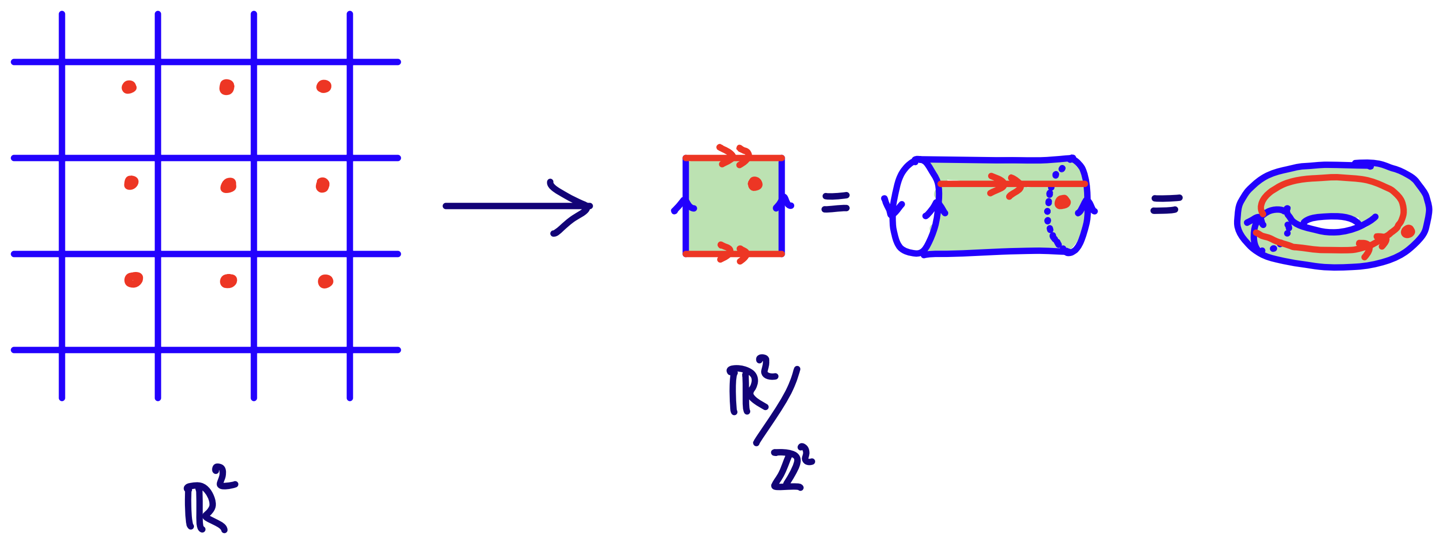
Lecture 6: Arnol'd's Cat Map

<u>Overview</u>: Arnol'd's "cat map" is a classical example of a chaotic dynamical system. We'll

- Define it
- Discuss its <u>symbolic dynamical</u> representation
- Use symbolic dynamics to introduce <u>Ruelle's operator</u>

Arnol'd's Cat Map

<u>The Torus</u>: $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2 = \left\{ \binom{x}{y} + \mathbb{Z}^2 : x, y \in \mathbb{R} \right\}$



<u>The "Cat Map"</u>: $T_A : \mathbb{T}^2 \to \mathbb{T}^2$, $T_A\left[\binom{x}{y} + \mathbb{Z}^2\right] = A\binom{x}{y} + \mathbb{Z}^2$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a matrix s.t.:

(1) $a, b, c, d \in \mathbb{Z}$

(2) $\det(A) = 1$

(3) <u>Hyperbolicity</u>: Two eigenvalues λ^u, λ^s s.t. $|\lambda^u| > 1$, $0 < |\lambda^s| < 1$

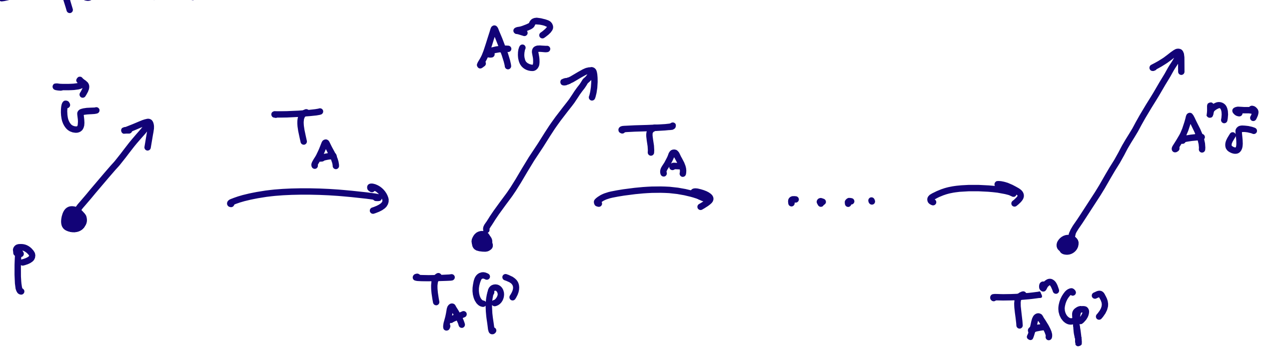
Example: $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

