Lecture 9: Introduction to Hyperbolic Phenomena

The Stability Problem: Recall that $T^n(x)$ models the state of a system at time n, when x is the initial state, and T = law of motion.

In reality, we do not know x or T with absolute precision. This raises the following questions:

- How sensitive is the model to errors in initial conditions x?
- How sensitive is the model to errors in the map T?
  - parameters (physical constants, control parameters)
  - functional form (approximations, neglections, etc.)
  - roundoff errors/noise interfering with each application of T.

Exponential Sensitivity to Initial Conditions (ESIC): This is a very bad, but common, scenario, where:

"In order to know the state $T^n x$ up to time n with error $< r$, we must know the initial condition x with exponential precision."

Def.: $T: M \to M$ has exponential sensitivity to initial conditions at x if $\exists r > 0$ $\exists \chi > 0$ s.t. $\forall n$ $\exists y$ s.t. $d(y, x) \le e^{-\chi n}$, but $d(T^k y, T^k x) \ge r$ for some $0 \le k \le n$.

(Popular science books call this "the butterfly effect".)

Hyperbolicity is a dynamical phenomenon which causes stable exponential sensitivity to initial conditions.

"Stability": the existence of x with ESIC survives perturbations of T, and is thus a "true" feature of the model.

A Naive Model for ESIC: A continuous map $T: X \to X$ on a metric space (X,d) is called *expanding* if $\exists r_0 > 0$ $\exists \lambda > 1$ s.t. for all $x,y \in X$ s.t. $d(x,y) < r_0$,

$$d(Tx, Ty) \geq \lambda d(x,y).$$

Example: $T(z) = z^2$ on S^1.

Exercise: What is the "best" r_0 in this example?

Unfortunately, the framework of expanding maps is too narrow to be interesting:

Theorem: There are no volume preserving expanding diffeomorphisms on smooth Riemannian manifolds.

Proof: We give the proof for $T: M \to M$, M = open subset of $\mathbb{R}^d$. Readers with basic knowledge of differential geometry will have no problems to generalize the argument to the general case.

Assume by contradiction that $T: U \to V$, $U, V \subseteq \mathbb{R}^d$ open is volume preserving, and $d(Tx, Ty) > \lambda d(x,y)$ for $\lambda > 1$ and all $x, y \in U$ sufficiently close.

Since T is volume preserving, the Jacobian of T $|\det(dT_x)|$ equals one.

Necessarily $\exists$ unit tangent vector $\underline{v}$ s.t.

$$\|(dT)_x \underline{v}\| < \lambda$$

(Linear operators which expand all vectors by $\lambda > 1$ have determinant with modulus $\geq \lambda^d$, because they map a ball of radius one to an ellipsoid containing a ball of radius $\lambda > 1$)

But this means that

$$\|T(\underline{x} + \varepsilon \underline{v}) - T(\underline{x})\| = \|(dT)_x \cdot \varepsilon \underline{v} + o(\varepsilon)\|$$
$$\leq |\varepsilon| \|(dT)_x \underline{v}\| + o(|\varepsilon|) < \lambda|\varepsilon| + o(\varepsilon)$$
$$\leq \lambda \|\varepsilon \underline{v}\| \text{ for all } \varepsilon \text{ small.}$$

We obtained a contradiction to expansion. $\square$

<u>Exercise</u> (for People who Know Riemannian Manifolds). Let (M, g) be a compact differentiable Riemannian manifold. Show that there are no expanding volume preserving diffeomorphisms on M.

<u>Exercise</u>: Construct an expanding differentiable (<u>non invertible</u>) onto map on $\mathbb{T}^2$.

Hyperbolicity

The problem with the expansion property is that it assumes that errors in <u>any</u> direction explode exponentially.

But for exponential sensitivity to initial conditions it's sufficient that errors in <u>some</u> direction explode.

"Hyperbolicity", roughly, is a scenario where

- errors in <u>some</u> directions grow exponentially
- errors in the <u>other</u> directions shrink exponentially.

