THE HOROCYCLE FLOW AND THE LAPLACIAN ON HYPERBOLIC SURFACES OF INFINITE GENUS

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ABSTRACT. Consider a complete hyperbolic surface which can be partitioned into countably many pairs of pants whose boundary components have lengths less than some constant. We show that any infinite ergodic invariant Radon measure for the horocycle flow is either supported on a a single horocycle associated with a cusp, or corresponds canonically to an extremal positive eigenfunction of the Laplace–Beltrami operator.

CONTENTS

1. STATEMENT OF RESULTS

Suppose Γ is a torsion–free Fuchsian group of the first kind, which is not geometrically finite. We study the invariant measures for the horocycle flow on the

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unit tangent bundle of the hyperbolic surface $\Gamma \setminus \mathbb{D}$. Ratner theory identifies all finite invariant measures (they are all supported on periodic horocycles surrounding cusps [R]). We study the infinite invariant Radon measures.

The geometrically finite case is well-understood thanks to the works of Furstenberg [F], Dani [Da] and Dani & Smillie [DS]: the ergodic invariant Radon measures are either proportional to the volume measure, or are supported on a periodic horocycle surrounding a cusp.[1] But if Γ is not geometrically finite, then there may be many other measures, as first discovered by Babillot & Ledrappier [BL2] in the particular case when $\Gamma \lhd \Gamma_0$, Γ_0 is a uniform lattice in $\mathrm{PSL}(2,\mathbb{R})$, and $\Gamma_0/\Gamma \cong \mathbb{Z}^d$ (i.e. when $M = \Gamma \setminus \mathbb{D}$ is a $\mathbb{Z}^d$–cover of a compact hyperbolic surface).

Motivated by the work of Sullivan, M. Babillot has suggested a connection between the abundance of ergodic horocycle invariant Radon measures and the abundance of positive eigenfunctions for the Laplace–Beltrami operator Δ of $\Gamma \setminus \mathbb{D}$ [Ba]. We explain the connection.

Every positive eigenfunction on $\Gamma \setminus \mathbb{D}$ lifts to a Γ–invariant positive eigenfunction for the Laplace–Beltrami operator of the hyperbolic disc. It is known that such functions can always be represented in the form

$$F(z) = \int_{\partial\mathbb{D}} P(e^{i\theta}, z)^{\alpha} d\nu(e^{i\theta}),$$

where ν is a finite positive measure on $\partial\mathbb{D}$, $P(e^{i\theta}, z) = (1-|z|^2)/|e^{i\theta}-z|^2$ is Poisson's kernel, and $\alpha \geq 1/2$ (Karpelevich [Kar], see also [GLT]). If the Poincaré exponent of Γ is larger than or equal to $1/2$, then this representation is unique ([Su], theorem 2.17, see also [Ba]).

$T^1(\mathbb{D})$ can be identified with $\partial\mathbb{D} \times \mathbb{R} \times \mathbb{R}$ via $(e^{i\theta}, s, t) \leftrightarrow (h^t \circ g^s)(\omega(e^{i\theta}))$, where h^t is the horocycle flow, g^s is the geodesic flow, and $\omega(e^{i\theta})$ is the element of $T^1(\mathbb{D})$ based at the origin, and pointing at $e^{i\theta}$ (these are the 'KAN–coordinates' for $T^1(\mathbb{D}) \cong \mathrm{PSL}(2,\mathbb{R})$). In these coordinates,

$$dm = e^{\alpha s} d\nu(e^{i\theta}) ds dt$$

is a Γ–invariant, horocycle invariant measure on $T^1(\mathbb{D})$. Thus a positive eigenfunction F gives rise to a horocycle invariant measure m.

Babillot has conjectured – at least in the case when $\{1\} \neq \Gamma \lhd \Gamma_0$, where Γ_0 is a uniform lattice such that Γ_0/Γ is nilpotent – that every invariant Radon measure arises this way [Ba]. This was later proved to be the case for all non-trivial normal subgroups of uniform lattices [LS].

The purpose of this paper is to prove Babillot's conjecture for a larger class of fuchsian groups of the first kind, which we call *tame*.

Tameness is a condition on the pants decomposition of the hyperbolic surface $\Gamma \setminus \mathbb{D}$. We review the relevant notions.

A hyperbolic surface with boundary is called a *pair of pants*, if it is homeomorphic to a sphere minus three disjoint closed discs or points. The *norm* of a pair of pants Y is the sum of the lengths of its boundary components, and is denoted by $\|Y\|$. Here and throughout a puncture is a boundary component of length zero.

The following result is classical [Hub]: Suppose Γ is a torsion free fuchsian group, then Γ is of the first kind iff $\Gamma \setminus \mathbb{D}$ can be partitioned into a countable collection of

[1] The case when Γ is a geometrically finite group of the second kind is also understood, see Burger [Bu], Roblin [Ro], and Schapira [Scha].

pants $\{Y_i\}$ which meet at boundary components of the same length. We call $\{Y_i\}$ a *pants decomposition* of M, and we call Y_i *pants components*.

Definition 1. *A fuchsian group is called* tame *if* $\Gamma\backslash\mathbb{D}$ *admits a pants decomposition* $\{Y_i\}$ *such that* $\sup\|Y_i\|<\infty$.

Non–trivial normal subgroups of uniform lattices are tame: the surfaces which they generate (regular covers of compact surfaces) admit infinite pants decompositions whose components fall into finitely many isometry classes.

There are many more examples: Whenever one glues a finite or countable collection of pants of bounded norm one to another in such a way that every boundary component is glued to some other boundary component of the same length, then the result is a complete hyperbolic surface.[2] If we represent this surface by $\Gamma\setminus\mathbb{D}$, then Γ is tame.

In fact our results hold under a weaker condition, which we now explain. We say that a geodesic ray on M *crosses* a pair of pants Y if it enters and leaves Y through two different boundary components. Any geodesic ray either crosses pairs of pants infinitely many times, or it gets trapped in a union of two adjacent pairs of pants. We call such unions *star pieces.*

Definition 2. *A torsion–free fuchsian group is called* weakly tame, *if* $\Gamma\setminus\mathbb{D}$ *admits a pants decomposition s.t. any geodesic ray which does not get trapped in a star piece crosses infinitely many pants components with norm bounded from above.*

We do not ask for the same bound for all geodesic rays.

An ergodic invariant Radon measure is called *trivial* if it is supported on a single horocycle made of unit tangent vectors whose forward geodesics tend to a cusp. A measure is called *quasi-invariant* under the geodesic flow g^s if $m\circ g^s\sim m$ for all s.

Theorem 1. *If* Γ *is weakly tame, then any non-trivial horocycle ergodic invariant Radon measure on* $T^1(\Gamma\setminus\mathbb{D})$ *is quasi-invariant under the geodesic flow.*

A positive eigenfunction is called *trivial*, if it is of the form

$$\sum_{g\in\Gamma/\mathrm{stab}_\Gamma(e^{i\theta})} P(g\cdot e^{i\theta},z)^\alpha$$

where $e^{i\theta}$ is a fixed point of some parabolic element of Γ, and $\mathrm{stab}_\Gamma(e^{i\theta})=\{g\in\Gamma: g(e^{i\theta})=e^{i\theta}\}$. A positive eigenfunction is called *minimal*, if it defines an extremal ray in the cone of positive eigenfunctions of the same eigenvalue.

Theorem 2. *If* Γ *is weakly tame, then the following map is a bijection between the non-trivial positive minimal eigenfunctions of the Laplace–Beltrami operator of* $\Gamma\backslash\mathbb{D}$*, and the non-trivial horocycle ergodic invariant Radon measures on* $T^1(\Gamma\backslash\mathbb{D})$*:*

$$\left[F(\Gamma z)=\int_{\partial\mathbb{D}}P(e^{i\theta},z)^\alpha d\nu(e^{i\theta})\right]\leftrightarrow\left[\begin{array}{l}\textit{The restriction of } dm=e^{\alpha s}d\nu(e^{i\theta})dsdt\\ \textit{to a fundamental domain of } \Gamma\end{array}\right].$$

Theorems 1 and 2 are false for general fuchsian groups of the first kind:

Theorem 3. *There exists a fuchsian group of the first kind* Γ *s.t. the horocycle flow on* $T^1(\Gamma\setminus\mathbb{D})$ *admits non-trivial invariant ergodic Radon measures which are not quasi-invariant under the geodesic flow.*

[2] Proof of completeness: The time it takes a geodesic to cross a pants whose norm is bounded from above is bounded from below, therefore any geodesic can be extended indefinitely.

Theorems 2 and 3 are consequences of theorem 1 and its proof. The proof of theorem 1 is based (as in [Sa] and [LS]) on a technical result called the *holonomy lemma*. Versions of this lemma were proved in [Sa] for $\mathbb{Z}^d$–covers of compact hyperbolic surfaces, and in [LS] or all regular covers of hyperbolic surfaces of finite area. The contribution of this paper is the proof of the holonomy lemma for all tame surfaces.

Notational conventions: $a = b \pm c$ means $b - c \leq a \leq a + c$, and $a = e^{\pm c} b$ means $e^{-c} b \leq a \leq e^c b$, $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Re} z > 0\}$, dist is the hyperbolic distance on $\mathbb{D}$, $\mathbb{H}$, or their quotients, and when we write $\mathrm{dist}(\omega_1, \omega_2)$ for two tangent vectors ω_1, ω_2, we mean the hyperbolic distance between the base points of these vectors.

