Lecture 5: Invariant Measures

Overview To develop a theory of dynamical systems $T: X \to X$, it is (very) useful to assume they preserve some structure on X:

- **Topological Dynamics**: Continuous maps T on metric spaces X. The invariant structure is the topology ($\varphi: X \to \mathbb{R}$ is continuous $\Rightarrow \varphi \circ T: X \to \mathbb{R}$ is continuous)

- **Differentiable (Smooth) Dynamics**: Differentiable maps T on manifolds M. The invariant structure is the differentiable structure ($\varphi: M \to \mathbb{R}$ diff $\Rightarrow \varphi \circ T$ diff)

- **Measure Theoretic Dynamics** ("Ergodic Theory"): Measure preserving maps T on measure spaces $(X, \mathcal{F}, \mu)$. (φ measurable $\Rightarrow \varphi \circ T$ measurable, $\int \varphi \circ T \, d\mu = \int \varphi \, d\mu$).

The next two lectures are about measure theoretic dynamics. Then we'll move to differentiable dynamics.

Review of Measure Theory

Intuitively, a "measure" μ on a space X is a consistent way of assigning

- a measure $\mu(E)$ to (some) $E \subseteq X$
- an integral $\int_X f \, d\mu$ to (some) $f: X \to \mathbb{R}$

The main difficulty is that we can't do this for all E, f.

σ-Algebras: A σ-algebra on X is a collection $\mathcal{F}$ of subsets of X s.t.

(a) $\emptyset, X \in \mathcal{F}$

(b) $E \in \mathcal{F} \implies E^c := X \setminus E := \{x \in X : x \notin E\} \in \mathcal{F}$

(c) $E_1, E_2, E_3, \ldots \in \mathcal{F} \implies$

$$\bigcup_{i=1}^{\infty} E_i := \{x \in E : \exists i \in \mathbb{N} \;\; x \in E_i\} \in \mathcal{F}$$

$$\bigcap_{i=1}^{\infty} E_i := \{x \in E : \forall i \in \mathbb{N} \;\; x \in E_i\} \in \mathcal{F}$$

The elements of $\mathcal{F}$ are called $\mathcal{F}$-measurable sets.

Important: We only impose (c) for countable collections of sets. Nothing is asserted for uncountable unions/intersect. Such as $\bigcup_{t \in [0,1]} E_t$, $\bigcap_{t \in [0,1]} E_t$.

The Borel σ-algebra: Suppose (X, d) is a metric space. Borel's σ-algebra is the smallest σ-algebra which contains all open sets:

$$\mathcal{B} := \mathcal{B}(X) := \bigcap \left\{ \mathcal{F} : \begin{array}{l} \mathcal{F} \text{ is a } \sigma\text{-algebra which} \\ \text{contains all open sets} \end{array} \right\}$$

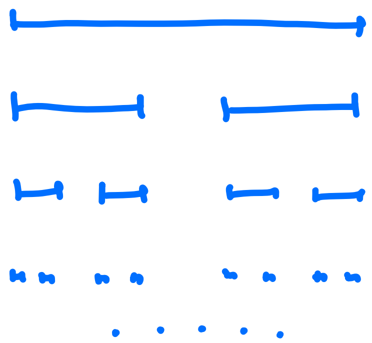
There is no (useful) explicit description of Borel sets. But it's usually easy to check Borel measurability for explicit sets

Example

- $\mathbb{Q} \in \mathcal{B}(\mathbb{R})$:

$$\mathbb{Q} = \bigcup_{n\in\mathbb{Z}} \bigcup_{m\in\mathbb{Z}\setminus\{0\}} \left\{\frac{n}{m}\right\} = \underbrace{\bigcup_{n\in\mathbb{Z}} \bigcup_{m\in\mathbb{Z}\setminus\{0\}}}_{\text{countable unions}} \overbrace{\big(\underbrace{\mathbb{R}\setminus\{\tfrac{m}{n}\}}_{\text{open}}\big)^c}^{\text{complement}}$$

- <u>Cantor Set</u>: countable intersection of finite unions of closed intervals.