This is compatible with volume preservation:

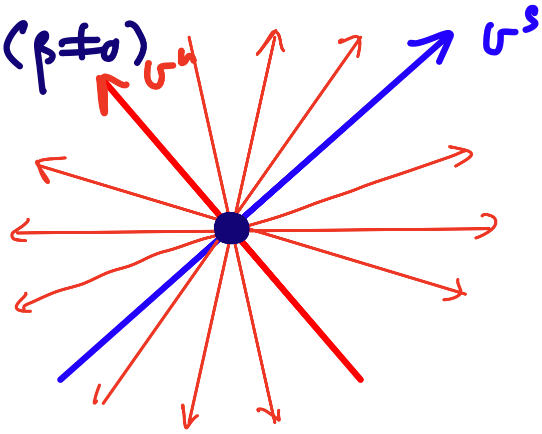
<u>Example</u>: The "cat map" $T_A\left[\binom{x}{y}+\mathbb{Z}^2\right] = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\binom{x}{y}+\mathbb{Z}^2$

$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ has two eigenvectors $\vec{v}^u$ (expanded) and $\vec{v}^s$ (contracted). The map

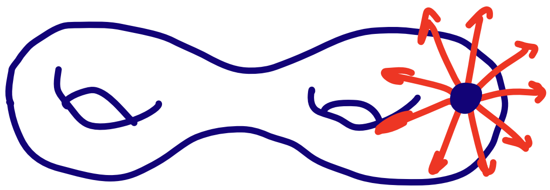
- <u>expands</u> errors in directions $\alpha\vec{v}^u + \beta\vec{v}^s$ $(\alpha \neq 0)$
- <u>contracts</u> errors in directions $\beta\vec{v}^s$ $(\beta \neq 0)$
- <u>preserves area</u> $(\because \det A = 1)$

red directions: expanded

blue direction: contracted



In order to speak of the "direction" of the error, we need to work with differentiable Riemannian manifolds. The "direction" of the error is modeled by a tangent vector.



directions of perturbing x

<u>Setup</u>: M is a compact Riemannian manifold

$T: M \to M$ is a C^1 diffeomorphism

(i.e. $T, T^{-1}: M \to M$ are differentiable)

<u>Def$^{\underline{n}}$</u>. A compact invariant set $\Lambda \subseteq M$ is called hyperbolic, if $\exists$ $\lambda \in (0,1)$, $C > 0$ an families of linear subspaces $E^u(x), E^s(x) \subseteq T_x M$ $(x \in \Lambda)$ s.t.

- $T_x M = E^u(x) \oplus E^s(x)$ for all $x \in M$
- <u>Invariance</u>: $(dT)_x (E^u(x)) = E^u(T(x))$

  $(dT)_x (E^s(x)) = E^s(T(x))$
- <u>Uniform Contraction / Expansion</u>: $\forall x \in \Lambda$ $\forall n \geq 0$

$$\|(dT^n)_x(\vec{v})\| \leq C\lambda^n \|\vec{v}\| \qquad (\vec{v} \in E^s(x))$$

$$\|(dT^{-n})_x(\vec{v})\| \leq C\lambda^n \|\vec{v}\| \qquad (\vec{v} \in E^u(x)).$$

$E^u(x), E^s(x)$ are called unstable (resp stable) spaces.

<u>Note</u>: It follows that $\forall x \in \Lambda$, $\forall n \geq 0$

$$\|(dT^n)_x \vec{v}\| \geq C^{-1} \lambda^{-n} \|v\| \qquad (v \in E^u(x))$$

Proof: The chain rule says that $(d(T_1 \circ T_2))_x = (dT_2)_{T_1(x)} (dT_1)_x$.

Thus $\|\vec{v}\| = \| \underbrace{(dT^{-n})_{T^n x} (dT^n)_x}_{= d(T^{-n} \circ T) = id} \vec{v} \| \leq C \lambda^n \| (dT^n)_x \vec{v} \|$

So $\|(dT^n)_x \vec{v}\| \geq C^{-1} \lambda^{-n} \|\vec{v}\| \longrightarrow \infty$ exponentially fast. $\square$

Exercise: Suppose Λ is a hyperbolic set and $x \in \Lambda$. Then

- $E^s(x) = \{ \vec{v} \in T_x M : \|(dT^n)_x \vec{v}\| \xrightarrow[n \to \infty]{} 0 \}$
- $E^u(x) = \{ \vec{v} \in T_x M : \|(dT^{-n})_x \vec{v}\| \xrightarrow[n \to \infty]{} 0 \}$

But $E^u(x) \subsetneqq \{ \vec{v} \in T_x M : \|(dT^n)_x \vec{v}\| \xrightarrow[n \to \infty]{} \infty \}$.

This is the reason why E^u is characterized using contraction in the past, instead of the more natural, but weaker property of expansion in the future.

Notice that hyperbolicity is stated on the infinitesimal level: It's a property of $(dT)_x$, not of T.
Eventually, we will prove the following:

Theorem. Suppose $T: M \to M$ is a C^2 diffeomorphism with a hyperbolic set Λ. Then T has exponential sensitivity to initial conditions at any $x \in \Lambda$.