Basic Properties:

- **Well-Defined**: $\binom{x}{y} + \mathbb{Z}^2 = \binom{x'}{y'} + \mathbb{Z}^2 \Rightarrow T_A\left[\binom{x}{y} + \mathbb{Z}^2\right] = T_A\left[\binom{x'}{y'} + \mathbb{Z}^2\right]$
  (because $a, b, c, d \in \mathbb{Z}$)

- **Area preserving**, because $\det(A) = 1$
- **Invertibile**, with inverse $T_{A^{-1}}$
  ($\det A = 1 \Longrightarrow A^{-1}$ is also integer-valued)

- **Two Lyapunov Exponents**: $\chi^u = \underbrace{\log|\lambda^u|}_{\text{pftv}}$, $\chi^s = \underbrace{\log|\lambda^s|}_{\text{ngtv}}$
  with Oseledets decomposition
  $$T_p(\mathbb{T}^2) = \operatorname{Span}\{\vec{v}^u\} \oplus \operatorname{Span}\{\vec{v}^s\}$$
  where $\vec{v}^t$ are the eigenvectors with eigenvalues λ^t ($t = u, s$)
  Indeed, at every point p, the linearization of T_A acts as follows:



$$\|A^n \vec{v}^u\| = |\lambda^u|^n \cdot \|\vec{v}^u\| \sim \exp(n\chi^u), \quad \chi^u > 0$$
$$\|A^n \vec{v}^s\| = |\lambda^s|^n \, \|\vec{v}^s\| \sim \exp(n\chi^s) \quad \chi^s < 0.$$

In particular: We have <u>exponential sensitivity to initial conditions everywhere</u> $\longrightarrow$ unstable numerics!

Symbolic Dynamics

Overview: Symbolic dynamics is a change of coordinates which "simplifies the dynamics":

- space of orbits $\longrightarrow$ space of paths on a finite graph
- periodic orbits $\longrightarrow$ loops on this graph
- $T_A \longrightarrow \sigma =$ left shift map (easy to iterate)
- In our special case, area measure $\longrightarrow$ Markov measure

Naïve Idea (which doesn't work):

(1) Fix a partition $\alpha = \{R_1, \dots, R_s\}$ of $\mathbb{T}^2$, and build the dynamical graph G_α with

- vertices $R_1, \dots, R_s$
- edges $R_i \to R_j$ whenever $T_A(R_i) \cap R_j \neq \emptyset$

(2) The itinerary of $p \in \mathbb{T}^2$ is $(\cdots R_{x_{-1}}, R_{x_0}, R_{x_1}, \cdots)$ s.t. $T^k(p) \in R_{x_k}$ $(k \in \mathbb{Z})$.

- for "good" partitions, the itinerary determines p
- T_A acts on itineraries by the left shift:

$$\text{Itinerary}(p) = \underline{x} \Rightarrow \text{Itinerary}(T_A^k(p)) = \sigma^k(\underline{x})$$

- Every itinerary is a path on the dynamical graph.

(3) Let $\Sigma = \{\text{paths on } G_\alpha\} = \{(\cdots, R_{x_{-1}}, R_{x_0}, R_{x_1} \cdots) : R_{x_i} \to R_{x_{i+1}}\}$

This is a subshift of finite type (see prev. lecture).

The Difficulty: Some paths on G_α may not be itineraries of genuine initial condition, because

$$\begin{pmatrix} R_1 \to R_2 \\ R_2 \to R_3 \\ \vdots \\ R_{n-1} \to R_n \end{pmatrix} \iff \begin{pmatrix} \exists x_1 \in R_1 \text{ s.t. } T_A(x_1) \in R_2 \\ \exists x_2 \in R_2 \text{ s.t. } T_A(x_2) \in R_3 \\ \vdots \\ \exists x_{n-1} \in R_{n-1} \text{ s.t. } T_A(x_{n-1}) \in R_n \end{pmatrix},$$

but we don't know that $\exists x_1 \in R_1$ s.t. $T_A^i(x_1) \in R_{i+1}$ $(i = 1, \ldots, n-1)$.