- Every closed interval is Borel because it's the complement of an open set

- Finite unions of Borel sets are Borel, because

$$\bigcup_{i=1}^{N} E_i = \bigcup_{i=1}^{\infty} E_i', \quad E_i' = \begin{cases} E_i & i \le N \\ \emptyset & i > N. \end{cases}$$

Measurable Functions: $f: X \to \mathbb{R}$ is <u>$\mathcal{F}$-measurable</u> if for all $E \in \mathcal{B}(\mathbb{R})$, $f^{-1}(E) := \{x \in X : f(x) \in E\} \in \mathcal{F}$.

<u>Exercise</u>: It's sufficient to check that $[f>t] := \{x \in X : f(x) > t\}$ are measurable for all $t \in \mathbb{R}$.

Measures & Measure Spaces: These are triplets $(X, \mathcal{F}, \mu)$ where X is a set, $\mathcal{F}$ is a σ-algebra on X and μ, called a **measure**, is a function $\mu : \mathcal{F} \to [0, \infty]$ s.t.

(a) $\mu(\emptyset) = 0$

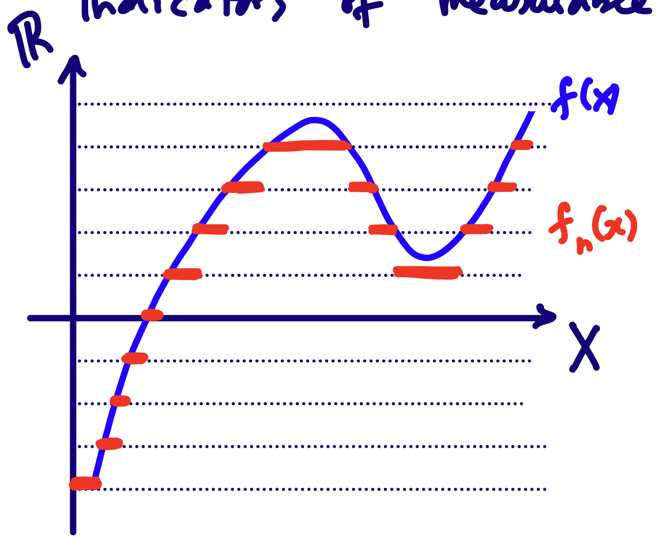
(b) σ-additivity: If $E_1, E_2, E_3, \ldots \in \mathcal{F}$ are pairwise disjoint then $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$.

- If $\mu(X) = 1$, μ is called a **probability measure** and $(X, \mathcal{F}, \mu)$ is called a **probability space**.
- A measure on a metric space equipped with its Borel σ-algebra is called a **Borel measure**.
- The existence of such objects is not trivial, and is the subject of measure theory.

The Integral: Suppose $\mu(X) = 1$. We wish to assign a number $\int_X f \, d\mu$ to each $f : X \to \mathbb{R}$ bounded and measurable, so that

- $f \geq 0 \Rightarrow \int_X f \, d\mu \geq 0$
- $\int_X (\alpha f + \beta g) \, d\mu = \alpha \int_X f \, d\mu + \beta \int_X g \, d\mu \qquad (\alpha, \beta \in \mathbb{R})$
- $\left| \int_X f \, d\mu \right| \leq \int_X |f| \, d\mu$
- $\int_X 1_E \, d\mu = \mu(E) \qquad \left(1_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases} \right)$
- $|f_n| \leq \text{const}, \; f_n(x) \xrightarrow[n \to \infty]{} f(x) \Rightarrow \lim_{n \to \infty} \int_X f_n \, d\mu = \int_X \lim_{n \to \infty} f_n \, d\mu$

<u>Sketch of Construction</u>: Suppose $|f(x)| \leq M$ for all x. Fix n large. We approximate f <u>uniformly</u> up to precision $1/n$ by finite linear combinations of indicators of measurable sets:



$$f_n(x) := \sum_{k=-\lceil Mn \rceil}^{\lceil Mn \rceil} \frac{k}{n} \mathbb{1}_{\left[\frac{k}{n} \leq f < \frac{k+1}{n}\right]}(x)$$

$$\mathbb{1}_{\left[\frac{k}{n} \leq f < \frac{k+1}{n}\right]}(x) = \begin{cases} 1 & \frac{k}{n} \leq f < \frac{k+1}{n} \\ 0 & \text{else} \end{cases}$$