Examples of Hyperbolic Sets

(1) **Hyperbolic Fixed Points**: Suppose $T: M \to M$ is a diffeo and $p \in M$ is a point where $dT_p: T_pM \to T_pM$ is a linear map

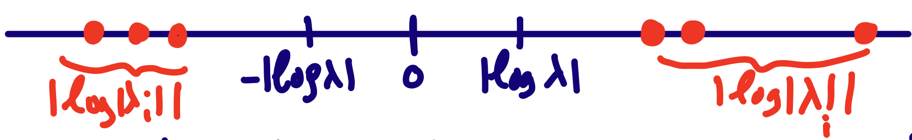
- with some eigenvalues of modulus >1
- — " — <1
- no eigenvalues of modulus 1.

Theorem: $\Lambda = \{p\}$ is a hyperbolic set

Proof. Working with the Jordan block decomposition, we can decompose $T_pM = E^u \oplus E^s$ where

- $(dT_p)(E^u) = E^u$ and all e.v.'s of $dT_p|_{E^u}$ have modulus >1
- $(dT_p)(E^s) = E^s$ and all e.v. of $dT_p|_{E^s}$ — " — <1

Claim: Choose $0<\lambda<1$ s.t. for each eigenvalue λ_i if $|\lambda_i|<1$, then $|\lambda_i|<\lambda$, and if $|\lambda_i|>1$, then $|\lambda_i|>\lambda^{-1}$.



Then $\exists C>0$ s.t. $\|(dT^n)_p\vec{v}\| \le C\lambda^n\|\vec{v}\|$ on E^s

$$\|(dT^{-n})_p\vec{v}\| \le C\lambda^n\|\vec{v}\| \text{ on } E^u.$$

Proof. E^s breaks to the sum of invariant spaces E_i s.t. $(dT)_p: E_i \to E_i$ is similar to a Jordan block

$$J = \begin{pmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & 1 & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{pmatrix} \Big\updownarrow m_i = \dim E_i, \quad |\lambda_i|<1.$$

Let's write $J = \lambda_i I + N$, where N is the nilpotent matrix

$$N = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix} \updownarrow m_i$$

Note that $N^{m_i} = 0$ and $\|N\| \leq 1$.

Therefore $J^n = (\lambda_i I + N)^n = \sum_{j=0}^{\min(n,m_i)} \binom{n}{j} \lambda_i^{n-j} N^j$

$$= \lambda_i^n \sum_{j=0}^{\min(n,m_i)} \binom{n}{j} \lambda_i^{-j} N^j$$

$$= \lambda_i^n (I + D_n), \quad \|D_n\| \leq \text{const.}\ n^{m_i} \text{ as } n \to \infty$$

(because $\binom{n}{j} = O(n^j)$ and $j \leq m_i$).

Thus $\|J^n\| \leq |\lambda_i|^n \cdot \text{const.}\ n^{m_i}$. Since $|\lambda_i| < \lambda < 1$, there's a constant s.t. $\|J^n\| < C\lambda^n$ for all $n \geq 0$ and we get uniform exponential contraction on each E_i^s whence on their sum E^s.

(2) Hyperbolic Periodic Orbits: Suppose $T^k(p) = p$ and $(dT^k)_p : T_pM \to T_pM$ is a linear operator

- with some eigenvalues of modulus > 1
- ———— " ———— < 1
- no eigenvalues of modulus 1

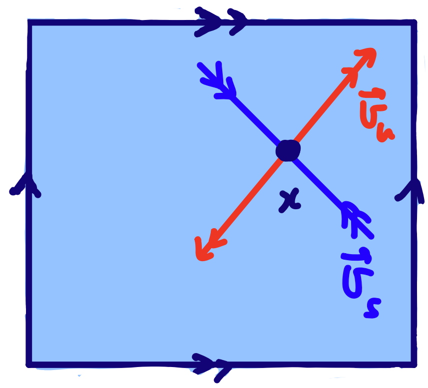
Then $\Lambda = \{p, T(p), \dots, T^{k-1}(p)\}$ is a hyperbolic set.

Exercise: prove this.

(3) The Cat Map: $T_A : \mathbb{T}^2 \to \mathbb{T}^2$, $T_A\left[\binom{x}{y} + \mathbb{Z}^2\right] = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\binom{x}{y} + \mathbb{Z}^2$.

Recall that $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ has two eigenvalues $\lambda_u > 1$, $0 < \lambda_s < 1$. Let $\vec{v}_u, \vec{v}_s$ be the corresponding eigenvectors.

$\mathbb{T}^2$



At each point x, let

$$E^u(x) = \operatorname{Span}\{\vec{v}^u\}$$
$$E^s(x) = \operatorname{Span}\{\vec{v}^s\}$$

Then $T_xM = E^u(x) \oplus E^s(x)$ satisfies the axioms of hyperbolic sets, so $\Lambda = \mathbb{T}^2$ is a hyperbolic set.