Markov Partition: A special partition $\{R_1, \ldots, R_s\}$ s.t. every path $(\cdots R_{x_{-1}}, R_{x_0}, R_{x_1}, \cdots)$ on the dynamical graph is the itinerary of some p "up to closures":

$$(*) \quad \exists p \text{ s.t. } \left(T_A^k(p) \in \overline{R}_{x_k} \text{ for all } k\right)$$

Thm (Adler-Weiss): Arnol'd's cat map has a finite Markov partition α. For this partition

(1) every itinerary is a path on the dynamical graph G_α

(2) every path $(\cdots R_{x_{-1}}, R_{x_0}, R_{x_1}, \cdots)$ on G_α determines a unique $p = \pi(\underline{x})$ s.t. $T_A^k(p) \in \overline{R}_{x_k}$ for all $k \in \mathbb{Z}$

(3) $\pi: \Sigma \to \mathbb{T}^2$ is Hölder, finite-to-one, and $\pi \circ \sigma = T_A \circ \pi$.

(4) If μ is the area measure on $\mathbb{T}^2$, then $\mu \circ \pi^{-1}$ is a Markov measure: $\exists$ stoch matrix $(p_{ij})_{d\times d}$ and a prob. vector $(p_j)_\alpha$ s.t.

$$\mu\{p : T_A^k(p) \in R_{a_k} \ (k = 0, \ldots, n-1)\} = p_{a_0} p_{a_0 a_1} \cdots p_{a_{k-1} a_k}$$

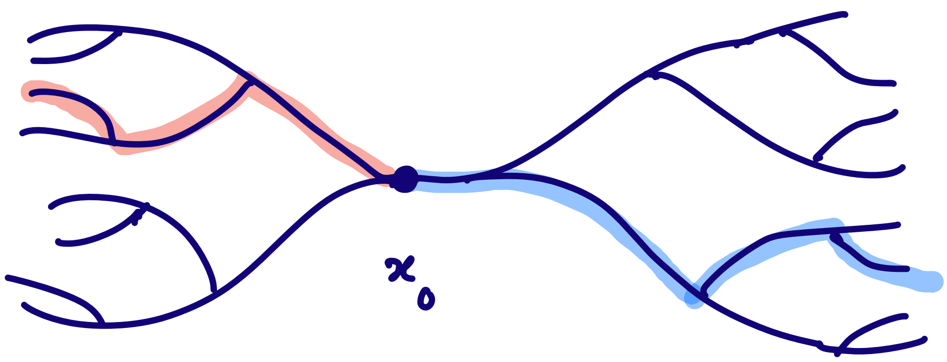
Let $\Sigma = \{$ paths on $G_\alpha \}$. If $\underline{x}, \underline{y} \in \Sigma$ and $x_0 = y_0$,

then $(\underbrace{\cdots x_{-2} x_{-1}}_{\text{past of } \underline{x}}\, x_0\, \underbrace{y_1 y_2 \cdots}_{\text{future of } \underline{y}}) \in \Sigma$

__Corollary 1__: Suppose $T^k(p) \in R_{x_k}$, $T^k(q) \in R_{y_k}$ $(k \in \mathbb{Z})$.

If $R_{x_0} = R_{y_0}$, then $\exists z$ s.t.

- $\text{past}(z) = \text{past}(p) = (\cdots R_{x_{-2}} R_{x_{-1}} R_{x_0})$
- $\text{future}(z) = \text{future}(q) = (R_{y_0}, R_{y_1}, R_{y_2}, \ldots)$



"Given the present, the past and future are combinatorially independent"

__Corollary 2__: Let $B =$ transition matrix of G_α. We saw in the previous lecture that

$$\#\left\{ \begin{matrix} \text{loops of length} \\ n \text{ on } G \end{matrix} \right\} = \operatorname{tr}(B^n) \sim \text{const.}\, \lambda^n$$

where $\lambda =$ Perron-Frobenius eigenvector of B.

If $\sigma^n(\underline{x}) = \underline{x}$, then $T_A^n(\pi(\underline{x})) = \pi(\sigma^n(\underline{x})) = \pi(\underline{x})$

Thus $\#\{p \in \mathbb{T}^2 : T_A^n(p) = p\} \geq \text{const.}\ \lambda^n$.

(in fact, it's $\sim \text{const}\ \lambda^n$).