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2. The Holonomy Lemma

2.1. **The Radon-Nikodym Cocycle and the Holonomy Lemma.** Let Γ be a torsion–free Fuchsian group and $M = \Gamma \backslash \mathbb{D}$ the corresponding surface. Let $\mathrm{Fix}(\Gamma) := \{e^{i\theta} \in \partial\mathbb{D} : \exists id \neq g \in \Gamma \text{ s.t. } g(e^{i\theta}) = e^{i\theta}\}$. Define an equivalence relation $\mathfrak{G}$ on $\partial\mathbb{D} \setminus \mathrm{Fix}(\Gamma)$ by $(e^{i\theta}, e^{i\theta'}) \in \mathfrak{G} \iff \exists g \in \Gamma \text{ s.t. } e^{i\theta'} = g(e^{i\theta})$.

The *Radon-Nikodym cocycle* is $R : \mathfrak{G} \to \mathbb{R}$ given by $R(e^{i\theta}, e^{i\theta'}) = -\log |g'(e^{i\theta})|$ for the unique $g \in \Gamma$ s.t. $e^{i\theta'} = g(e^{i\theta})$.

A Γ–holonomy is a bi-measurable invertible map $\kappa : \mathrm{dom}(\kappa) \to \kappa(\mathrm{dom}\,\kappa)$ defined on $\mathrm{dom}\,\kappa \subset \partial\mathbb{D}$ such that for every $e^{i\theta} \in \mathrm{dom}\,\kappa$, $\exists g \in \Gamma$ s.t. $\kappa(e^{i\theta}) = g(e^{i\theta})$. Let $\mathrm{Par}(\Gamma)$ denote the (countable) collection of fixed points of parabolic elements of Γ.

Lemma 2.1.1 (Holonomy Lemma)**.** *Let Γ be a weakly tame fuchsian group, and suppose m is a Radon measure on $\partial\mathbb{D} \times \mathbb{R}$ which is ergodic and invariant under the action $g : (e^{i\theta}, s) \mapsto (g(e^{i\theta}), s - \log |g'(e^{i\theta})|)$ $(g \in \Gamma)$. Suppose $m[\mathrm{Par}(\Gamma) \times \mathbb{R}] = 0$.*

(1) $\exists \alpha_0 \neq 0, M_0 > 0$ *s.t.* $\forall \varepsilon > 0$, $n \in \mathbb{N}$, *there exists a Γ–holonomy κ for which* $m[(\mathrm{dom}\,\kappa \times \mathbb{R})^c] = 0$ *and* $|\kappa(e^{i\theta}) - e^{i\theta}| < \varepsilon$, $\left|R(e^{i\theta}, \kappa(e^{i\theta})) - n\alpha_0\right| \leq M_0$ *for a.e.* $(e^{i\theta}, s)$.

(2) $\forall c_0 > 0\ \exists M(c_0) > 0$, $0 < \varepsilon(c_0) < 1$ *s.t.* $\forall \varepsilon > 0$ *there exists a Γ–holonomy κ for which* $m[(\mathrm{dom}\,\kappa \times \mathbb{R})^c] = 0$, $|\kappa(e^{i\theta}) - e^{i\theta}| < \varepsilon$, $\left|R(e^{i\theta}, \kappa(e^{i\theta}))\right| \leq M(c_0)$, *and* $\mathrm{dist}\left(R(e^{i\theta}, \kappa(e^{i\theta})), c_0\mathbb{Z}\right) > \varepsilon(c_0)$ *for a.e.* $(e^{i\theta}, s)$.

Versions of this lemma for smaller classes of groups can be found in [Sa] (normal subgroups of uniform lattices with quotient isomorphic to $\mathbb{Z}^d$), and in [LS] (all normal subgroups of lattices).

2.2. **The holonomy lemma implies theorem 1.** This is the same as in [LS], barring some easy modifications, but we give the details for completeness.

Step 1 [LS]. Reduction to the following problem: Suppose m is an ergodic invariant Radon measure for the Γ action on $\partial\mathbb{D} \times \mathbb{R}$ given by $g : (e^{i\theta}, s) \mapsto (g(e^{i\theta}), s - \log |g'(e^{i\theta})|)$ $(g \in \Gamma)$. Then either $m[\mathrm{Par}(\Gamma) \times \mathbb{R}] \neq 0$, or the group

$$H_m := \{s \in \mathbb{R} : m \circ g^s \sim m\}, \text{ where } g^s : (e^{i\theta}, s') \mapsto (e^{i\theta}, s' + s)$$

is equal to $\mathbb{R}$.

Explanation: Every horocycle invariant measure on $T^1(M)$, $M = \Gamma \setminus \mathbb{D}$, lifts to a horocycle invariant, Γ–invariant, measure on $T^1(\mathbb{D})$. Recall that $T^1(\mathbb{D})$ can be identified with $\partial\mathbb{D}\times\mathbb{R}\times\mathbb{R}$ via $(e^{i\theta}, s, t) \leftrightarrow (h^t \circ g^s)(\omega(e^{i\theta}))$, where h^t is the horocycle flow, g^s is the geodesic flow, and $\omega(e^{i\theta})$ is the element of $T^1(\mathbb{D})$ based at the origin, and pointing at $e^{i\theta}$.

In these coordinates, the horocycle flow is the translation on the last coordinate, so every horocycle invariant measure must be of the form $dm(e^{i\theta}, s)dt$. Γ–invariance and horocycle–ergodicity translate to invariance and ergodicity of m under the action $g : (e^{i\theta}, s) \mapsto (g(e^{i\theta}), s - \log|g'(e^{i\theta}))$ (see [LS] for details).

If $m[\mathrm{Par}(\Gamma) \times \mathbb{R}] \neq 0$, then ergodicity considerations imply that $\exists(e^{i\theta}, s_0) \in \mathrm{Par}(\Gamma) \times \mathbb{R}$ such that m is supported on $\{g(e^{i\theta_0}, s_0) : g \in \Gamma\}$. This means that our original measure is supported on the projection of the horocycle tangent to $\partial\mathbb{D}$ at the parabolic fixed point $e^{i\theta_0}$. Such a measure is trivial.

If $H_m = \mathbb{R}$, then m is quasi–invariant under the translation flow on the second coordinate. This means that $dmdt$ is quasi–invariant under the flow $\varphi^s(e^{i\theta'}, s', t') = (e^{i\theta'}, s' + s, t'e^{-s})$. This is the geodesic flow in our system of coordinates, so we get quasi–invariance under the geodesic flow.

The dichotomy "$H_m = \mathbb{R}$ or $m(\mathrm{Par}(\Gamma)\times\mathbb{R}) \neq 0$" thus translates to the dichotomy "quasi-invariant or trivial".

From now on we assume that m is a Radon measure on $\partial\mathbb{D}\times\mathbb{R}$, which is ergodic and invariant for the Γ action mentioned above, such that $m[\mathrm{Par}(\Gamma) \times \mathbb{R}] = 0$. Our aim is to show $H_m = \mathbb{R}$.

Step 2. If $m[\mathrm{Par}(\Gamma) \times \mathbb{R}] = 0$, then m is non-atomic.

Proof. Suppose the contrary, then $\exists(e^{i\theta_0}, s_0)$ such that $m\{(e^{i\theta_0}, s_0)\} \neq 0$. Use the first part of the holonomy lemma with fixed $n_0 > 10M_0/\alpha_0$ and a shrinking sequence $\varepsilon = 1/j$ $(j \geq 1)$ to construct holonomies κ_j such that

- $|\kappa_j(e^{i\theta_0}) - e^{i\theta_0}| \xrightarrow[j\to\infty]{} 0$;
- $R(e^{i\theta_0}, \kappa_j(e^{i\theta_0})) \neq 0$;
- $n_0\alpha_0 - M_0 \leq R(e^{i\theta_0}, \kappa_j(e^{i\theta_0})) \leq n_0\alpha_0 + M_0$.

The first two properties imply that $A := \{(\kappa_j(e^{i\theta_0}), s_0 + R(e^{i\theta_0}, \kappa_j(e^{i\theta_0})) : j \geq 1\}$ is infinite. Therefore it has infinite measure. But the third property implies that A is pre-compact, so we get a contradiction to the Radon property of m.

Step 3. $\exists$ a Borel function $u : \partial\mathbb{D} \to \mathbb{R}$ such that if $\vartheta : (e^{i\theta}, s) \mapsto (e^{i\theta}, s - u(e^{i\theta}))$, then $m \circ \vartheta^{-1}$ is supported on $\partial\mathbb{D} \times H_m$.

Proof: [Sa], Theorem 2.

Step 4. The u in step 3 can be chosen so that $\operatorname{ess\,sup}|u| < \infty$.

Proof. H_m is a subgroup of $\mathbb{R}$, and this subgroup is closed.[3] The only possibilities are $H_m = \mathbb{R}, c_0\mathbb{Z}$, and $\{0\}$. In the first case we can take $u \equiv 0$, and in the second $u \bmod |c_0|$. It remains to treat the case $H_m = \{0\}$.

If $H_m = \{0\}$, then m is carried by the graph of $u : \partial\mathbb{D} \to \mathbb{R}$. We claim that $\operatorname{ess\,sup}|u|$ is finite. Assume by way of contradiction that $\operatorname{ess\,sup}|u| = \infty$.