Def$\underline{\text{n}}$: A diffeomorphism $T: M \to M$ s.t. M is a hyperbolic set is called an Anosov diffeomorphism.

(4) Transverse Homoclinic Points: This is perhaps the most common and important mechanism for producing hyperbolic sets. Suppose p is a hyperbolic fixed point.

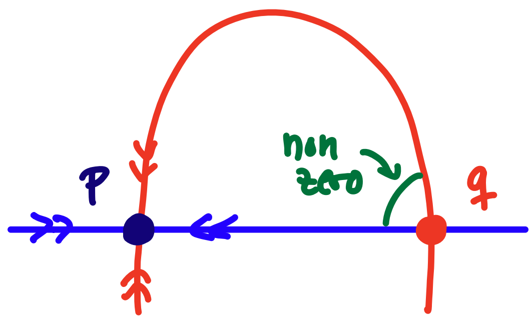
Fact ("Stable Manifold Thm"): $\exists\, \epsilon > 0$ s.t.

$$W^s(p) := \{x : d(T^n x, p) \xrightarrow[n \to +\infty]{} 0\}$$
$$W^u(p) := \{x : d(T^{-n} x, p) \xrightarrow[n \to -\infty]{} 0\}$$

are C^1 submanifolds of M of dimensions $\dim E^s(p), \dim E^u(p)$.

(We'll prove a more general result later.)

Def$^{\underline{n}}$. A <u>transverse homoclinic point</u> is an intersection point $q \in W^u(p) \cap W^s(p)$, where the angle between W^u, W^s is not zero.



- "transverse" : $\measuredangle(W^u, W^s) \neq 0$
- "homoclinic":
$$d(T^n q, T^n p) = d(T^n q, p) \xrightarrow[n\to\infty]{} 0$$
$$d(T^{-n} q, T^{-n} p) = d(T^{-n} q, p) \xrightarrow[n\to\infty]{} 0$$

<u>Smale–Birkhoff–Poincaré "Homoclinic Theorem"</u>: Suppose a hyperbolic fixed point p has a transverse homoclinic intersection. Then T has a compact invariant set Λ s.t.:

(a) Λ is a hyperbolic set

(b) $\Lambda = R_0 \cup R_1$ where R_0, R_1 are disjoint closed sets, and $\exists N$ s.t. for <u>every</u> sequence of zeroes and ones $(x_n)_{n\in\mathbb{Z}}$, there is a unique $x \in \Lambda$ s.t. $T^{kN}(x) \in R_{x_k}$ $(k \in \mathbb{Z})$.

The proof requires many preparations. <u>Maybe</u> we'll have time to give it, in later lectures.

<u>Meaning of (b)</u>: Define $\varphi : \Lambda \to \{0,1\}$, $\varphi(x) = \begin{cases} 0 & x \in R_0 \\ 1 & x \in R_1 \end{cases}$.
(b) says that nothing can be said of the sequence
$$\varphi(T^{Nk}(x)) \qquad (k \in \mathbb{Z})$$
because <u>any</u> possible sequence can be realized for some initial condition.

Even worse: Knowing the full past $\{\varphi(T_x^{N_k})\}_{k\le 0}$ gives <u>no</u> information on the future $\{\varphi(T_x^{N_k})\}_{k>0}$ because any two half sequences

$$(\cdots x_{-2}, x_{-1}, x_0) \quad ; (y_1, y_2, y_3, \ \cdots)$$

can be concatenated to a sequence

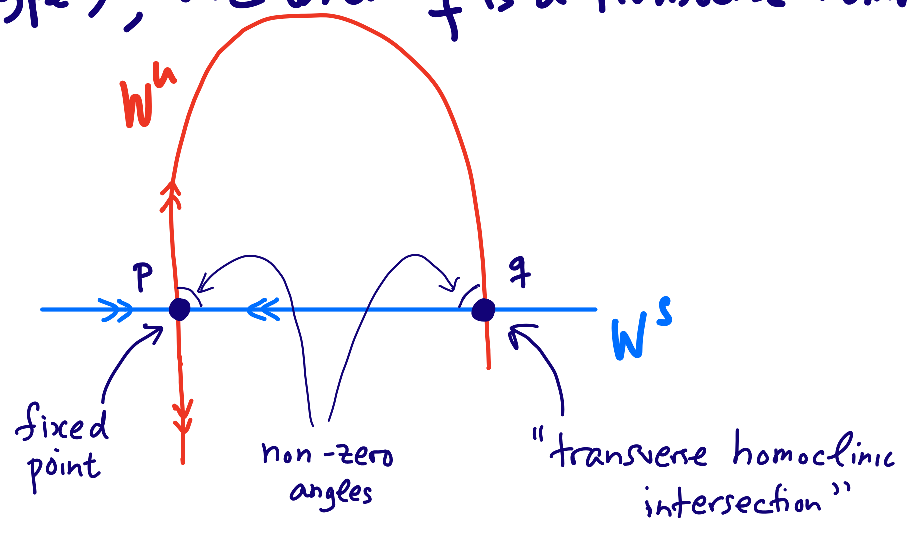
$$(\cdots x_{-2} x_{-1} x_0 y_1 y_2 y_3 \ \cdots)$$

realized by some initial condition.