Now we define

$$\int_X f_n \, d\mu := \sum_{k=-\lceil Mn \rceil}^{\lceil Mn \rceil} \frac{k}{n} \mu\left\{x : \frac{k}{n} \leq f(x) < \frac{k+1}{n}\right\}$$

$$\int_X f \, d\mu := \lim_{n \to \infty} \int_X f_n \, d\mu$$

$$:= \lim_{n \to \infty} \sum_{k=-Mn}^{Mn} \frac{k}{n} \mu\left\{x : \frac{k}{n} \leq f(x) < \frac{k+1}{n}\right\}$$

The existence of the limits, the independence of all non-canonical choices (e.g. of M), and the properties of the integral are proved in measure theory textbooks (e.g. Royden: Real Analysis)

Prop$^{\underline{n}}$. Suppose T is a continuous map on a compact metric space (X,d), and μ is a Borel probability measure. Then $\mu\circ T^{-1}$ is a Borel probability measure, and

$$\int_X f \, d\mu\circ T^{-1} = \int_X (f\circ T)\, d\mu \quad \text{for any bounded measurable } f: X\to\mathbb{R}.$$

Proof. First we must show that $E \mapsto (\mu\circ T^{-1})(E)$ is well-defined.

(1) $\underline{T \text{ is Measurable}}$, i.e. $\forall E\in\mathcal{B}(X)$, $T^{-1}E := \{x : T(x)\in E\}$ is in $\mathcal{B}(X)$. Proof: Let

$$\mathcal{G} := \{E\subseteq X : T^{-1}(E)\in\mathcal{B}(X)\}$$

- $\underline{\text{Every open set is in } \mathcal{G}}$, because by continuity if E is open, then $T^{-1}(E)$ is open, whence in $\mathcal{B}(X)$.
- $\underline{E\in\mathcal{G} \implies E^c\in\mathcal{G}}$: If $E\in\mathcal{G}$, then $T^{-1}E\in\mathcal{B}$, so
$$T^{-1}(E^c) = \{x : T(x)\notin E\} = \{x : T(x)\in E\}^c$$
$$= (\underbrace{T^{-1}E}_{\in\mathcal{B}})^c \in\mathcal{B} \quad (\because \mathcal{B} \text{ is a } \sigma\text{-algebra})$$
- $\underline{E_1, E_2, E_3, \ldots \in\mathcal{G} \implies \bigcup_{n=1}^\infty E_n, \bigcap_{n=1}^\infty E_n \in\mathcal{G}}$:
$$T^{-1}\Big(\bigcup_{n=1}^\infty E_n\Big) = \{x : \exists n\ T(x)\in E_n\} = \bigcup_{n=1}^\infty T^{-1}E_n \in\mathcal{B}(X)$$
$$T^{-1}\Big(\bigcap_{n=1}^\infty E_n\Big) = \{x : \forall n\ T(x)\in E_n\} = \bigcap_{n=1}^\infty T^{-1}E_n \in\mathcal{B}(X)$$

We see that $\mathcal{G}$ is a σ-algebra which contains the open sets; Necessarily $\mathcal{G} \supseteq \mathcal{B}(X)$. It follows that $\forall E \in \mathcal{B}(X)$, $T^{-1}E \in \mathcal{B}(X)$, and $\mu(T^{-1}E)$ is well defined.