Ruelle's Operators

Symbolic dynamics allows us to replace $T_A : \mathbb{T}^2 \to \mathbb{T}^2$ by the <u>subshift of finite type</u> $\sigma : \Sigma \to \Sigma$ where

- $\Sigma = \{$two-sided paths on the dynamical graph$\}$
- $\sigma : \Sigma \to \Sigma$ is the <u>left shift map</u>

$$\sigma(\underline{x}) = \underline{y}, \text{ where } y_k = x_{k+1}.$$

<u>One-Sided Functions</u>: $f : \Sigma \to \mathbb{R}$ measurable s.t. $f(\underline{x}) = f(x_0, x_1, x_2, \ldots)$ only depends on x_k where $k \geq 0$.

Let
$$\mathcal{H}^+ := \{f \in L^2(\mu) : f \text{ is one-sided}\}$$

Then <u>Koopman's Operator</u> $U_T f = f \circ T$ preserves $\mathcal{H}^+$:

$$U_T(\mathcal{H}^+) \subseteq \mathcal{H}^+$$

(but $U_T(\mathcal{H}^+) \neq \mathcal{H}^+$, so U_T is <u>not</u> invertible on $\mathcal{H}^+$).

The <u>dual operator</u> $U_T^* : \mathcal{H}^+ \to \mathcal{H}^+$ is the unique operator s.t. $\langle U_T^* f, g \rangle = \langle f, U_T g \rangle = \int f \cdot g \circ T \, d\mu$.

Fact: $\exists$ one-sided function $g_\mu(x_0, x_1, x_2, \dots)$ called the g-function of μ s.t. $0 \le g_\mu \le 1$, $\sum_{p: p \to x_0} g_\mu(p, \underline{x}) = 1$ and

$$\boxed{(U_T^* f)(x_0, x_1, \dots) = \sum_{p: p \to x_0} g_\mu(p, x_0, x_1, \dots) f(p, x_0, x_1, \dots)}$$

Roughly, $g_\mu(p, x_0, x_1, \dots)$ " $= \lim_{n\to\infty} \frac{\mu[p; x_0, x_1, \dots, x_n]}{\mu[x_0, x_1, x_2, \dots, x_n]}$ ".

Notice that U_T^* is an averaging operator. This gives it "good" properties.

D. Ruelle introduced the following construction:

- $\Sigma^+ = \{$ one-sided paths on the dynamical graph $\}$.

  We think of $(x_0, x_1, \dots) \in \Sigma^+$ as of the "configuration" of a 1D lattice gas model

- fix a one-sided potential $\phi(x_0, x_1, \dots)$ s.t.

$$|\phi(\underline{x}) - \phi(\underline{y})| \le \text{Const.} \exp\left[-\gamma \min\{n : x_n \ne y_n\}\right]$$

  We think of $\phi = -\frac{1}{k_B T} U$ where

$$U(x_0 | x_1, \dots) = \text{"Interaction potential" between } x_0 \text{ and } x_1, x_2, x_3 \dots$$

- Ruelle's Operator: $L_\phi : C(\Sigma^+) \to C(\Sigma^+)$ ← continuous one-sided functions

$$(L_\phi f)(x_0, x_1, \dots) = \sum_{p: p \to x_0} e^{\phi(p\underline{x})} f(p\underline{x})$$

<u>Ruelle's Perron-Frobenius Thm</u>: Suppose Σ^+ is a subshift of finite type of a connected, aperiodic graph. Suppose $\phi : \Sigma^+ \to \mathbb{R}$ is as above. Then there exist $\lambda > 0$, $h(x_0, x_1, \ldots)$ positive and continuous, and a prob. measure ν s.t.

$$(\lambda^{-n} L_\phi^n f)(\underline{x}) \xrightarrow[n\to\infty]{} h(\underline{x}) \int f \, d\nu$$

In addition:

(1) $L_\phi h = \lambda h$, $L_\phi^* \nu = \lambda \nu$, $\int h \, d\nu = 1$

(2) $dm_\phi = h \, d\nu$ is an <u>invariant</u> prob. measure on Σ

(3) m_ϕ is the unique measure which minimizes the "<u>free energy</u>" $\int \sigma \, d\mu - k_B T \, h_\mu(\tau)$ (recall: $\phi = -\frac{1}{k_B T} \sigma$).

(4) the value of the minimal free energy is $-k_B T \cdot \log \lambda$.

Various choices of ϕ lead to interesting measures m_ϕ. For example, $\phi = \text{const.}$ leads to the measure of maximal entropy.

Anosov Diffeomorphisms

Take some non-area preserving, non-linear, small perturbation of Arnol'd's cat map.

Sinai: If the perturbation is small, there's still a Markov partition. (It looks very different from the Adler–Weiss partition.)

Since the perturbation is non-volume preserving, the area is no longer invariant. For "most" perturbation there's no invariant density.