Such behavior is called colloquially "chaotic" (but professionals avoid this, because there are several non equivalent definitions of "chaos".)

<u>Heuristics: The Complicated Structure of Stable and Unstable Manifolds Near a Transverse Homoclinic Intersection</u>

Let's build, step-by-step, a qualitative picture of the structure of $W^u(p), W^s(p)$ when p is a hyperbolic fixed point (of saddle type), and when q is a transverse homoclinic point.



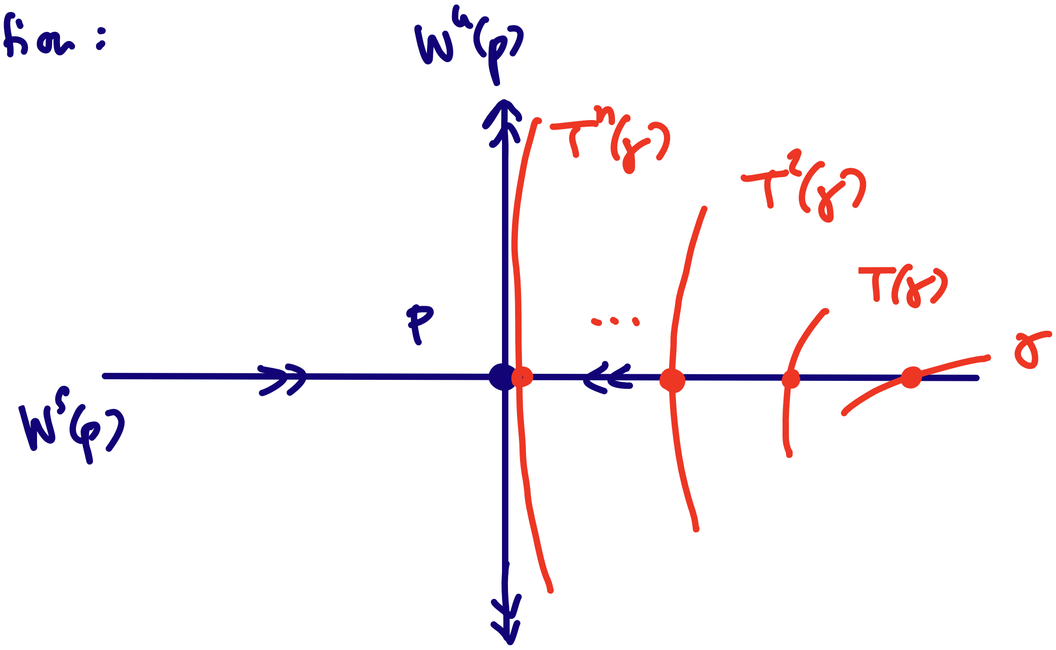
We'll use the following assumptions:

- dim $M=2$ (can be relaxed)
- $W^u(p), W^s(p)$ are C^1 non self-intersecting <u>invariant</u> curves, i.e.
  $$T(W^s(p)) = W^s(p) \text{ and } T(W^u(p)) = W^u(p)$$
  (This follows from "stable manifold theorem" we will prove later)
- $q \in W^u(p) \cap W^s(p)$, $T^n(q) \xrightarrow[n\to\infty]{} p$, $T^{-n}(q) \xrightarrow[n\to\infty]{} p$
- $W^u(p)$ intersects $W^s(p)$ at q with non-zero angle

