omega_2, \omega_3, \dots) := \exp\left(2\pi i \sum_{i=1}^{\infty} \omega_i / 2^i\right) \qquad \frac{1}{2\pi}\operatorname{Arg}(z)$$

In these coordinates, T acts by the left shift map

$$\sigma(\omega_1, \omega_2, \omega_3, \dots) = (\omega_2, \omega_3, \omega_4, \dots), \quad \text{because}$$

$$T(\pi(\underline{\omega})) = \left[\exp\left(2\pi i \sum_{i=1}^{\infty} \frac{\omega_i}{2^i}\right)\right]^2 = \exp\left(2\pi i \cdot 2 \sum_{i=1}^{\infty} \frac{\omega_i}{2^i}\right)$$

$$= \exp\left(2\pi i \left(\underset{\underset{\mathbb{Z}}{\cap}}{\omega_1} + \sum_{i=1}^{\infty} \frac{\omega_{i+1}}{2^i}\right)\right) = \exp\left(2\pi i \sum_{i=1}^{\infty} \frac{\omega_{i+1}}{2^i}\right) = \pi(\sigma(\underline{\omega}))$$

Thus $T^n(\pi(\underline{\omega})) = \pi(\sigma^n(\underline{\omega}))$. The gain: $\sigma^n(\underline{\omega})$ is easy to find, $\sigma^n(\underline{\omega}) = (\omega_{n+1}, \omega_{n+2}, \dots)$.

In summary: Let

- $\Sigma^+ = \{(\omega_1, \omega_2, \omega_3, \dots) : \omega_i = 0 \text{ or } 1\}$ with the metric
  $d(\underline{\omega}, \underline{\omega}') = \exp(-\min\{n : \omega_n \neq \omega_n'\})$ or zero if $\underline{\omega} = \underline{\omega}'$
- $\pi : \Sigma^+ \to \{z : |z| = 1\}$, $\pi(\underline{\omega}) = \exp\left[2\pi i \sum_{n=0}^{\infty} \frac{\omega_{n+1}}{2^n}\right)$
  Note that π is continuous.
- $\sigma : \Sigma^+ \to \Sigma^+$ the left shift map: $\sigma(\omega_0, \omega_1, \dots) = (\omega_1, \omega_2, \dots)$

Then the diagram

$$\begin{tikzcd}
\Sigma^+ \arrow[r, "\sigma"] \arrow[d, "\pi"'] & \Sigma^+ \arrow[d, "\pi"] \\
S^1 \arrow[r, "z^2"'] & S^1
\end{tikzcd}$$

commutes.

(1) "Anything Is Possible:" Suppose we divide

$$S^1 = A \cup B, \quad \begin{array}{l} A = \{z \in S^1 : \operatorname{Arg}(z) \in [0, \pi)\} \\ B = \{z \in S^1 : \operatorname{Arg}(z) \in [\pi, 2\pi)\} \end{array}$$

Given $z \in S^1$, let's mark the arc visited by $T^n(z)$:

$$A, A, B, A, B, B, \ldots$$

Is there any regularity whatsoever in the sequence we get? No! $\longrightarrow$ "unpredictable behavior"

Fact. Any sequence of A's and B's which doesn't terminate in A^∞ or B^∞ is realized by some z.

Proof. Take any sequence $\underline{\omega} \in \Sigma^+$ which does not end with A^∞ or B^∞. Let $z = \pi(\underline{\omega})$. Then

$$T^n(z) = \pi(\sigma^n(\underline{\omega})) = \pi(\omega_{n+1}, \omega_{n+2}, \ldots) \in \begin{cases} A & \omega_{n+1} = 0 \\ B & \omega_{n+1} = 1 \end{cases}.$$

Exercise: Which sequences ending in A^∞ or B^∞ are realized?