However, if we use Ruelle's Perron–Frobenius for the "potential" $\phi(\underline{x}) := -\log \| Df|_{E^s(\pi(\underline{x}))} \|$ ← Oseledets space,
then we obtain an important measure μ_{SRB}, called the Sinai–Bowen–Ruelle measure with the following property:

$$\text{area}\left\{ p \in \mathbb{T}^2 \;\middle|\; \begin{array}{l} \forall f: \mathbb{T}^2 \to \mathbb{R} \text{ continuous} \\ \frac{1}{N}\sum_{k=0}^{N-1} f(T^k_p) \longrightarrow \int f \, d\mu_{SRB} \end{array} \right\} > 0$$

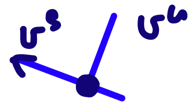
μ_{SRB} sits on an attractor, which typically has zero area. But it captures the behavior of positive area of initial condition.

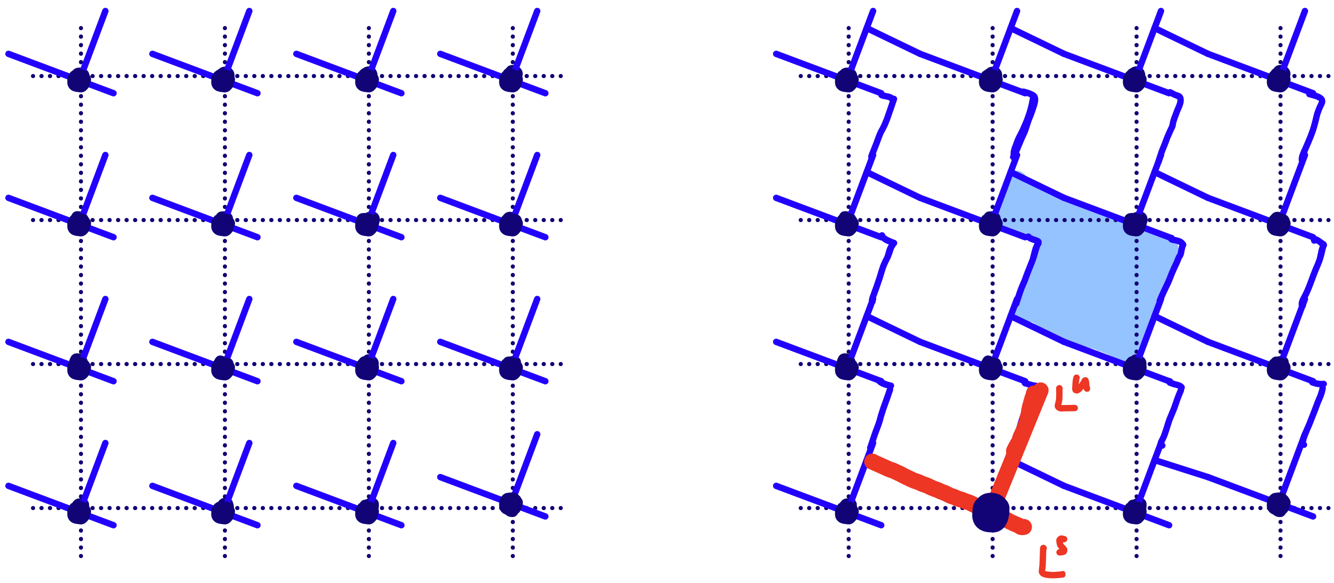
"strange attractor"

Construction of Adler-Weiss Partition

Recall that A has two eigenvectors $\vec{v}^u$, $\vec{v}^s$ with e.v. λ^u, λ^s s.t. $|\lambda^u| > 1$, $0 < |\lambda^s| < 1$.

Step 1: Find a new fundamental domain of $\mathbb{R}^2/\mathbb{Z}^2$ with sides parallel to $\vec{v}^u, \vec{v}^s$