(2) <u>$\mu \circ T^{-1}$ is σ-additive</u>: Given a <u>countable</u> <u>pairwise disjoint</u> family of measurable sets $E_1, E_2, \ldots$, $T^{-1}(\bigcup_{n=1}^{\infty} E_n) = \bigcup_{n=1}^{\infty} T^{-1}E_n$, and $T^{-1}E_n$ are pairwise disjoint (check). So

$$(\mu \circ T^{-1})(\bigcup_{n=1}^{\infty} E_n) = \mu\left[T^{-1}(\bigcup_{n=1}^{\infty} E_n)\right] = \mu(\bigcup_{n=1}^{\infty} T^{-1}E_n)$$

because $T^{-1}E_n$ are pairwise disjoint and μ is a measure

$$= \sum_{n=1}^{\infty} \mu(T^{-1}E_n) \equiv \sum_{n=1}^{\infty} (\mu \circ T^{-1})(E_n).$$

(3) <u>$\forall f : X \to \mathbb{R}$ bounded and measurable</u>, $\int_X f \, d\mu \circ T^{-1} = \int_X f \circ T \, d\mu$.

Fix some M s.t. $|f(x)| \leq M$ for all x.

Then

$$\int_X f \, d\mu \circ T^{-1} = \lim_{n \to \infty} \sum_{k=-\lfloor Mn \rfloor}^{\lceil Mn \rceil} \frac{k}{n} (\mu \circ T^{-1}) \{x : \frac{k}{n} \leq f(x) < \frac{k+1}{n}\}$$

$$= \lim_{n \to \infty} \sum_{k=-\lfloor Mn \rfloor}^{\lceil Mn \rceil} \frac{k}{n} \mu(T^{-1}\{x : \frac{k}{n} \leq f(x) < \frac{k+1}{n}\})$$

$$= \lim_{n \to \infty} \sum_{k=-\lfloor Mn \rfloor}^{\lceil Mn \rceil} \frac{k}{n} \mu\{x : \frac{k}{n} \leq f(Tx) < \frac{k+1}{n}\}$$

$$= \int_X f \circ T \, d\mu. \qquad \square$$

Borel Measures and Functionals

Suppose (X, d) is a compact metric space. Let

- $C(X) := \{ f : X \to \mathbb{R} : f \text{ is continuous} \}$
- $\|f\|_\infty = \sup_{x \in X} |f(x)| \quad (f \in C(X))$

Separability Thm: $C(X)$ is separable, i.e $\exists$ countable collection of $f_n \in C(X)$ s.t. $\forall f \in C(X)$ $\forall \epsilon > 0$, $\exists n$ s.t. $\|f_n - f\|_\infty < \epsilon$

Exercise: Use the Stone–Weierstrass thm to show this.

Def^n. A bounded linear functional on $C(X)$ is a linear function $\varphi : C(X) \to \mathbb{R}$ s.t. for some $M > 0$,

$$|\varphi(f)| \le M \|f\|_\infty \quad \text{for all } f \in C(X)$$

- The smallest possible M is called the norm of φ. Equivalently,

$$\|\varphi\| := \sup \left\{ \frac{|\varphi(f)|}{\|f\|_\infty} : f \in C(X) \setminus \{0\} \right\}$$

- A bounded linear functional φ is called positive, if $\varphi(f) \ge 0$ for all non-negative $f \in C(X)$.

- The space of all bounded linear functionals is denoted by $C(X)^*$, and is called the dual of $C(X)$.

Principal Example: Suppose μ is a Borel probability measure on X. Let

$$\varphi_\mu(f) := \int_X f \, d\mu \qquad (f \in C(X))$$

Then φ_μ is a positive bounded linear functional on X s.t. $\varphi_\mu(1) = 1$. [The integral is well-defined because continuous functions are bounded and Borel measurable.]

Riesz Representation Thm: Suppose X is a compact metric space. Then any positive bounded linear functional φ s.t. $\varphi(1)=1$ equals φ_μ for some unique Borel probability measure.

Example: Every continuous function on $[0,1]$ is Riemann integrable, therefore we can define

$$\varphi(f) = \int_0^1 f(t)\,dt \qquad (f \in C[0,1]).$$

Clearly $\varphi \in C[0,1]^*$ and $\varphi \geq 0$. (What's the norm?)
By the Riesz Representation thm, $\exists$ Borel measure μ on $[0,1]$ s.t. $\int_{[0,1]} f \, d\mu = \int_0^1 f(t)\,dt$ for all f continuous.
This measure is called Lebesgue's Measure (on $[0,1]$).
What did we gain? $\int_X f \, d\mu$ is defined for many non-Riemann integrable functions, for which $\varphi(f)$ and $\int_0^1 f(t)\,dt$ in Riemann's sense are meaningless.