[3]Proof [ANSS]: m, $m \circ g^s$ are two Γ–invariant ergodic Radon measures. They are equivalent iff they are proportional. Saying that $s \in H_m$ is saying that $\exists c(s) \in (0, \infty)$ such that $m(F) = c(s)m(F \circ g^s)$ for all F continuous with compact support. This a closed condition on s.

Let M_0 and α_0 be the constants in the first part of the holonomy lemma. There must exist two closed intervals $I, J \subset \mathbb{R}$ and a natural number n such that

(a) $I \cap J = \varnothing$;
(b) $N_{M_0}(I + n\alpha_0) \subset J$ (here and throughout $N_{M_0}(\cdot)$ is the M_0–neighborhood);
(c) $m(\partial\mathbb{D} \times I), m(\partial\mathbb{D} \times J) \neq 0$.

To see this write $\partial\mathbb{D}\times\mathbb{R} = \biguplus_{k\in\mathbb{Z}} \partial\mathbb{D}\times[k\alpha_0, (k+1)\alpha_0)$, and note that $\exists k_1, k_2$ such that $(k_2-k_1)\alpha_0 > 2M_0+\alpha_0$ and $m\big[\partial\mathbb{D}\times[k_i\alpha_0, (k_i+1)\alpha_0)\big] \neq 0$ $(i = 1, 2)$ (otherwise the support of m is compact, in contradiction to $\operatorname{ess\,sup}|u| = \infty$). Choose $n = k_2 - k_1$, $I := [k_1\alpha_0, (k_1 + 1)\alpha_0]$, and $J := [k_2\alpha_0 - M_0, (k_2 + 1)\alpha_0 + M_0]$.

Our definition of I, J means that $\mu_I(E) := m(E \times I)$, $\mu_J(E) := m(E \times J)$ are finite mutually singular measures on $\partial\mathbb{D}$. They are mutually singular, because $u \in I$ μ_I–a.e., $u \in J$ μ_J–a.e., and $I \cap J = \varnothing$.

Since $\mu_I \perp \mu_J$, there exists a closed arc $A \subset \partial\mathbb{D}$ such that[4]

$$0 \neq 2\mu_J(A) < \mu_I(A).$$

Since A is closed, $\exists\varepsilon > 0$ such that

$$\mu_J[N_\varepsilon(A)] < \frac{3}{2}\mu_J(A).$$

Use the first half of the holonomy lemma to construct a Γ–holonomy κ s.t. $m[(\mathrm{dom}(\kappa) \times \mathbb{R})^c] = 0$, and

$$|\kappa(e^{i\theta}) - e^{i\theta}| < \varepsilon \text{ and } |R(e^{i\theta}, \kappa(e^{i\theta})) - n\alpha_0| \leq M_0 \ m\text{–almost everywhere.}$$

Since κ is a Γ–holonomy, the map $\kappa_R : (e^{i\theta}, s) \mapsto \big(\kappa(e^{i\theta}), s + R(e^{i\theta}, \kappa(e^{i\theta}))\big)$ is measure preserving. Therefore

$$\begin{aligned}
2\mu_J(A) &\leq \mu_I(A) \equiv m(A \times I) = (m \circ \kappa_R)(A \times I) \\
&\leq m[N_\varepsilon(A) \times N_{M_0}(I + n\alpha_0)] && \text{(choice of } \kappa) \\
&\leq m[N_\varepsilon(A) \times J] \equiv \mu_J[N_\varepsilon(A)] \\
&\leq \frac{3}{2}\mu_J(A) && \text{(choice of } \varepsilon.)
\end{aligned}$$

This contradiction shows the impossibility of $\operatorname{ess\,sup}|u| = \infty$.

Step 5. Let u be as in the previous step. There is a number λ and a finite measure ν on $\partial\mathbb{D}$, such that $dm \circ \vartheta^{-1} = e^{\lambda s} d\nu(e^{i\theta}) dm_{H_m}(s)$, where m_{H_m} is a Haar measure for H_m.

Proof. Let $u : \partial\mathbb{D} \to \mathbb{R}$ be the function constructed in the previous two steps. The change of coordinates $\vartheta : (e^{i\theta}, s) \mapsto (e^{i\theta}, s - u(e^{i\theta}))$ transforms the measure m into a measure $m \circ \vartheta^{-1}$ supported on $\partial\mathbb{D} \times H_m$. The essential boundedness of u guarantees that $m \circ \vartheta^{-1}$ is Radon.

By the definition of H_m, and the ergodicity and invariance of m, there exists a constant λ such that $m \circ g^s = e^{\lambda s} m$ for all $s \in H_m$. This means that $e^{-\lambda s} m \circ \vartheta^{-1}$ is invariant under the translation of the second coordinate by any element of H_m.

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[4]Proof: Otherwise $\mu_I(A) \leq 2\mu_J(A)$ for all closed arcs. Suppose $N \subset \partial\mathbb{D}$ satisfies $\mu_J(N) = 0$. N can be covered by a countable collection of open arcs A_i s.t. $\sum \mu_J(A_i) < \varepsilon$. Find two sequence of points in each A_i, none of which equal to an atom of μ_J, such that one sequence tends to the right endpoint of A_i, and the other to the left endpoint of A_i. Use these sequences to partition A_i into a countable union of *closed* arcs A_{ij}, which are pairwise disjoint modulo the measure μ_J. We get $N \subset \bigcup_{i,j} A_{ij}$, and since A_{ij} are closed, $\sum_{i,j} \mu_I(A_{ij}) \leq 2\sum_{i,j} \mu_J(A_{ij}) = 2\sum_i \mu_J(A_i) < 2\varepsilon$. This argument shows that $\mu_J(N) = 0 \Rightarrow \mu_I(N) = 0$, in contradiction to $\mu_I \perp \mu_J$.

This means that $(m \circ \vartheta^{-1}) \circ g^s = e^{\lambda s} d\nu(e^{i\theta}) dm_{H_m}(s)$ where ν is a finite measure on $\partial\mathbb{D}$, and m_{H_m} is the Haar measure of H_m. The Γ–invariance of m forces

$$\frac{d\nu \circ g}{d\nu} = |g'|^\lambda \exp(\lambda u \circ g - \lambda u).$$

Step 6. $H_m = \mathbb{R}$.

Proof. Since H_m is a closed subgroup of $\mathbb{R}$, it is enough to show that $H_m \not\subseteq c_0\mathbb{Z}$ for any $c_0 > 0$. Assume by way of contradiction that $H_m \subseteq c_0\mathbb{Z}$ for some $c_0 > 0$.

We construct a Γ–holonomy κ with $m[(\mathrm{dom}\,\kappa \times \mathbb{R})^c] = 0$ s.t. the set

$$E := \{e^{i\theta} : R(e^{i\theta}, \kappa(e^{i\theta})) + u(e^{i\theta}) - u(\kappa(e^{i\theta})) \notin c_0\mathbb{Z}\}$$

has positive ν–measure. This will give us the required contradiction, because if we define the measure preserving $\kappa_R : (e^{i\theta}, s) \mapsto \big(\kappa(e^{i\theta}), s + R(e^{i\theta}, \kappa(e^{i\theta}))\big)$, then $(\vartheta \circ \kappa_R \circ \vartheta^{-1})(E \times \{0\}) \subset \partial\mathbb{D} \times (\mathbb{R} \setminus c_0\mathbb{Z}) \subseteq (\partial\mathbb{D} \times H_m)^c$, so

$$\begin{aligned} 0 \neq \nu(E) m_{H_m}(\{0\}) &= (m \circ \vartheta^{-1})(E \times \{0\}) \\ &= [(m \circ \vartheta^{-1}) \circ (\vartheta \circ \kappa_R \circ \vartheta^{-1})](E \times \{0\}) \leq (m \circ \vartheta^{-1})[(\partial\mathbb{D} \times H_m)^c] = 0. \end{aligned}$$

Here is the construction. Recall the constants $\varepsilon(c_0), M(c_0)$ from part 2 of the holonomy lemma. Fix some $u_0 \in \mathbb{R}$ such that $A := \{e^{i\theta} : |u(e^{i\theta}) - u_0| < \frac{1}{4}\varepsilon(c_0)\}$ has positive ν–measure. Fix $\delta > 0$ so small that

$$(1-\delta)(1-2\delta)(1 + \exp[-\lambda M(c_0) - 2\lambda\, \mathrm{ess\,sup}\,|u|]) > 1. \tag{2.1}$$

Choose a closed arc $B \subset \partial\mathbb{D}$ such that[5]

$$\nu(A \cap B) > (1-\delta)\nu(B),$$

and using the fact that B is closed, choose $\varepsilon > 0$ so small that

$$\nu(B) \geq (1-2\delta)\nu[N_\varepsilon(B)].$$

Having chosen $\varepsilon > 0$, we now use the second half of the holonomy lemma to construct a Γ–holonomy κ s.t. $m[(\mathrm{dom}\,\kappa \times \mathbb{R})^c] = 0$, and s.t. for a.e. $(e^{i\theta}, s)$,

$$|\kappa(e^{i\theta}) - e^{i\theta}| < \varepsilon, \big|R(e^{i\theta}, \kappa(e^{i\theta}))\big| \leq M(c_0), \text{ and } \mathrm{dist}\big(R(e^{i\theta}, \kappa(e^{i\theta})), c_0\mathbb{Z}\big) > \varepsilon(c_0).$$

Define $C := A \cap B$. We claim that $C \cap \kappa^{-1}(C)$ is a subset of E, and that this subset has positive ν–measure.