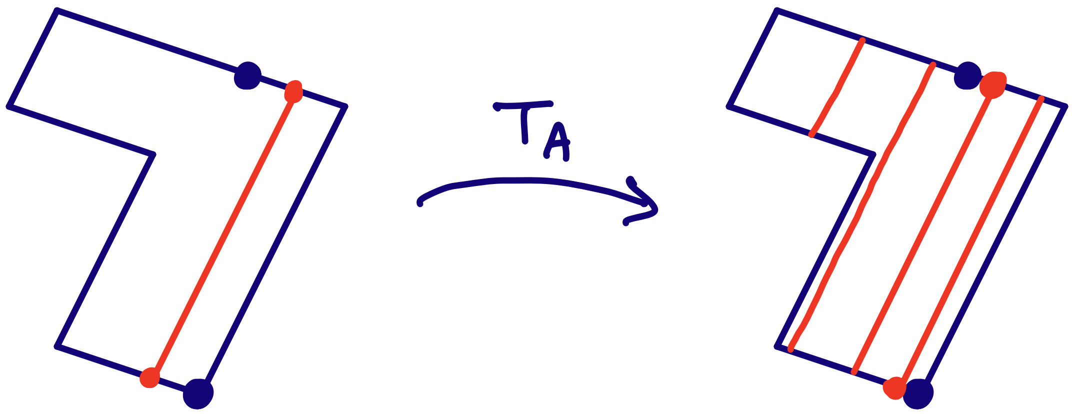
(a) (b)

Notice: All sides are are on L^u, L^s where L^t are linear segments in direction $\vec{v}^t$ $(t = u, s)$, and passing through the fixed point $\bullet = \binom{0}{0} + \mathbb{Z}^2$.

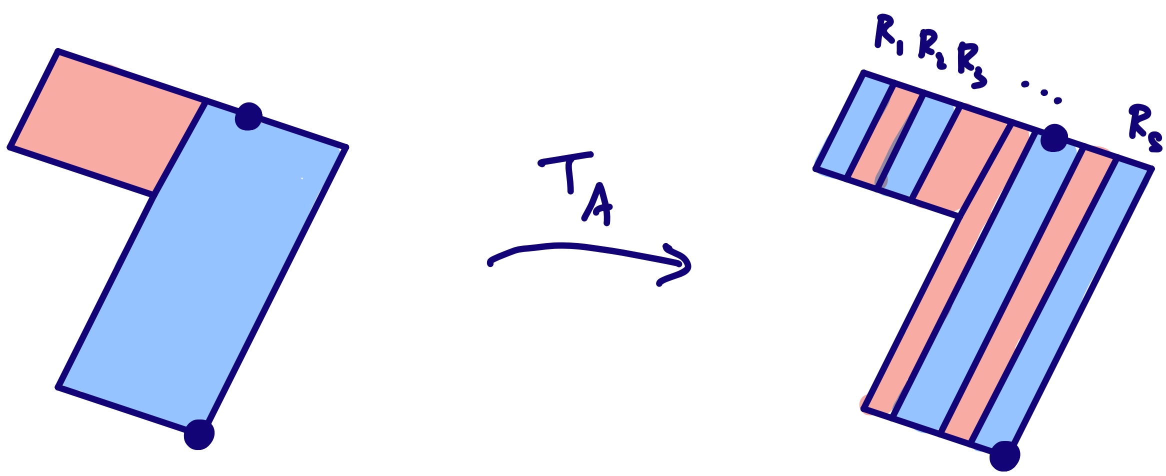
Terminology: A u-fibre is a line segment in direction $\vec{v}^u$, and endpoints on L^s.

Fact: $T_A(L^s) \subseteq L^s$ (because L^s contains a fixed point, and is in direction $\vec{v}^s$).

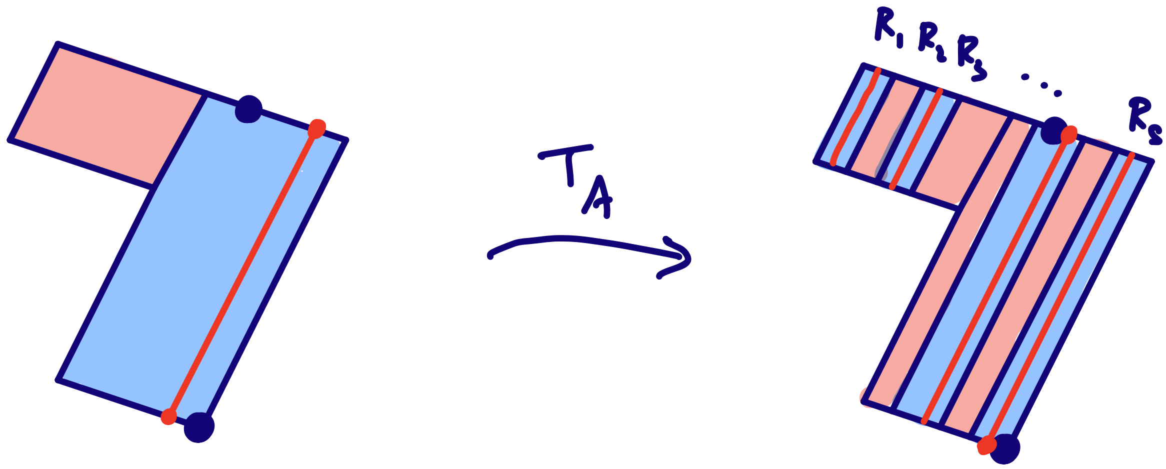
Corollary: $T_A(u\text{-fibre}) =$ union of u-fibres.