Weak Star Convergence: Suppose X is a compact metric space

Def^{n}. A sequence $\varphi_n \in C(X)^*$ converges weak star to $\varphi \in C(X)$ if $\varphi_n(f) \xrightarrow[n\to\infty]{} \varphi(f)$ for all $f \in C(X)$.

If we view Borel probability measures as functionals, we are led to the following definition:

Def^{n}. A sequence of Borel probability measures μ_n converges weak star to a Borel probability measure μ, if $\int_X f \, d\mu_n \xrightarrow[n\to\infty]{} \int_X f \, d\mu$ for each continuous $f: X \to \mathbb{R}$.

CAUTION: This does not mean that $\mu_n(E) \to \mu(E)$ for sets E, or that $\int_X f \, d\mu_n \to \int_X f \, d\mu$ for discontinuous $f: X \to \mathbb{R}$.

Example: $X = [-1, 1]$, $x_n \in [-1,1]$, $x_n \to 1/2$. Dirac's measure at x is $\delta_x(E) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$. Then

(a) $\delta_{1/n} \longrightarrow \delta_0$ weak star, but

(b) $\delta_{1/n}((0,1]) \not\longrightarrow \delta_0((0,1])$

(c) $\int_{[-1,1]} f \, d\delta_{1/n} \not\longrightarrow \int_{[-1,1]} f \, d\delta_0$ for $f(x) = \mathrm{sgn}(x) = \begin{cases} +1 & x>0 \\ 0 & x=0 \\ -1 & x<0 \end{cases}$.

[**Portmanteau Thm**. In a compact metric space, if $\mu_n \to \mu$ weak star, then $\mu_n(E) \to \mu(E)$ for any measurable set E s.t. $\mu(\partial E) = 0$, where

$$\partial E = \left\{ x : \begin{array}{l} \text{Any open ball centered at } x \\ \text{intersects } E \text{ and } E^c \end{array} \right\}.$$
]

Banach Alaoglu Thm: Suppose X is a compact metric space, then any sequence of $\varphi_n \in C(X)^*$ s.t. $\|\varphi_n\| \le M$ has a subsequence which converges weak star.

Proof. In the setup of the theorem, $C(X)$ is separable, so $\exists\, g_1, g_2, \dots$ which is dense in $C(X)$. Let

$$A := \text{Span}_{\mathbb{Q}}\{g_1, g_2, \dots\} = \left\{ \sum_{i=1}^{n} \alpha_i g_i : n \in \mathbb{N},\ \alpha_i \in \mathbb{Q} \right\}.$$

This set is countable. Enumerate it: $A = \{f_1, f_2, f_3, \dots\}$.

- $|\varphi_n(f_1)| \le \|f_1\|_\infty$, so $\exists\, n_k(1)$ s.t. $\varphi_{n_k(1)}(f_1) \xrightarrow[k\to\infty]{} L_1$.
- $|\varphi_{n_k(1)}(f_2)| \le \|f_2\|_\infty$ so $\exists\, n_k(2)$, a subsequence of $n_k(1)$ s.t. $\varphi_{n_k(2)}(f_2) \longrightarrow L_2$

Continuing in this way we extract subsequence $n_k(\ell+1)$ from $n_k(\ell)$ s.t. $\varphi_{n_k(\ell+1)}(f_{\ell+1}) \xrightarrow[k\to\infty]{} L_{\ell+1}$.