To see that $C \cap \kappa^{-1}(C)$ has positive measure we first observe that

$$\frac{d\nu \circ \kappa}{d\nu} = \exp[-\lambda R(e^{i\theta}, \kappa(e^{i\theta})) + \lambda u \circ \kappa - \lambda u] \geq e^{-\lambda M(c_0) - 2\lambda\, \mathrm{ess\,sup}\,|u|}.$$

Using the identity $\nu[\kappa(C)] = \int_C \frac{d\nu \circ \kappa}{d\nu} d\nu$ we see that

$$\begin{aligned} \nu[\kappa(C)] &= \int_C \frac{d\nu \circ \kappa}{d\nu} d\nu \geq e^{-\lambda M(c_0) - 2\lambda\, \mathrm{ess\,sup}\,|u|}\nu(C) \\ &= e^{-\lambda M(c_0) - 2\lambda\, \mathrm{ess\,sup}\,|u|}\nu(A \cap B) \geq e^{-\lambda M(c_0) - 2\lambda\, \mathrm{ess\,sup}\,|u|}(1-\delta)\nu(B) \\ &\geq e^{-\lambda M(c_0) - 2\lambda\, \mathrm{ess\,sup}\,|u|}(1-\delta)(1-2\delta)\nu[N_\varepsilon(B)], \end{aligned}$$

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[5]To see that such a closed arc exists normalize ν to have measure one, and construct a nested sequence of partitions α_n of $\partial\mathbb{D}$ into arcs of length $\leq 2^{-n}$ whose endpoints are not atoms of ν. Let $\mathcal{F}_n$:=sigma–algebra generated by α_n. By the martingale theorem, $\mathbb{E}(1_A|\mathcal{F}_n) \xrightarrow[n\to\infty]{} 1_A$ ν–almost surely. Since $\mathbb{E}(1_A|\mathcal{F}_n)(x) = \nu(A|B)$ for the atom $B \in \mathcal{F}_n$ which contains x, there must be some element like this with $\nu(A|B) > 1 - \delta$. By construction the atoms of $\mathcal{F}_n$ are equal modulo ν to closed arcs, so we are done.

whence by (2.1)

$$\nu(C)+\nu[\kappa(C)]\geq(1+e^{-\lambda M(c_0)-2\lambda\operatorname{ess\,sup}|u|})(1-\delta)(1-2\delta)\nu[N_\varepsilon(B)]>\nu[N_\varepsilon(B)].$$

But by construction $C\subseteq B$ and $\kappa(C)\subset N_\varepsilon(C)\subseteq N_\varepsilon(B)$, so

$$\nu[C\cup\kappa(C)]\leq\nu[N_\varepsilon(B)]<\nu(C)+\nu[\kappa(C)].$$

This means that $\nu[C\cap\kappa(C)]\neq 0$. Since $\nu\circ\kappa^{-1}\sim\nu$, $\nu[C\cap\kappa^{-1}(C)]\neq 0$.

We finish the proof by showing that $E\supseteq C\cap\kappa^{-1}(C)\bmod\nu$: If $e^{i\theta},\kappa(e^{i\theta})\in C\cap\operatorname{dom}\kappa$, then

$$\operatorname{dist}\big(R(e^{i\theta},\kappa(e^{i\theta})),c_0\mathbb{Z}\big)\geq\varepsilon(c_0) \qquad \text{by the choice of } \kappa,$$

$$|u(e^{i\theta})-u(\kappa(e^{i\theta}))|\leq 2\cdot\frac{\varepsilon(c_0)}{4}=\frac{\varepsilon(c_0)}{2} \qquad \text{by the choice of } A,$$

so $R(e^{i\theta},\kappa(e^{i\theta}))+u(e^{i\theta})-u(\kappa(e^{i\theta}))\notin c_0\mathbb{Z}$, and $e^{i\theta}\in E$. □

2.3. **The proof of theorem 1 implies theorem 2.** This was demonstrated by Babillot in the special case when Γ is a normal subgroup of a uniform lattice with nilpotent quotient [Ba], but her argument works verbatim for all fuchsian groups whose Poincaré exponent is larger than or equal to 1/2 (see [LS] for details).

We show that every weakly tame fuchsian group Γ has Poincaré exponent larger than or equal to 1/2.

Suppose that Γ has Poincaré exponent less than 1/2. Patterson has proved that in this case

$$\sum_{g\in\Gamma}|g'(e^{i\theta})|<\infty \text{ Lebesgue almost surely in } \partial\mathbb{D}, \tag{2.2}$$

([Pa1], proof of theorem 3, equation 18).

Let $m:=e^s d\theta ds$. This is a Γ–invariant Radon measure on $\partial\mathbb{D}\times\mathbb{R}$. Let $m=\int m_y d\pi(y)$ be its ergodic decomposition with respect to action of Γ (see [Schm] for a discussion of ergodic decompositions for non-singular group actions). Evidently, for almost every ergodic component m_y,

- m_y is Radon (because the topology of $\partial\mathbb{D}\times\mathbb{R}$ has a countable basis made of precompact open sets),
- $m_y[\operatorname{Fix}(\Gamma)\times\mathbb{R}]=0$,
- $\sum_{g\in\Gamma}|g'(e^{i\theta})|<\infty$ m_y–almost surely.

In particular, $u(e^{i\theta}):=\log\sum_{g\in\Gamma}|g'(e^{i\theta})|$ makes sense m_y–almost surely. We calculate and find that for m_y–a.e. $e^{i\theta}$ and all $g\in\Gamma$,

$$R(e^{i\theta},g\cdot e^{i\theta})+u(e^{i\theta})-u(g\cdot e^{i\theta'})=0.$$

This equation implies that $F(e^{i\theta},s):=s-u(e^{i\theta})$ is Γ–invariant, whence equal to come constant c_y m_y–almost surely, for π–almost every y. This means that almost every ergodic component m_y is carried by a graph $\{(e^{i\theta},c_y+u(e^{i\theta})):e^{i\theta}\in\partial\mathbb{D}\}$. It follows that for π–a.e. y, $m_y\circ g^s\perp m_y$ for all $s\neq 0$, whence $H_{m_y}=\{0\}$.

But this is impossible, because the proof of theorem 1 shows that if Γ is weakly tame, then every Γ–ergodic invariant Radon measure μ on $\partial\mathbb{D}\times\mathbb{R}$ such that $\mu[\operatorname{Par}(\Gamma)\times\mathbb{R}]=0$ satisfies $H_\mu=\mathbb{R}$. The contradiction shows that the Poincaré exponent of a weakly tame group is no less than 1/2, whence theorem 2. □

2.4. **The proof of theorem 1 implies theorem 3.** Take a fuchsian group of the first kind whose Poincaré exponent is less than 1/2 (such groups exist, see Patterson [Pa2]). Let m be the volume measure. As we just saw, almost every ergodic component m_y of m satisfies $H_{m_y} = \{0\}$ and $m_y[\mathrm{Par}(\Gamma) \times \mathbb{R}] = 0$, and thus corresponds to non-trivial horocycle ergodic invariant Radon measures which is not quasi–invariant under the geodesic flow (and therefore does not arise via Babillot's bijection from eigenfunctions of the laplacian). This proves theorem 3. □

3. Geometric preparations to the proof of the holonomy lemma

3.1. **Pairs of pants and hyperbolic octagons.** The material in this subsection is standard, see e.g. [Se1] or [Hub]. Suppose Y is a pair of pants with boundary components of non-zero lengths a, b and c. The *seams* of Y are defined to be the shortest geodesics connecting its boundary components. These meet the boundary components at right angles. If we cut Y along two of its seams (say those from b to a, c), then we get a right angled hyperbolic octagon O_Y (figure 1).

Denote the sides of O_Y, counterclockwise, by $\alpha, a, \overline{\alpha}, b_2, \overline{\beta}, c, \beta, b_1$, where $a, b = b_1 \cup b_2, c$ are the boundary components; $\alpha, \overline{\alpha}$ are the seam from a to $b := b_1 \cup b_2$; and $\beta, \overline{\beta}$ are the seam from c to b.

Let ℓ denote hyperbolic length. Clearly,

$$\ell(\alpha) = \ell(\overline{\alpha}) \text{ and } \ell(\beta) = \ell(\overline{\beta}).$$

The seam from a to c divides the octagon into two right–angled hexagons. The 'hexagon formula' ([Hub], exercise 3.5.7) implies that

$$\ell(b_1) = \ell(b_2),$$

and that if ℓ is the hyperbolic length of the geodesic segment connecting boundary components with lengths ℓ_1, ℓ_2, and ℓ_3 is the length of the other boundary component, then

$$\cosh \ell = \frac{\cosh \frac{\ell_3}{2} + \cosh \frac{\ell_1}{2} \cosh \frac{\ell_2}{2}}{\sinh \frac{\ell_1}{2} \sinh \frac{\ell_2}{2}}. \tag{3.1}$$

In particular, O_Y, whence Y, is determined up to isometry by $\ell(a), \ell(b), \ell(c)$.