$\Big\}$ i.e. q is a transverse homoclinic intersection

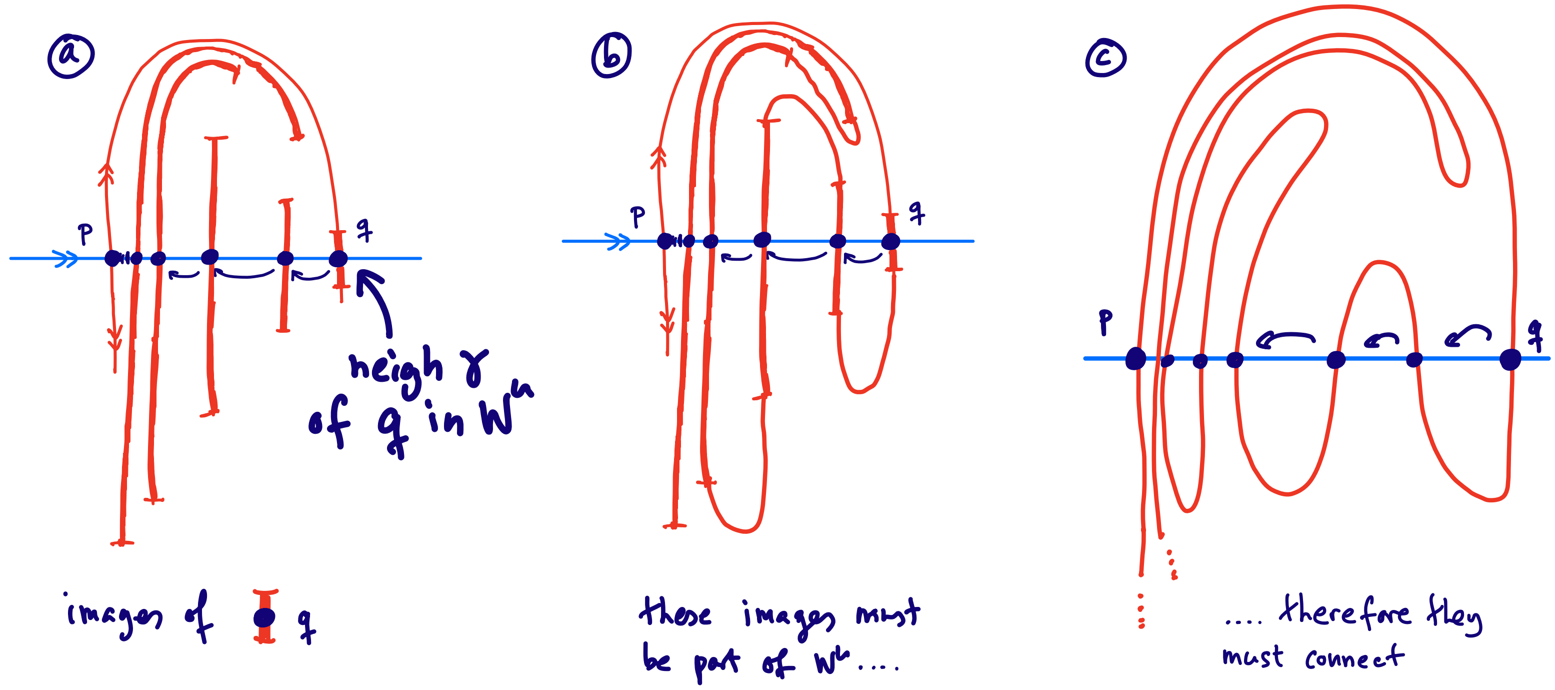
- "<u>Inclination Lemma</u>:" If γ is a C^1 smooth curve which intersects $W^s(p)$ transversely, then $T^n(\gamma)$ converges, as $n \to +\infty$ to $W^u(p)$.

For T <u>linear</u> on a neigh of p this is clear because T stretches the y-direction, and contracts the x-direction:

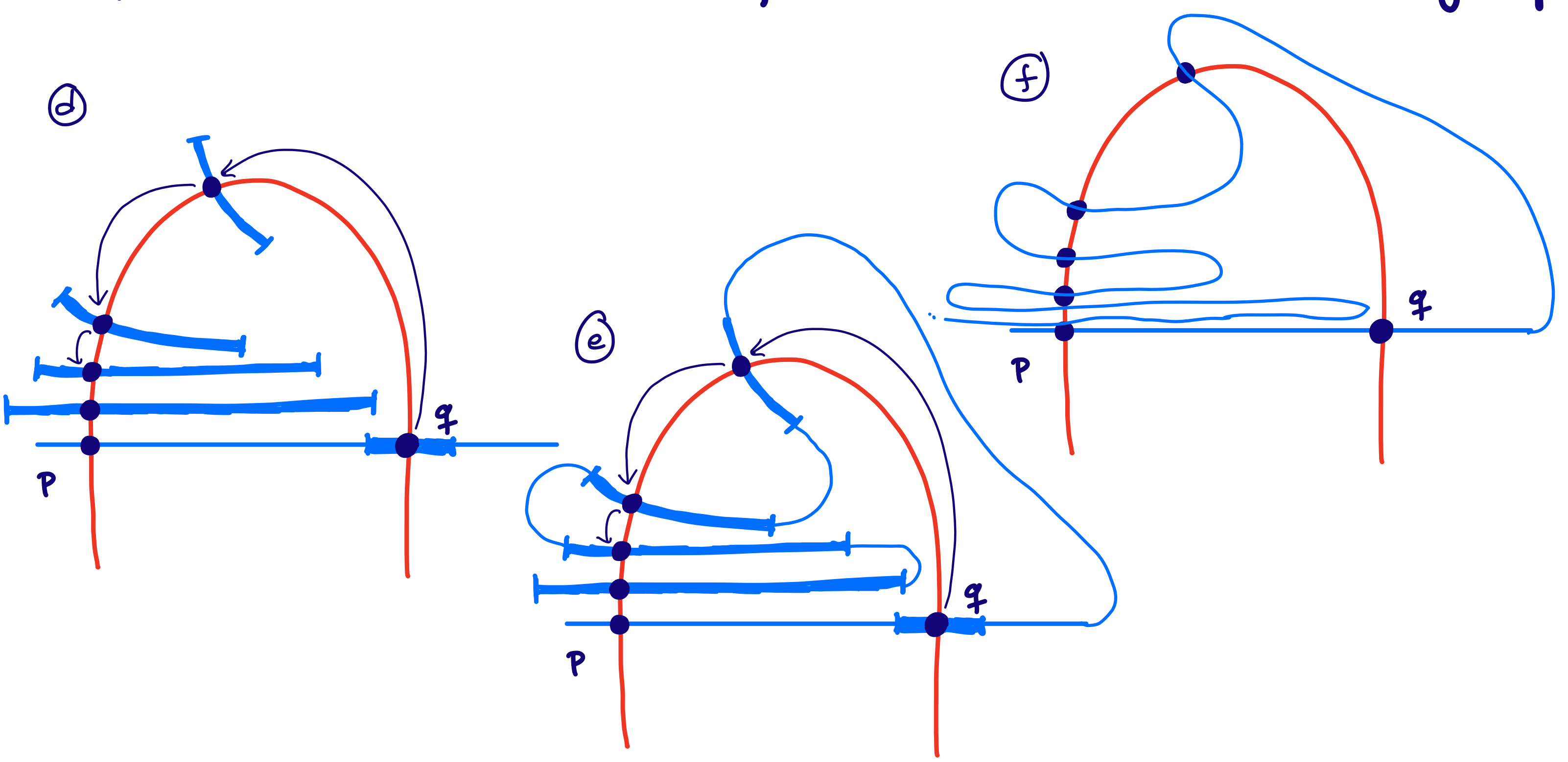


In the general case, this is a theorem called the "inclination lemma" or the "λ-lemma".

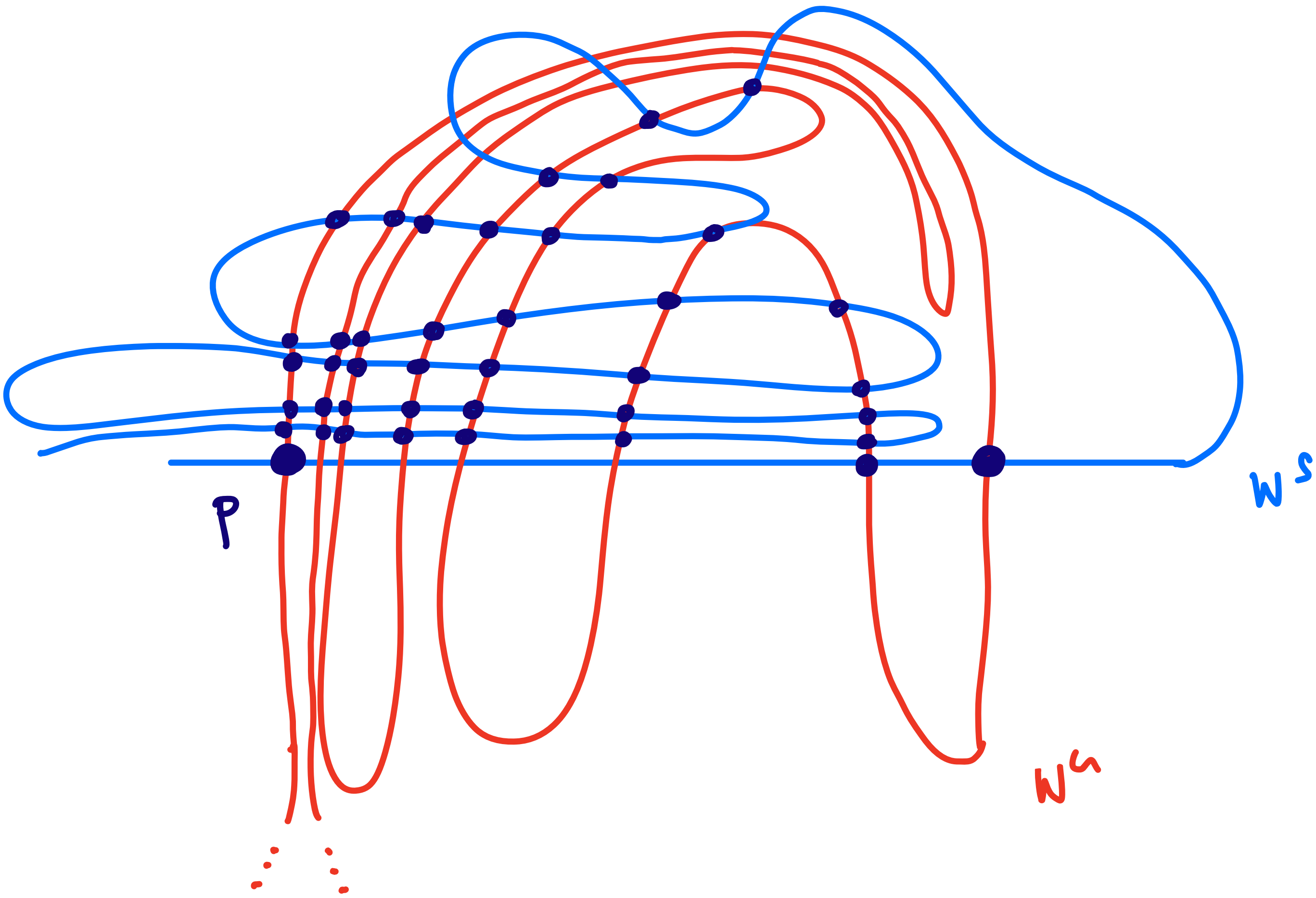
Lets draw the images of a small neighborhood γ of q in W^u