(2) <u>Construction of z with ω-limit set S^1</u>:

Let $\underline{w}_1, \underline{w}_2, \underline{w}_3, \ldots$ be a list of all finite words with letters $0, 1$:

$$0,\ 1,\ 00,\ 01,\ 10,\ 11,\ 000,\ 001,\ \ldots\ldots$$

Let $\underline{\omega}^0 = (\underline{w}_1, \underline{w}_2, \underline{w}_3, \ldots)$.

<u>Lemma</u>: $\{\sigma^n(\underline{\omega}^0) : n \geq 0\}$ is dense in Σ^+:

<u>Proof</u>: Suppose $\underline{x} = (x_0, x_1, \ldots) \in \Sigma^+$. Take n_k s.t. $\underline{w}_{n_k} = (x_0, \ldots, x_k)$. By construction, $\exists m_k$ s.t.

$$\sigma^{m_k}(\underline{\omega}^0) = (\underline{w}_{n_k}, \underline{w}_{n_k+1}, \ldots)$$

Then $d(\sigma^{m_k}(\underline{\omega}^0), \underline{x}) \leq \frac{1}{2^k} \longrightarrow 0$.

<u>Corollary</u>: Let $z_0 := \pi(\underline{\omega}^0)$. Then $\{T^n(z_0) : n \geq 0\}$ is dense in S^1.

<u>Proof</u>. For each $z \in S^1$, write $z = e^{2\pi i \alpha}$ and let $\underline{x}$ = binary expansion of α.

Choose $m_k \to \infty$ s.t. $\sigma^{m_k}(\underline{\omega}^0) \longrightarrow \underline{x}$ in Σ^+.
Since π is continuous, $\pi(\sigma^{m_k}(\underline{\omega}^0)) \longrightarrow \pi(\underline{x}) = z$.
By the commutation relation $\pi \circ \sigma^{m_k} = T^{m_k} \circ \pi$,

$$T^{m_k}(z) = T^{m_k}(\pi(\underline{\omega}^0)) = \pi(\sigma^{m_k}(\underline{\omega}^0))$$

Since $\sigma^{m_k}(\underline{\omega}^0) \to \underline{x}$ and π is continuous,

$$T^{m_k}(z) \longrightarrow \pi(\underline{x}) \equiv z. \qquad \square$$

(3) Orbits with Fractal ω-limit sets

: Let $\underline{w}_1, \underline{w}_2, \underline{w}_3, \ldots$ be a list of concatenations of 00 and 11:

00, 11, 0000, 0011, 1100, 1111, ... Let $\underline{\omega}^0 = (\underline{w}_1, \underline{w}_2, \underline{w}_3, \ldots)$

The ω-limit set of $\underline{\omega}^0$ in Σ^+ is $F = A \cup \sigma(A)$

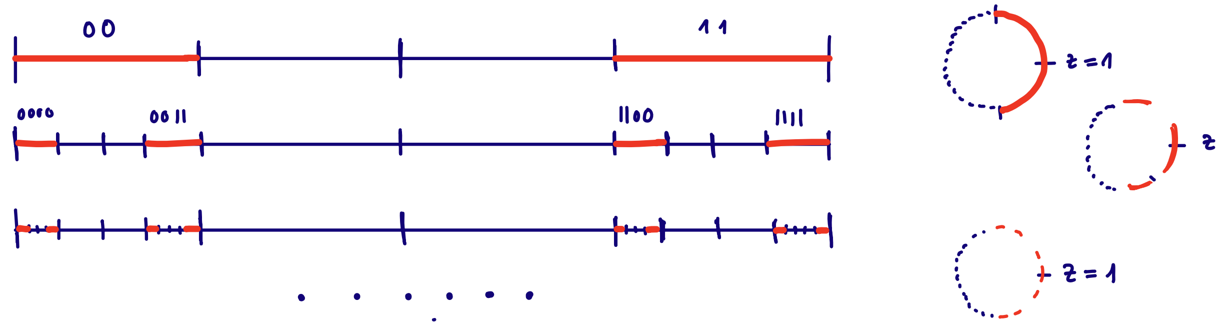
where $A =$ all infinite concatenations of 01, 11

The ω-limit set of $z_0 := \pi(\underline{\omega}^0)$ is $\boxed{\omega(z_0) = \pi(F)}$ (exercise

Observe that $F = A \cup \{(0\,\underline{y}) : \underline{y} \in A\} \cup \{(1\,\underline{y}) : \underline{y} \in A\}$.