Place O_Y in the hyperbolic disc $\mathbb{D}$. Each of the complete geodesics determined by the sides of O_Y divides $\mathbb{D}$ into two hyperbolic half–spaces. We say that a point in $\mathbb{D}$ is *under* geodesic x, ($x \in \{\alpha, \beta, \overline{\alpha}, \overline{\beta}\}$) if it lies in the hyperbolic half space determined by x which does *not* contain O_Y. Let

$$H(x) := \text{open hyperbolic half space under } x.$$

This is a *geodesically convex* set: if two points belong to it, so does the geodesic segment connecting them.

Recall that in hyperbolic geometry, the angles of a triangle add up to less than 180°. Thus if two infinite geodesics have a common geodesic perpendicular, then they do not intersect.

It follows that if x, y are two sides of O_Y, then $H(x) \cap H(y) \neq \varnothing$ iff x, y are adjacent. In particular, the geodesic extension of any side of O_Y can only intersect the geodesic extensions of the sides which are adjacent to it (figure 1).

Set $\mathcal{S} := \{\alpha, \overline{\alpha}, \beta, \overline{\beta}\}$, and define for $x \in \mathcal{S}$ the Möbius transformation φ_x which maps side x onto side $\overline{x}$ (with the convention $\overline{(\overline{x})} = x$). Then $\varphi_{\overline{x}} = \varphi_x^{-1}$.

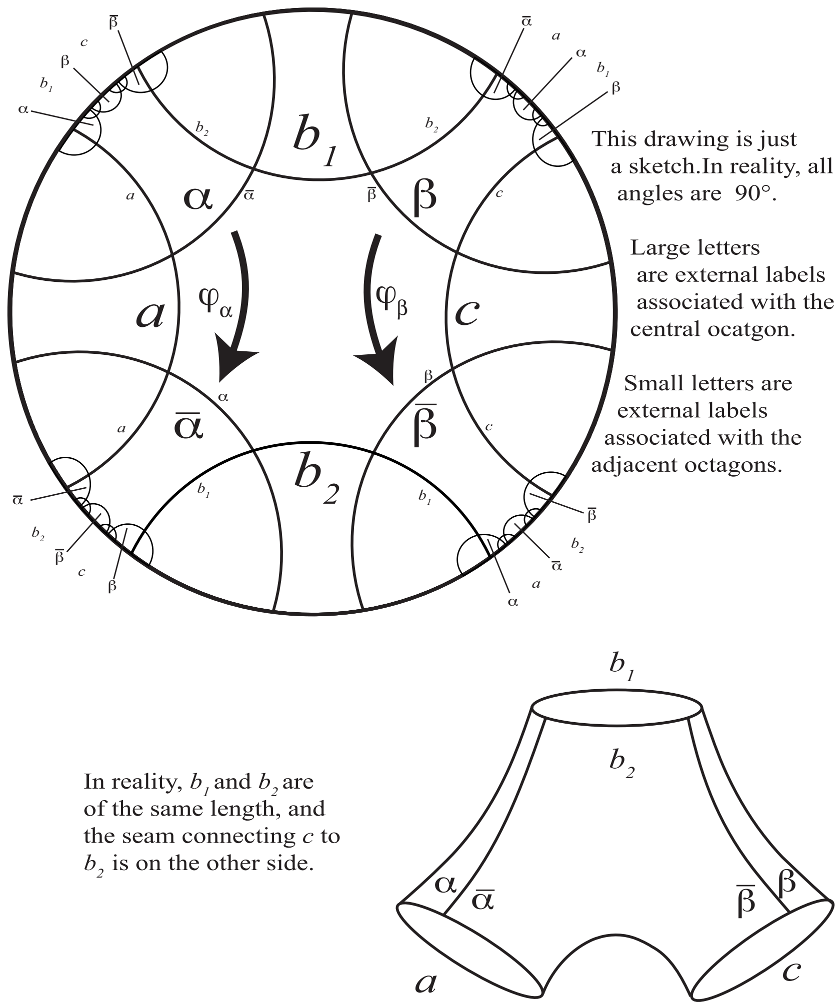

FIGURE 1. A pair of pants (with an indication – out of scale – of the image under φ_β)

The combinatorics of the action of $\{\varphi_x : x \in \mathcal{S}\}$ on the sides of O_Y and the half spaces they determine is summarized pictorially in figure 1 (see also lemma 4.6.2 in the appendix). The reader can verify it by noting that Möbius transformations preserve orientation and angles.

An immediate conclusion ('Klein's ping–pong argument') is that $\varphi_\alpha, \varphi_\beta$ generate a free group $\Gamma(O_Y)$, with fundamental domain

$$\widetilde{O}_Y := O_Y \cup \bigcup_{x=a,b_1,b_2,c} Q(x),$$

where $Q(x) :=$ relative closure in $H(x)$ of $H(x) \setminus \bigcup_{x \neq z \in \mathcal{S} \cup \{a,b_1,b_2,c\}} H(z)$. The complete hyperbolic surface $\Gamma(\widetilde{O}_Y) \setminus \mathbb{D}$ is the hyperbolic surface obtained from Y by extending it across a, b, c to 'funnels', represented by $Q(a), Q(b_1) \cup Q(b_2), Q(c)$.

A pair of pants with cusps is obtained in a similar way from a 'degenerate' right–angle hyperbolic 'octagon' with a, b or c collapsed to a point. It is convenient to abuse terminology and call these polygons right-angled octagons as well.

3.2. **Collars and cores.** A hyperbolic surface C is called a *hyperbolic cylinder* if it is isometric to $\Gamma \setminus \mathbb{H}$ where $\mathbb{H}$ is the hyperbolic upper-half plane, and $\Gamma = \langle g \rangle$ with g a hyperbolic isometry. Such a surface admits a unique closed geodesic γ_p (p for 'periodic'), whose length α is the translation length of g. The *η–collar* of γ_p is defined to be the set

$$C_\eta(\gamma_p) := \{q \in C : \operatorname{dist}(q, \gamma_p) < \eta\},$$

where dist denotes the hyperbolic metric.

More generally, a simple closed geodesic γ on a general complete hyperbolic surface is said to have an *η–collar*, if $\{q \in M : \operatorname{dist}(q, \gamma) < \eta\}$ is isometric to the η–collar of the the unique closed geodesic of some hyperbolic cylinder (which is unique up to isometry).

Proposition 3.2.1. *Let M be a complete hyperbolic surface. Every simple closed geodesic of length α on M has an $\eta(\alpha)$–collar, where $\eta(\alpha) = \ln\coth\frac{\alpha}{4}$. If γ_i are disjoint simple geodesics of lengths α_i, then their $\eta(\alpha_i)$–collars are disjoint.*

See [Hub] for a proof.

The *core* of a pair of pants is what is left from it after the collars of the boundary components are removed.

Proposition 3.2.2. *There is a function $M_{core} : \mathbb{R}^+ \to \mathbb{R}^+$ s.t. the diameter of the core of a pair of pants Y is less than $M_{core}(\|Y\|)$, where $\|Y\| := \max\{\ell(a), \ell(b), \ell(c)\}$.*

The proof is by direct application of proposition 3.2.1 and the hexagon formula.

The previous results persist in the limit $\alpha \to 0^+$: Any cusp in a complete hyperbolic surface X is encircled by a unique closed horocycle of length two, and the region bounded by this horocycle is isometric to $\{z \in \mathbb{H} : \operatorname{Im}(z) > \frac{1}{2}\}/\langle z \mapsto z+1\rangle$. We call this the *collar* of the cusp. The collar of a cusp is disjoint from the collars of all other cusps of simple closed geodesics ([Hub], proposition 3.8.9).

3.3. **Cutting sequences.** Suppose γ is a finite geodesic segment inside Y, whose endpoints are not on the projections of α, β. We wish to associate to γ a word on the alphabet $\{\alpha, \overline{\alpha}, \beta, \overline{\beta}\}$ which captures the way the segment winds inside Y.

Represent Y by a hyperbolic octagon O_Y, and lift γ to $\mathbb{D}$ in such a way that the lift $\widetilde{\gamma}$ starts at a point inside O_Y. Since γ does not leave Y, $\widetilde{\gamma}$ is completely contained in $\bigcup_{g \in \Gamma(O_Y)} g(O_Y)$.

Label the sides of O_Y from the outside as in figure 1, and call these labels *external labels.* We define the *internal label* of a side with external label x to be $\overline{x}$ (where as always $\overline{(\overline{x})} = x$). There is a unique $\Gamma(O_Y)$–invariant way of extending this labeling system to all $g(O)$, $g \in \Gamma(O_Y)$.

Definition 3 (Artin). *The* cutting sequence *of γ is the ordered list $(w_1, \ldots, w_k)$ of external labels of the sides of $g(O_Y)$, $g \in \Gamma$ that $\widetilde{\gamma}$ cuts on the way* in.

There is a very useful alternative characterization of $(w_1, \ldots, w_k)$, see [Se1, Se2]: $(w_1, \ldots, w_k)$ is the word of minimal length such that the endpoint of $\widetilde{\gamma}$ is in $(\varphi_{w_1} \cdots \varphi_{w_k})(O_Y)$. In particular, $(w_1, \ldots, w_k)$ does not contain combinations of the form $x\overline{x}$ or $\overline{x}x$.

Words on the alphabet $\mathcal{S}$ not containing pairs of the form $x\overline{x}$ or $\overline{x}x$ are called *reduced.* It is well known that every reduced word can appear as the cutting sequence of a geodesic segment lying on a geodesic which never leaves Y (see [Se1]).