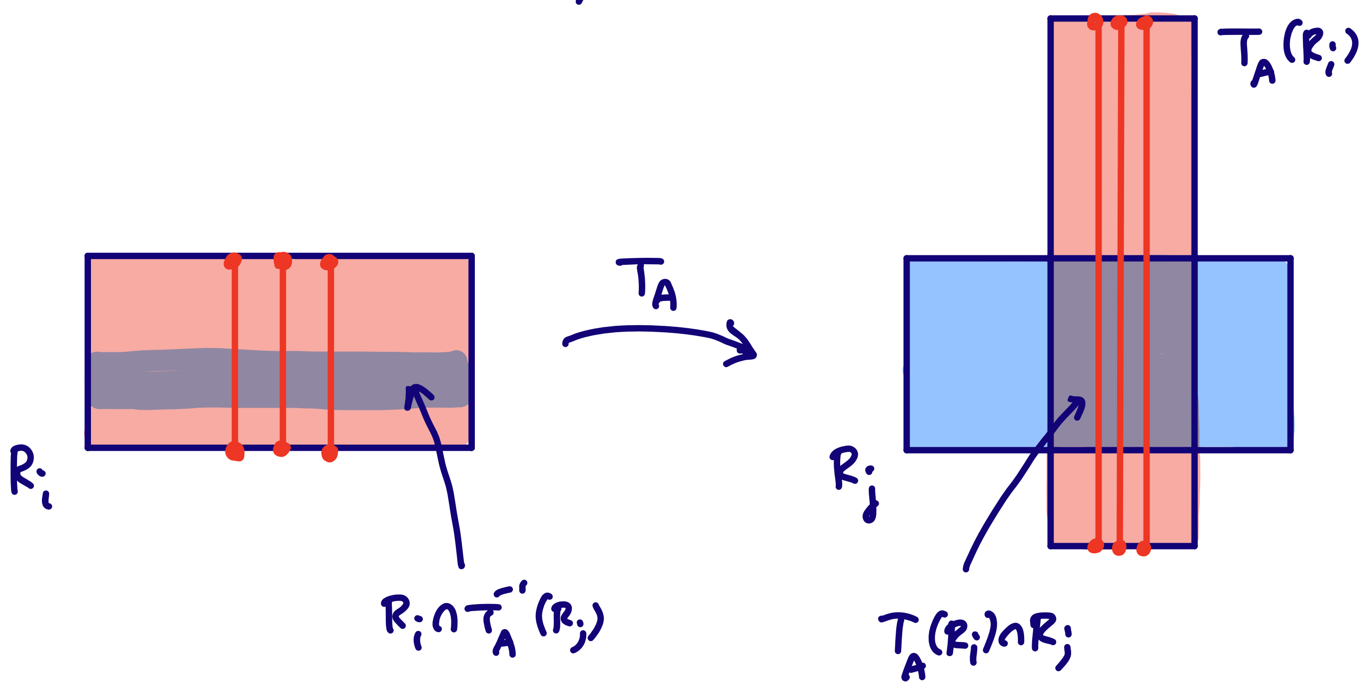
Adler Weiss Partition: $\{R_1, \ldots, R_s\}$ obtained from



Crucial Property: The image of each u-fibre intersects each R_i at one full u-fibre, or not at all