Each of these sets is a Cantor set:

$\pi(A)$



Topological Dynamics

The previous examples show that the behavior of orbits of specific initial conditions is often intractable

"Modern Approach to Dynamical Systems": Use soft methods to understand the behavior of "many" or "some" initial conditions, at the following price – no explicit info on these initial conditions.

We'll see two examples:

- existence of recurrent orbits (today)
- existence of dense orbits
- applications to combinatorics

} future

Existence of Recurrent Orbits

Def$^{\underline{n}}$. Let $T: X \to X$ be a continuous map on a metric space. A point $x \in X$ is called recurrent if $\exists\, n_k \to \infty$ s.t. $T^{n_k}(x) \xrightarrow[k\to\infty]{} x$.

We will show:

Birkhoff's Thm: Every continuous map on a compact metric space has at least one recurrent orbit.

(But the proof will not tell us how to find it.)

Preparations to the Proof:

Def$^{\underline{n}}$. A minimal set is a closed nonempty set F s.t. $T(F) \subseteq F$, and so that there is no $G \subsetneq F$ closed non-empty s.t. $T(G) \subseteq G$.

Exercise: Suppose F is a minimal set of a continuous map $T: X \to X$ on a metric space X. Then

(a) $\forall x \in F$, $\overline{\{T^n(x) : n \geq 1\}} = F$; Therefore

(b) every $x \in F$ is recurrent.

Prop$^{\underline{n}}$. Any continuous map T on a compact metric space X admits at least one minimal set

Proof. The proof uses Zorn's Lemma (see below).

Let $\mathcal{F} := \left\{ A \subseteq X : \begin{array}{l} A \text{ is closed, non-empty} \\ \text{and } T(A) \subseteq A \end{array} \right\}$.

$\mathcal{F}$ is not empty, because $X \in \mathcal{F}$.

Put a partial order $\preccurlyeq$ on $\mathcal{F}$ by declaring

$$A \preccurlyeq B \iff A \supseteq B.$$

This is a partial order: It's reflexive ($A \preccurlyeq A$); antisymmetric ($A \preccurlyeq B$ and $B \preccurlyeq A \Rightarrow A = B$); and transitive ($A \preccurlyeq B, B \preccurlyeq C \Rightarrow A \preccurlyeq C$).

But it's not a total order: some A, B are not comparable.

Terminology:

- A chain is a collection of sets $\mathcal{C} \subseteq \mathcal{F}$ s.t. $\forall C_1, C_2 \in \mathcal{C}$, either $C_1 \leq C_2$ or $C_2 \leq C_1$.
- An upper bound of a chain $\mathcal{C}$ is an element B s.t. $B \geq A$ for all $A \in \mathcal{C}$.

Claim: In $(\mathcal{F}, \leq)$, every chain has at least one upper bound.

Proof. Suppose $\mathcal{C}$ is a chain and let

$$B := \bigcap_{A \in \mathcal{C}} A = \{x : x \in A \text{ for each } A \in \mathcal{C}\}.$$

Clearly $B \geq A$ (i.e. $B \subseteq A$) for all $A \in \mathcal{C}$. To see that B is an upper bound, we just need to see that $B \in \mathcal{F}$.

(1) B is closed (any intersection of closed sets is closed)

(2) $T(B) \subseteq B$: $T(B) \subseteq \bigcap_{A \in \mathcal{C}} T(A) \underset{A \in \mathcal{F}}{\subseteq} \bigcap_{A \in \mathcal{C}} A = B$

(3) $B \neq \emptyset$: Proof below.