The next proposition discusses the realizability of reduced words as cutting sequences of geodesics which *cross* a pair of pants Y, entering through boundary component a and leaving through c or b.

Proposition 3.3.1. *Let O_Y be a hyperbolic octagon representing a pair of pants Y, suppose $\underline{w} = (w_1, \ldots, w_k)$ is a reduced word on the alphabet $\mathcal{S}$, and let γ be a geodesic in $\mathbb{D}$.*

(1) *If $\gamma(-\infty) \in \varphi_\alpha^k[Q(a)]$, $\gamma(\infty) \in \varphi_\alpha^\ell \varphi_{w_1} \cdots \varphi_{w_k} \varphi_\beta^m [Q(c)]$, $w_1 \notin \{\alpha, \overline{\alpha}\}$, $w_k \notin \{\beta, \overline{\beta}\}$, then*
   (a) *$\gamma^* := \varphi_\alpha^{-\ell}[\gamma]$ enters O_Y through one of the sides $\alpha, a, \overline{\alpha}$;*
   (b) *$\gamma^\# := (\varphi_\alpha^\ell \varphi_{w_1} \cdots \varphi_{w_k})^{-1}[\gamma]$ leaves O_Y through one of the sides $\beta, c, \overline{\beta}$;*
   (c) *the cutting sequence of the projection of γ in Y contains the word $\underline{w}$.*
(2) *If $\gamma(-\infty) \in \varphi_\alpha^k[Q(a)]$, $\gamma(\infty) \in \varphi_\alpha^\ell \varphi_{w_1} \cdots \varphi_{w_k} (\varphi_\beta \varphi_{\overline{\alpha}})^m [Q(b_2) \cup \varphi_\beta Q(b_1)]$, $k \geq 2$, $|\ell|, |m|$ are maximal*[6]*, and $(\alpha^\ell, w_1, \ldots, w_k, (\beta\overline{\alpha})^m)$ is reduced, then*
   (a) *$\gamma^* := \varphi_\alpha^{-\ell}[\gamma]$ enters O_Y through one of the sides $\alpha, a, \overline{\alpha}$;*
   (b) *$\gamma^\# := (\varphi_\alpha^\ell \varphi_{w_1} \cdots \varphi_{w_k})^{-1}[\gamma]$ leaves O_Y through one of the sides $\overline{\alpha}, b_2, \overline{\beta}$;*
   (c) *the cutting sequence of the projection of γ in Y contains the word $\underline{w}$.*
(3) *If $\gamma(-\infty) \in \varphi_\alpha^k[Q(a)]$, $\gamma(\infty) \in \varphi_\alpha^\ell \varphi_{w_1} \cdots \varphi_{w_k} (\varphi_{\overline{\beta}} \varphi_\alpha)^m [Q(b_1) \cup \varphi_\beta^{-1} Q(b_2)]$, $k \geq 2$, $|\ell|, |m|$ are maximal, and $(\alpha^\ell, w_1, \ldots, w_k, (\overline{\beta}\alpha)^m)$ is reduced, then*
   (a) *$\gamma^* := \varphi_\alpha^{-\ell}[\gamma]$ enters O_Y through one of the sides $\alpha, a, \overline{\alpha}$;*
   (b) *$\gamma^\# := (\varphi_\alpha^\ell \varphi_{w_1} \cdots \varphi_{w_k})^{-1}[\gamma]$ leaves O_Y through one of the sides α, b_1, β;*
   (c) *the cutting sequence of the projection of γ to Y contains the word $\underline{w}$.*

The proof is delegated to the appendix.

3.4. **Shadowing constants.** Let γ_p be a closed geodesic in a pair of pants Y. We say that a geodesic segment γ in Y *ε–shadows γ_p during a time interval of length* T if there are unit tangent vectors $\omega \in \gamma$, $\omega_p \in \gamma_p$ with lifts $\widetilde{\omega}, \widetilde{\omega}_p \in O_Y \subset \mathbb{D}$ s.t.

$$\mathrm{dist}(g^s \widetilde{\omega}, g^s \widetilde{\omega}_p) \leq \varepsilon \text{ for all } 0 \leq s \leq T, \tag{3.2}$$

where $g^s : T^1(\mathbb{D}) \to T^1(\mathbb{D})$ is the geodesic flow. We remind the reader that dist mean the distance between the base points of the vectors.

We say that a geodesic segment is *rich* if its cutting sequence includes at least one of the words $\alpha\beta, \beta\alpha, \overline{\alpha}\overline{\beta}, \overline{\beta}\overline{\alpha}$.

[6]With the understanding that x^0 is the empty word, and $x^k = \overline{x}^{|k|}$ where $k < 0$.

Proposition 3.4.1. *$\exists \varepsilon_{sh}, t_{sh}, \ell_{sh} : \mathbb{R}^+ \to \mathbb{R}$ such that the following holds for all $L > 1$: Suppose Y is a pair of pants whose boundary components have lengths in $[L^{-1}, L]$, and let γ_p be a rich closed geodesic in Y with cutting sequence $\underline{w}$ and length $\ell(\gamma_p)$. Let γ be a geodesic which crosses Y.*

(a) *If γ $\varepsilon_{sh}(L)$–shadows γ_p during a time interval of length T, then the cutting sequence of γ in Y contains the word $\underline{w}^n$ with $n > T/\ell(\gamma_p) - 10$;*
(b) *If the cutting sequence of γ contains the word $\underline{w}^N$ with N so large that $(N-3)\ell(\gamma_p) > \ell_{sh}(L)$, then γ $\frac{1}{2}\varepsilon_{sh}(L)$–shadows γ_p in Y during a time interval of length $T > (N-3)\ell(\gamma_p) - t_{sh}(L)$.*

It is possible to make this construction with $\varepsilon_{sh}(\cdot)$ arbitrarily small.

Proof. Fix $L > 0$ and a pair of pants Y whose boundary components have lengths in $[L^{-1}, L]$. Let O_Y be the hyperbolic octagon which represents Y. We claim that there exists $\delta(L) > 0$ such that for every $(w_1, w_2) \in \{(\alpha, \beta), (\beta, \alpha), (\overline{\alpha}, \overline{\beta}), (\overline{\beta}, \overline{\alpha})\}$, the distance from side w_1 to side $\overline{w}_2$ is at least $\delta(L)$.

For every triplet of non-negative real numbers (ℓ_a, ℓ_b, ℓ_c) there exists a unique right angled hyperbolic octagon $O(\ell_a, \ell_b, \ell_c)$ such that (in the labeling of figure 1),

- $\ell(a) = \ell_a$, $\ell(b_1) = \ell(b_2) = \frac{1}{2}\ell_b$, $\ell(c) = \ell_c$;
- $H(\alpha) = \{z \in \mathbb{D} : \mathrm{Re}(z) < 0\}$;
- $H(a) = \{z \in \mathbb{D} : \mathrm{Im}(z) < 0\}$.

The vertices of $O(\ell_a, \ell_b, \ell_c)$ are continuous functions of (ℓ_a, ℓ_b, ℓ_c). So are the end-points of of the geodesic extensions of the sides of $O(\ell_a, \ell_b, \ell_c)$.

Fix $(w_1, w_2) \in \{(\alpha, \beta), (\beta, \alpha), (\overline{\alpha}, \overline{\beta}), (\overline{\beta}, \overline{\alpha})\}$. The geodesic extensions of w_1 and $\overline{w}_2$ cannot intersect or even touch at infinity, thus the distance them is a positive continuous function of the endpoints of these geodesics, whence a positive continuous function of (ℓ_a, ℓ_b, ℓ_c). Let $\delta_{w_1 w_2}(L)$ be the minimum of this positive continuous function on the compact set $\{(\ell_a, \ell_b, \ell_c) : L^{-1} \le \ell_a, \ell_b, \ell_c \le L\}$, and define

$$\delta(L) := \min\{\delta_{w_1 w_2}(L) : (w_1, w_2) = (\alpha, \beta), (\beta, \alpha), (\overline{\alpha}, \overline{\beta}), (\overline{\beta}, \overline{\alpha})\}.$$

We prove part (a). Choose $0 < \varepsilon_{sh}(t) < \frac{1}{4}\delta(t)$. Suppose γ $\varepsilon_{sh}(L)$–shadows a rich closed γ_p during a time interval of length T, then $\exists \widetilde{\omega}, \widetilde{\omega}_p \in T^1(O_Y)$ which project to tangent vectors to γ, γ_p, s.t. $\mathrm{dist}(g^s(\widetilde{\omega}), g^s(\widetilde{\omega}_p)) < \varepsilon_{sh}(L)$ for all $0 \le s \le T$.

The arc $\{g^s(\widetilde{\omega}_p)\}_{s=0}^T$ projects to γ_p, so if the sequence of copies of O_Y it intersects is $\varphi_{u_1}[O_Y], \varphi_{u_1}\varphi_{u_2}[O_Y], \ldots, (\varphi_{u_1} \cdots \varphi_{u_n})[O_Y]$, then $(u_1, \ldots, u_n) \in \mathcal{S}^n$ is reduced, and $(u_1, \ldots, u_n)$ contains $\underline{w}^N$ for some $N \ge T/\ell(\gamma_p) - 2$. We use the shadowing property to show that the cutting sequence of $\{g^s(\widetilde{\omega})\}_{s=0}^T$ contains the word $(u_{2+|\underline{w}|}, \ldots, u_{n-|\underline{w}|-1})$, whence the word $\underline{w}^{N-6}$ for $N \ge T/\ell(\gamma_p) - 2$.