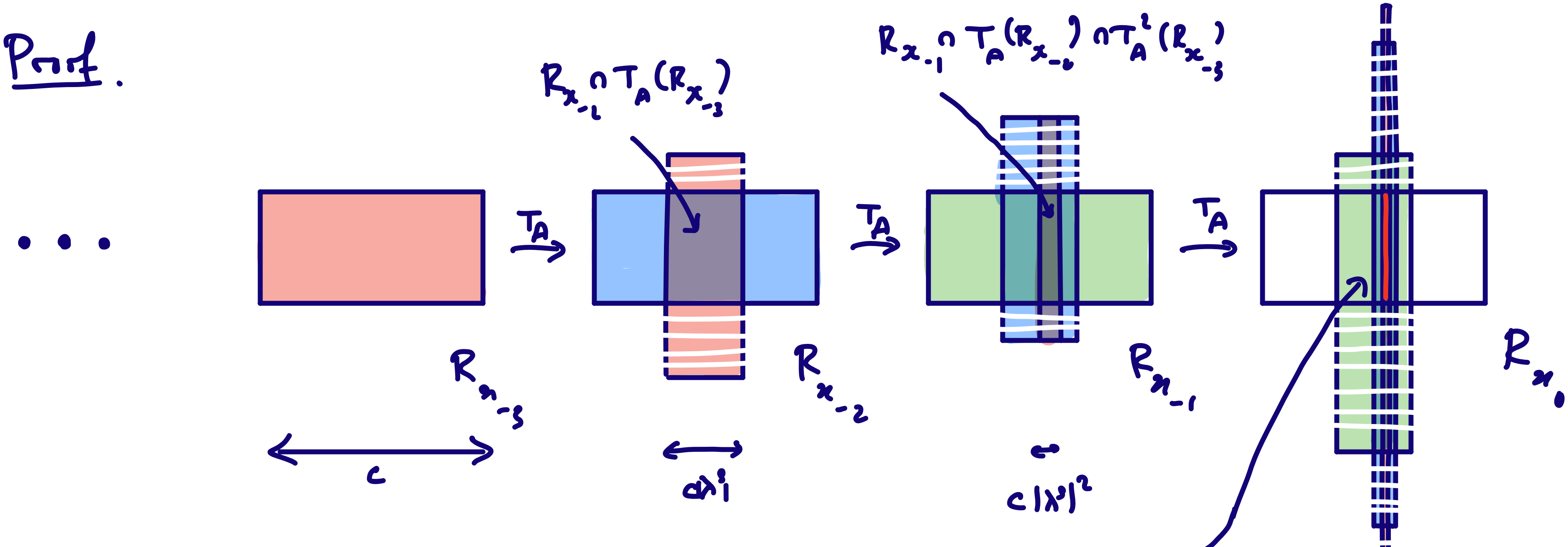


Thus, if $T_A(R_i) \cap R_j$, then the intersections look like this:



Claim: Suppose $(\cdots R_{x_{-2}}, R_{x_{-1}}, R_{x_0})$ is a backward infinite sequence on the dynamical graph of the Adler–Weiss partition. Then $\{p : T_A^{-k}(p) \in \overline{R_{x_{-k}}} \ (k \geq 0)\}$ is a u-fibre in R_{x_0}.

Proof.



This picture represents the set $R_{x_0} \cap T_A(R_{x_{-1}}) \cap T_A^2(R_{x_{-2}}) \cap \dots \cap T_A^k(R_{x_{-k}})$

It reveals that this set is a rectangle of u-fibres in R_{x_0} of width $\sim |\lambda^s|^k$

Since $|\lambda^s| \in (0,1)$, in the limit we get a single u-fibre. $\square$

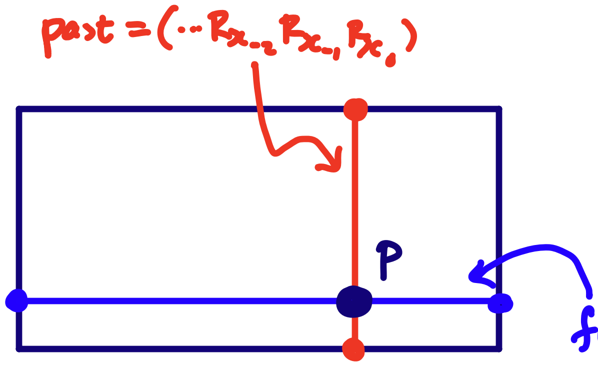
Similarly, if $(R_{x_0}, R_{x_1}, \dots)$ is a forward infinite path on the dynamical graph, then

$$\{p \in \mathbb{T}^2 : T_A^k(p) \in \overline{R}_{x_k} \ (k \geq 0)\} = s\text{-fibre in } R_{x_0}$$

Thus, for any doubly infinite path $(\dots R_{x_{-1}}, R_{x_0}, R_{x_1} \dots)$

$$\{p \in \mathbb{T}^2 : T_A^k(p) \in \overline{R}_{x_k} \ (k \in \mathbb{Z})\}$$

$$= (u\text{-fibre}) \cap (s\text{-fibre}) = \text{single point } p.$$



This point satisfies

$$T_A^k(p) \in \overline{R}_{x_k} \quad \forall k \in \mathbb{Z}.$$

In summary, every path on the dynamical graph is the itinerary "up to closures" of some genuine initial condition.

The Space $\mathcal{H}^{-1}$

The construction shows that $\{p : T_A^k(p) \in R_{x_k} \ (k \geq 0)\}$ is an s-fibre. Therefore, the one-sided functions are exactly the functions which are constant on <u>u-fibres</u>.