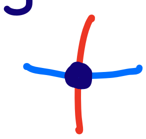


Similarly, if we draw the T^{-n}-images of a small neighborhood of q in W^s we get from the symmetric version of the inclination lemma for T^{-1} and curves intersecting $W^s(p)$:

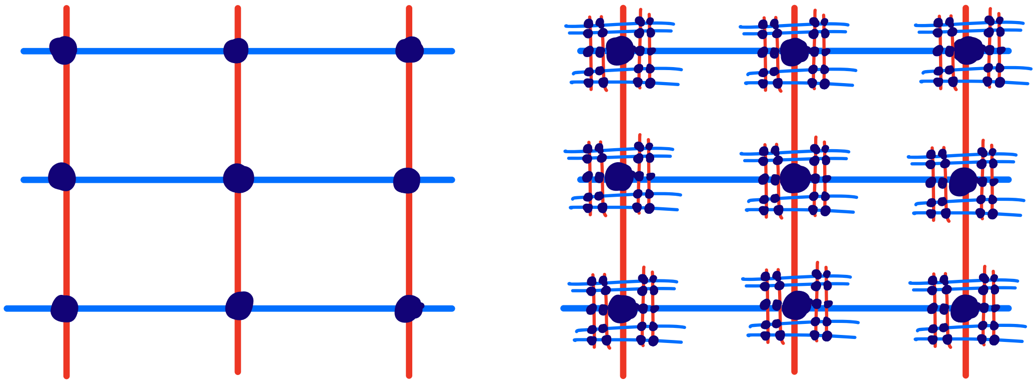


If we super-impose these pictures e, f we obtain a "web" intersecting at "second generation" transverse homoclinic intersections (all homoclinic to p)



- The "new" intersections are <u>homoclinic</u> to p, because they are all on W^u, W^s
- They are all <u>transverse</u>, because they are images of  by a diffeomorphism, and diffeomorphisms cannot map non-zero angles to zero angles (their differentials are invertible linear maps).

<u>Later Generation THP</u>: Each 2^{nd} gen THI gives rise to 3^{rd} generation THI etc.



"When one tries to imagine the figure formed by these two curves and their infinitely many intersections, each corresponding to a doubly asymptotic solution, these intersections form a kind a kind of lattice, web, or network with infinitely many tight loops; neither of the two curves must ever intersect itself, but it must bend in such a complex fashion that it intersects all the loops of the network infinitely many times.

One is struck by the complexity of this figure which I am not even trying to draw. Nothing can give us a better idea of the complexity of the three-body problem and of all problems of dynamics where there is no analytic integral and Bohlin's series diverges."

Henri Poincaré

Les méthodes nouvelles de la mécanique céleste, vol 3, p. 389

See: K.G. Andersson, Poincaré's discovery of homoclinic points, Archive for History of Exact Sciences 48, (1994), p. 133–147.