Let $n_0 = |\underline{w}|$ and suppose $0 < s_1 < \cdots < s_n < T$ are the times when $g^{s_i}(\widetilde{\omega})$ crosses side u_i of $O_i := \varphi_{u_1} \cdots \varphi_{u_i}[O_Y]$.

Fix $1 + n_0 < i < n - n_0$. We claim that there are $1 \le j_1, j_2 \le n_0$ s.t. the $\varepsilon_{sh}(L)$–neighborhoods of the base points of $g^{s_{i-j_1}}(\widetilde{\omega}_p), g^{s_{i+j_2}}(\widetilde{\omega}_p)$ lie on two different sides of the geodesic extension of side u_i of O_i.

To construct j_1, use the assumption that γ_p is rich to find $1 \le j_1 \le n_0$ such that $(u_{i-j_1}, u_{i-j_1+1}) \in \{(\alpha, \beta), (\beta, \alpha), (\overline{\alpha}, \overline{\beta}), (\overline{\beta}, \overline{\alpha})\}$. The hyperbolic distance between sides u_{i-j_1} and $\overline{u}_{i-j_1+1}$ of O_{i-j_1} is at least $\delta(L) > 4\varepsilon_{sh}(L)$, thus the $\varepsilon_{sh}(L)$–neighborhood of $g^{s_{i-j_1}}(\widetilde{\omega}_p)$ is below side u_{i-j_1+1} of O_{i-j_1+1} . Therefore it is below side u_i of O_i. In the same way one constructs $1 \le j_2 \le n_0$ such that the $\varepsilon_{sh}(L)$–neighborhood of $g^{s_{i+j_2}}(\widetilde{\omega}_p)$ is above side u_i of O_i.

The geodesic segment $[g^{s_{i-j_1}}(\widetilde{\omega}), g^{s_{i+j_2}}(\widetilde{\omega})]$ connects these neighborhoods, therefore it crosses the geodesic extension of side u_i of O_i. The crossing point is on side u_i, not just its geodesic extension, because our geodesic segment lies inside Y.

In summary, the geodesic $\{g^s(\widetilde{\omega})\}_{s=0}^T$ moves from O_{i-1} to O_i through side u_i of O_i for every $1+n_0 < i < n-n_0$. Since a geodesic in $\mathbb{D}$ can only enter a hyperbolic octagon once, this forces the cutting sequence of γ to contain the word $(u_{2+n_0}, \ldots, u_{n-n_0-1})$, proving part (a).

We prove part (b). Suppose that the cutting sequence of γ contains the word $\underline{v}^N$ where $\underline{v}$ is the cutting sequence of a rich closed geodesic γ_p. Then the cutting sequence of γ contains the word $\underline{w}^{N-2}$ where $\underline{w}$ is the cutting sequence of γ_p, and $(w_1, w_2) \in \{(\alpha,\beta), (\beta,\alpha), (\overline{\alpha},\overline{\beta}), (\overline{\beta},\overline{\alpha})\}$. ($\underline{w}$ is a cyclic permutation of $\underline{v}$.)

Choose $\omega \in \gamma$ such that $\{g^s(\omega)\}_{s=0^-}^{T^+}$ has cutting sequence $\underline{w}^{N-2}$ and which hits side w_1 at time zero. Choose $\omega_p \in \gamma_p$ such that $\{g^s(\omega_p)\}_{s=0^-}^{[(N-2)\ell(\gamma_p)]^+}$ has cutting sequence $\underline{w}^{N-2}$ and which hits side w_1 at time zero. Lift ω, ω_p to $\widetilde{\omega}$, $\widetilde{\omega}_p \in O_Y$.

Every collar of a boundary geodesic in a pair of pants Y whose boundary components have lengths in $[L^{-1}, L]$ has diameter less than $d(L) := 2\eta(L^{-1}) + M_{core}(L)$.

Any geodesic whose cutting sequence starts with (w_1, w_2) must enter the core of O_Y at most $3d(L)$–units of time after hitting side w_1, because any geodesic which spends more than $3d(L)$ units of time at a collar of Y must have cutting sequence which starts with the word (x,x) for some $x \in \mathcal{S}$. But (w_1,w_2) is not of this form. This means that

(a) $\exists T', T'_p \in [0, 3d(L)]$ such that $g^{T'}(\widetilde{\omega}), g^{T'_p}(\widetilde{\omega}_p)$ are in the core of O_Y;
(b) $\exists T''_p \in [(N-3)\ell(\gamma_p), (N-3)\ell(\gamma_p) + 3d(L)]$ s.t. $g^{T''_p}(\widetilde{\omega}_p)$ is the core of $(\varphi_{w_1} \cdots \varphi_{w_{|\underline{w}|}})^{N-3}[O_Y]$;
(c) $\exists T'' \in [(N-3)\ell(\gamma_p) - M_{core}(L), (N-3)\ell(\gamma_p) + 3d(L) + M_{core}(L)]$ s.t. $g^{T''}(\widetilde{\omega})$ is in the core of $(\varphi_{w_1} \cdots \varphi_{w_{|\underline{w}|}})^{N-3}[O_Y]$.

In particular,

$$\mathrm{dist}(g^{T'}(\widetilde{\omega}), g^{T'_p}(\widetilde{\omega}_p)) < M_{core}(L) \text{ and } \mathrm{dist}(g^{T''}(\widetilde{\omega}), g^{T''_p}(\widetilde{\omega}_p)) < M_{core}(L);$$

$$\begin{aligned} |T''_p - T'_p| &\geq (N-3)\ell(\gamma_p) - 3d(L); \\ |T'' - T'| &\geq (N-3)\ell(\gamma_p) - [3d(L) + M_{core}(L)]. \end{aligned}$$

Part (b) now follows from the following (standard) fact from hyperbolic geometry: For every $\varepsilon > 0, c_0 > 0$ there are constants $\ell_0 = \ell_0(\varepsilon, c_0)$ and $t_0 = t_0(\varepsilon, c_0)$ such that if γ_1, γ_2 are two geodesic segments in $\mathbb{D}$ whose lengths are more than ℓ_0, and whose respective endpoints are c_0–close, then γ_1 contains a subsegment of length $\ell(\gamma_1) - t_0$ which ε–shadows a subsegment of γ_2. □

The following proposition treats collars without assuming that the length of the boundary component at their center is bounded from below.

Proposition 3.4.2. $\exists \varepsilon_{sh}^{col}, c_0^* > 0$ *s.t. the following holds for all collars C of cusps or of closed geodesics of length less than c_0^*: Suppose two geodesics γ_1, γ_2 enter C and ε_{sh}^{col}–shadow each other during their sojourn there. Let $k_i \geq 0$ be the lengths of the cutting sequences of γ_i in C; then $|k_1 - k_2| < 10$ or $k_1 = k_2 = \infty$.*

Proof. We begin with the case of a collar of a closed geodesic of length $c > 0$. We work in the upper half plane, using a lift which lifts the closed geodesic to the upper half of the y–axis. The collar lifts to $\{\xi + i\eta : \eta > s|x|\}$ where $s = s(c)$ is determined by $s = \tan\theta$, with θ the solution of $\mathrm{dist}(e^{i\theta}, i) = \eta(c)$. We may choose the lift in such a way that the seam of the collar lifts to the infinite family of curves $\{z : |z| = e^{kc}, k \in \mathbb{Z}\}$.

Using the identity $\sinh[\frac{1}{2}\mathrm{dist}(z,w)] = |z-w|/2\sqrt{\mathrm{Im}(z)\mathrm{Im}(w)}$ ([K], theorem 1.2.6), one checks that $s(c) = \tan\left[\arcsin\left(\frac{2\coth\frac{c}{4}}{1+\coth^2\frac{c}{4}}\right)\right] \sim \frac{1}{2}c$ as $c \to 0^+$.

Denote the geodesic from $x \in \partial\mathbb{H}$ to $y \in \partial\mathbb{H}$ by $\gamma[x,y]$. Suppose $\gamma[x,y]$ intersects the collar. Let $z \in \mathbb{H} \cup \{0,\infty\}$ be the entry point, and $w \in \mathbb{H} \cup \{\infty, 0\}$ the exit point. Then $z, w \in \{\xi + i\eta : \eta = s|\xi|\} \cup \{\infty, 0\}$, and the length k of the cutting sequence in the collar satisfies $\left|k - \left|\frac{1}{c}\ln|z/w|\right|\right| \le 3$, with the understanding that if $z = 0, \infty$ or $w = 0, \infty$, then $k = \infty$.

We compare $|z|$ to $|x|$. Write $z = \xi + i\eta \equiv \xi(1 \pm isx)$ (the sign depends on the sign of x). Let $\gamma[x, y_0]$ be semi-circle emanating from x, and tangent to the boundary of the collar. Denote its center by c_0, its radius by r_0, and let $z_0 = \xi_0 + i\eta_0$ be the point where it is tangent to the boundary of the collar. Since $\gamma[x,y]$ enters the collar, it lies above $\gamma[x,y_0]$, and this implies that $|z_0| \le |z|$ and

$$|x - \xi| \le |x - \xi_0| \le 2r_0 = 2|c_0 - z_0| = 2|z_0|\tan\theta, \quad \because \overrightarrow{c_0 z_0} \perp \overrightarrow{0 z_0} \text{ and } \measuredangle z_0 0 c_0 = \theta.$$

Since $|z_0| \le |z|$, $|x - \xi| \le 2s|z|$. Dividing by $|z|$, and recalling that $|\xi| = |z|/\sqrt{1+s^2}$, we see that $1/\sqrt{1+s^2} - 2s < |x/z| < 1/\sqrt{1+s^2} + 2s$. Now $s(c) \sim \frac{1}{2}c$ as $c \to 0^+$, so routine estimates show that $\exists c_0^*$ such that $c < c_0^* \Rightarrow |x| = e^{\pm 2c}|z|$. In the same way $c < c_0^* \Rightarrow |y| = e^{\pm 2c}|w|$.