$$\begin{array}{lllllll}
\varphi_{n_1(1)}(f_1), & \varphi_{n_2(1)}(f_1), & \varphi_{n_3(1)}(f_1), & \varphi_{n_4(1)}(f_1), & \cdots\cdots & \longrightarrow & L_1 \\
\varphi_{n_1(2)}(f_1), & \varphi_{n_2(2)}(f_1), & \varphi_{n_3(2)}(f_1), & \varphi_{n_4(2)}(f_1), & \cdots\cdots & \longrightarrow & L_2 \\
\varphi_{n_1(3)}(f_1), & \varphi_{n_2(3)}(f_1), & \varphi_{n_3(3)}(f_1), & \varphi_{n_4(3)}(f_1), & \cdots\cdots & \longrightarrow & L_3 \\
\varphi_{n_1(4)}(f_1), & \varphi_{n_2(4)}(f_1), & \varphi_{n_3(4)}(f_1), & \varphi_{n_4(4)}(f_1), & \cdots\cdots & \longrightarrow & L_4 \\
\vdots & \vdots & \vdots & \vdots & & &
\end{array}$$

The diagonal sequence $\{\varphi_{n_k(k)}\}_{k=1}^{\infty}$ has the following property:

$$\forall \ell \quad \{\varphi_{n_k(k)}(f_\ell)\}_{k=\ell}^{\infty} \text{ is a subsequence of } \{\varphi_{n_k(\ell)}(f_\ell)\}_{k=1}^{\infty}$$

$$\Rightarrow \lim_{k\to\infty} \varphi_{n_k(k)}(f_\ell) = \lim_{k\to\infty} \varphi_{n_k(\ell)}(f_\ell) = L_\ell \text{ for all } \ell.$$

(a) <u>Define</u> $\varphi : A \to \mathbb{R}$ by $\varphi(f_\ell) := L_\ell$. Then

- $|\varphi(f_i)| \le M\|f_i\|_\infty$ for all i ($\because |\varphi_{n_k(k)}(f_i)| \le M\|f_i\|_\infty$)
- $\forall \alpha, \beta \in \mathbb{Q}$, $\varphi(\alpha f_i + \beta f_j) = \alpha\varphi(f_i) + \beta\varphi(f_j)$ ($\because \alpha f_i + \beta f_j \in A$; $\varphi_{n_k(k)}$ are linear on A)

(b) <u>Extend</u> φ to $C(X)$ by defining

$$\varphi(f) := \lim_{j\to\infty} \varphi(f_{n_j}), \text{ whenever } \|f_{n_j} - f\|_\infty \longrightarrow 0$$

- Such a sequence f_{n_j} always exists, because $\{f_n\}$ is dense
- $\lim_{j\to\infty} \varphi(f_{n_j})$ exists, because if $\|f_{n_j} - f\|_\infty \to 0$, then

$$|\varphi(f_{n_j}) - \varphi(f_{n_k})| \le M\|f_{n_j} - f_{n_k}\|_\infty \le$$

$$\le M\left(\|f_{n_j} - f\|_\infty + \|f_{n_k} - f\|_\infty\right) \xrightarrow[j,k\to\infty]{} 0$$

So $\{\varphi(f_{n_j})\}$ is a Cauchy sequence

- $\lim_{j\to\infty}\varphi(f_{n_j}) = \lim_{k\to\infty} \varphi(f_{m_k})$ whenever $f_{n_j}, f_{m_k} \to f$ in $\|\cdot\|_\infty$ (exercise)

Thus the def$^{\underline{n}}$ of $\varphi(f)$ is proper.

We claim that $\varphi \in C(X)^*$ and $\varphi_{n_k(k)} \to \varphi$ weak star:

(1) <u>Boundedness</u>: $|\varphi(f)| \le M\|f\|_\infty$, because if $\|f_{n_k} - f\|_\infty \to 0$ then $\big|\|f_{n_k}\| - \|f\|\big| \le \|f_{n_k} - f\|_\infty \to 0$, so $\|f_{n_k}\|_\infty \to \|f\|_\infty$. So

$$|\varphi(f)| = \left|\lim_{k\to\infty} \varphi(f_{n_k})\right| \le \lim_{k\to\infty} M \cdot \|f_{n_k}\| = M\|f\|_\infty .$$

(2) <u>Linearity</u>: Suppose $\alpha, \beta \in \mathbb{R}$; $f, g \in C(X)$. Choose $\alpha_i, \beta_i \in \mathbb{Q}$, $f_{n_i}, f_{m_i} \in A$ s.t. $\alpha_i \to \alpha$, $\beta_i \to \beta$, $\|f_{n_i} - f\|_\infty \to 0$ and $\|f_{m_i} - f\|_\infty \to 0$.