To show that $\mathcal{B} \neq \phi$, we first show that $\mathcal{C}$ has the finite intersection property (FIP):

$$*) \quad \forall A_1, \dots, A_n \in \mathcal{C} \quad \left(\bigcap_{i=1}^{n} A_i \neq \phi \right).$$

Indeed, since $\mathcal{C}$ is a chain one can show by induction that $\exists$ permutation of $(1, \dots, n)$ s.t.

$$A_{i_1} \lesssim A_{i_2} \lesssim \cdots \lesssim A_{i_n}$$

and so $A_{i_1} \supseteq \cdots \supseteq A_{i_n}$, whence $\bigcap_{i=1}^{n} A_i = A_{i_n} \neq \emptyset$.

Cantor's Theorem: Any family of closed subsets of a compact space with FIP has non-empty intersection.*

It follows that $\mathcal{B} = \bigcap_{A \in \mathcal{C}} A \neq \emptyset$. So $\mathcal{B} \in \mathcal{F}$

Clearly, this is an upper bound for $\mathcal{C}$.

The claim is proved.

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* Proof: Assume by contradiction that $\bigcap_{A \in \mathcal{C}} A = \emptyset$. Then $\bigcup_{A \in \mathcal{C}} A^c = X$, so $\{ A^c : A \in \mathcal{C} \}$ is an open cover of X.

By compactness, $\exists$ finite sub cover $\bigcup_{i=1}^{n} A_i^c = X$. But this implies that $\bigcap_{i=1}^{n} A_i = \phi$, in contradiction to FIP.

Having proved the claim, we now invoke

Zorn's Lemma: Suppose $(\mathcal{F}, \leq)$ is a partially ordered set with the property that any chain has an upper bound. Then $\mathcal{F}$ contains at least one **maximal element** (i.e. $M \in \mathcal{F}$ s.t. there are no $A \in \mathcal{F}$ s.t. $A \geq M$, $A \neq M$.)

If M is such a "maximal element", then M is a minimal set, because

$$\left.\begin{array}{l} \bullet\ M \neq \emptyset \\ \bullet\ \overline{M} = M \\ \bullet\ T(M) \subseteq M \end{array}\right\} \quad \text{because } M \in \mathcal{F}$$

- No $M' \subsetneq M$ with these properties (by maximality). $\square$

Proof of Birkhoff's Thm: By the propⁿ, $T: X \to X$ has a minimal set $M \neq \emptyset$. By the exercise, $\forall x \in M$

- $\overline{\{T^n(x) : n \geq 1\}} = M$, and therefore
- $\forall x \in M\ \exists n_k \to \infty$ s.t. $T^{n_k}(x) \xrightarrow[k\to\infty]{} x$.

So every $x \in M$ is recurrent. $\square$

Exercises for Lecture 1

(not for submission)

(1) Consider the map $T(z) = z^N$ on $S^1 = \{z : |z| = 1\}$, where $N \geq 2$ is an integer. Show that there exists z_0 with orbit closure S^1.

(2) Give an example of a continuous map on a (non-compact) metric space without any recurrent orbits.

(3) Arnol'ds "Cat Map" is the map

$$T : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$$

$$T\left[\begin{pmatrix} x \\ y \end{pmatrix} + \mathbb{Z}^2\right] = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \mathbb{Z}^2$$

(a) Show that T is well defined.

(b) Show that T is invertible

(c) Show that every every $\begin{pmatrix} x \\ y \end{pmatrix} + \mathbb{Z}^2$ with $x, y \in \mathbb{Q}$ is T-periodic

(d) Show that to know $T^N\left[\begin{pmatrix} x \\ y \end{pmatrix} + \mathbb{Z}^2\right]$ up to precision ϵ, we must know $\begin{pmatrix} x \\ y \end{pmatrix} + \mathbb{Z}^2$ up to precision $\epsilon e^{-\lambda n}$. What is λ?