Clearly $\alpha_i f_{n_i} + \beta_i f_{m_i} \in A$, $\|(\alpha_i f_{n_i} + \beta_i f_{m_i}) - (\alpha f + \beta g)\|_\infty \to 0$.
So by the linearity of φ on A,

$$\varphi(\alpha f + \beta g) = \lim_{i\to\infty} \varphi(\alpha_i f_{n_i} + \beta_i f_{m_i}) = \lim_{i\to\infty} \big(\alpha_i \varphi(f_{n_i}) + \beta_i \varphi(f_{m_i})\big)$$
$$= \lim \alpha_i \lim \varphi(f_{n_i}) + \lim \beta_i \lim \varphi(f_{m_i})$$
$$= \alpha \varphi(f) + \beta \varphi(g).$$

(3) <u>$\varphi_{n_k(k)} \to \varphi$ weak star</u>: Fix $f \in C(X)$ and $\varepsilon > 0$. Choose $f_n \in A$ s.t. $\|f - f_n\|_\infty < \varepsilon/2M$. Then

$$|\varphi(f) - \varphi_{n_k(k)}(f)| \le |\varphi(f) - \varphi(f_n)| + |\varphi(f_n) - \varphi_{n_k(k)}(f_n)| + |\varphi_{n_k(k)}(f_n) - \varphi_{n_k(k)}(f)|$$
$$\le |\varphi(f - f_n)| + o(1) + |\varphi_{n_k(k)}(f_n - f)|$$
$$\le M\|f - f_n\|_\infty + o(1) + M\|f_n - f\|_\infty \le \varepsilon + o(1)$$

Thus, $\limsup_{k\to\infty} |\varphi(f) - \varphi_{n_k(k)}(f)| \le \varepsilon$. Since $\varepsilon > 0$ is arbitrary, the limsup is zero, and $\varphi_{n_k(k)}(f) \to \varphi(f)$. $\square$

Exercises

(1) Let $\mathcal{F}$ be a σ-algebra on X and suppose $f: X \to \mathbb{R}$ is a function s.t. $[f>t] := \{x \in X : f(x) > t\} \in \mathcal{F}$ for all t. Show that f is $\mathcal{F}$-measurable.
(Guidance: $\mathcal{G} := \{E \in \mathcal{B}(\mathbb{R}) : f^{-1}(E) \in \mathcal{F}\}$ is a σ-algebra which contains all open sets.)

(2) Show that any continuous function is Borel measurable

(3) Suppose $f_n : X \to \mathbb{R}$ are $\mathcal{F}$-measurable, bounded. Show:

(a) $f(x) := \sup_n f_n(x)$, $g(x) := \inf_n f_n(x)$ are $\mathcal{F}$-measurable

(b) $f(x) := \limsup_{n\to\infty} f_n(x)$, $g(x) := \liminf_{n\to\infty} f_n(x)$ are $\mathcal{F}$-meas.

Guidance: Start by showing the identities

$$\limsup_{n\to\infty} a_n = \inf_n \left(\sup \{a_k : k \geq n\} \right)$$

$$\liminf_{n\to\infty} a_n = \sup_n \left(\inf \{a_k : k \geq n\} \right)$$

(4) Suppose $(X, \mathcal{F}, \mu)$ is a measure space. Show

(a) $A, B \in \mathcal{F}$, $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$

(b) $A_1, A_2, A_3, \ldots \in \mathcal{F}$ (not necessarily pairwise disjoint)
$\Rightarrow \mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i)$

(Guidance: Start by rewriting $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$ where $B_i \subseteq A_i$ and B_i are pairwise disjoint).

(5) Look up and read the proof of the "Bounded Convergence Theorem"