We conclude that if $\gamma[x,y]$ enters the collar of a closed geodesic of length $c < c_0^*$, then the length k of its cutting sequence in the collar satisfies

$$\left|k - \left|\frac{1}{c}\ln|x/y|\right|\right| \le 3 + 4c. \tag{3.3}$$

Now suppose $\gamma_1 := \gamma[x_1, y_1]$ δ–shadows $\gamma_2 := \gamma[x_2, y_2]$ in the collar. Let $z_2 := \xi_2 + i\eta_2$ be the exit point of γ_2 from the collar. We know that γ_1, γ_2 pass through $B := \{z : \mathrm{dist}(z, z_2) < \delta\}$. Therefore γ_1, γ_2 enter the smallest hyperbolic half space which contains B, which is $H := \{z : |z - \xi_2| < e^\delta \eta_2\}$. This means that

$$\begin{aligned} |y_1| &= |\xi_2| \pm |y_1 - \xi_2| = |\xi_2| \pm e^\delta \eta_2 = [1 \pm s(c)e^\delta]|\xi_2|, \\ |y_2| &= [1 \pm s(c)e^\delta]|\xi_2|, \\ \therefore \quad \left|\ln\frac{|y_1|}{|y_2|}\right| &\le \ln\left|\frac{1 + s(c)e^\delta}{1 - s(c)e^\delta}\right| = e^\delta c + o(c), \quad \text{as } c \to 0^+. \end{aligned}$$

In the same way one shows that $\left|\ln\frac{|x_1|}{|x_2|}\right| \le e^\delta c + o(c)$. We see that

$$|k_1 - k_2| \le \left|\frac{1}{c}\ln\frac{|x_2 y_1|}{|y_2 x_1|}\right| + [6 + 8c] \le 2e^\delta c + 6 + o(1), \quad \text{as } c \to 0^+.$$

If we choose c_0^* and ε_{sh}^{col} sufficiently small, then $\delta < \varepsilon_{sh}^{col}$, $c < c_0^*$ imply $|k_1 - k_2| < 10$, which finishes the proof in the case of collars of short closed geodesics.

We indicate briefly the modifications needed for dealing with collars of cusps. Working in the upper half plane, we choose a lift so that the cusp is at ∞, the collar is $\{\xi + i\eta : \eta > \frac{1}{2}\}$, and the seam of the collar is the union of the geodesics $\xi = n$, $n \in \mathbb{Z}$.

Suppose $\gamma[x,y]$ enters and exists the collar through z,w; then the collar cutting sequence has length k where $|k-|z-w||\leq 2$ (if x or y are infinite, then $k=0$). We claim that $|x-z|,|y-w|<1$. Once this is shown we have $|k-|x-y||<4$, an analogue of (3.3), which allows us to continue as before.

Suppose $z=\xi+i\eta$ is the point of entry of γ to the collar, and let $c:=(x+y)/2$ and $r:=|y-x|/2$, the center and radius of the semi-circle $\gamma[x,y]$. Since $\gamma[x,y]$ enters the collar, $r>1/2$, and $\alpha:=\measuredangle xcz<\frac{\pi}{2}$. Now $\measuredangle xz\xi=\frac{\pi}{2}-\frac{\pi-\alpha}{2}=\frac{\alpha}{2}$ ($\triangle xcz$ is an isosceles). Thus $|x-z|=(1/2)/\cos\frac{\alpha}{2}\leq(2\cos\frac{\pi}{4})^{-1}=\frac{\sqrt{2}}{2}<1$. In the same way one shows that $|y-w|<1$. □

The proof shows that if we choose c_0^* to be small enough, then the following corollary holds

Corollary 3.4.1. *Suppose γ_1,γ_2 are two geodesics such that γ_2 enters the collar of a closed geodesic of length $0<c<c_0^*$, and γ_1 δ–shadows γ_2 during its sojourn in the collar. If $k_i\geq 0$ is the length of the cutting sequence of γ_i in the collar, then*

$$|k_1-k_2|\leq 2\ln\left|\frac{1+e^{\delta}s(c_0^*)}{1-e^{\delta}s(c_0^*)}\right|+8c_0^*+7,$$

where $s=s(c)$ is the solution of $s=\tan\theta$, $\mathrm{dist}(e^{i\theta},i)=\eta(c)$.

3.5. **Uniform non–arithmeticity.** Throughout this section, Y is a pair of pants whose boundary components are of *non-zero length*, and O_Y is its representing octagon, as in figure 1.

A *closed (directed) geodesic* in Y is a geodesic segment in Y whose endpoints are not on the projections of α,β, and such that direction vector at the beginning point is equal to the direction vector at the endpoint. A *symbolic period* is a word $\underline{w}$ on the alphabet $\{\alpha,\beta,\overline{\alpha},\overline{\beta}\}$ such that the concatenation $\underline{w}\,\underline{w}$ is reduced. A symbolic period is called *rich*, if it contains at least one of the sub-words $(\alpha,\beta),(\beta,\alpha),(\overline{\alpha},\overline{\beta}),(\overline{\beta},\overline{\alpha})$.

Lemma 3.5.1. *$\underline{w}$ is a symbolic period iff it is the cutting sequence of a closed directed geodesic in Y, and this geodesic is the projection of the axis of $\varphi_{\underline{w}}:=\varphi_{w_1}\cdots\varphi_{w_{|\underline{w}|}}$.*

Proof. See e.g. Series [Se1]. □

Define $p_Y(\underline{w})$ to be the length of the closed geodesic with cutting sequence $\underline{w}$ in Y. Note that our definition of a 'closed geodesic' is such that $p_Y(\underline{w}^n)=np_Y(\underline{w}^n)$. Note also that $p_Y(\underline{w})$ depends on the isometry class of Y, therefore it is a function of $\ell(a),\ell(b),\ell(c)$.

Lemma 3.5.2. *$(\ell(a),\ell(b),\ell(c))\mapsto p_Y(\underline{w})$ is continuous at any point $(\ell(a),\ell(b),\ell(c))$ where $\ell(a),\ell(b),\ell(c)\neq 0$.*

Proof. Suppose $\underline{w}=(w_1,\ldots,w_k)$ and $\varphi_{\underline{w}}:=\varphi_{w_1}\cdots\varphi_{w_k}$. The closed geodesic with cutting sequence $\underline{w}$ is the projection of the geodesic segment from P to $\varphi_{\underline{w}}(P)$ where P is some (any) point of the axis of $\varphi_{\underline{w}}$. Therefore

$$p_Y(\underline{w})=\mathrm{dist}(P,\varphi_{\underline{w}}(P))=\text{translation length of }\varphi_{\underline{w}}=2\cosh^{-1}(\tfrac{1}{2}\operatorname{tr}\varphi_{\underline{w}}).$$

In particular, $p_Y(\underline{w})$ depends continuously on the coefficients of the matrix representations of φ_x, $x\in\{\alpha,\overline{\alpha},\beta,\overline{\beta}\}$.

Fix a pair of pants Y with boundary components of lengths $\ell(a),\ell(b),\ell(c)\neq 0$. We claim that for every ε there exists a δ such that if Y' is a pair of pants with boundary lengths $(\ell(a'), \ell(b'), \ell(c'))$, and $|\ell(x) - \ell(x')| < \delta$ for $x = a, b, c$, then it is possible to construct right-angle octagons representing Y and Y' so that the matrix coefficients of the associated side pairing maps isometries are ε-close.

We begin by observing that for any two (directed) geodesic segments of the same length in $\mathbb{D}$ there is a unique element of Möb($\mathbb{D}$) mapping one to another, and this element depends continuously on the vertices of the two segments. Therefore it is enough to show that one can represent Y and Y' by two right–angle octagons whose respective vertices are close. Choose the octagons such that $H(\alpha) = \{z : \operatorname{Re} z < 0\}$ and $H(a) = \{z : \operatorname{Im} z < 0\}$, then the remaining vertices are determined in a continuous way by the lengths of the sides of the octagon. By (3.1), these lengths are are continuous functions of $\ell(a), \ell(b), \ell(c)$ whenever $\ell(a), \ell(b), \ell(c) \neq 0$. □

Proposition 3.5.1. *For all $L, c_0 > 0$ there is a finite collection of rich symbolic periods $\mathcal{W}(L, c_0)$ s.t. for all pairs of pants Y whose boundary components have lengths in $[L^{-1}, L]$ and all reduced $(x, y) \in \mathcal{S}^2$, $\exists \underline{w} \in \mathcal{W}(L, c_0)$ s.t. $p_Y(\underline{w}) \notin c_0\mathbb{Z}$ and $(w_1, w_2) = (x, y)$.*

Proof. Let $Y = Y(\ell_1, \ell_2, \ell_3)$ denote the pair of pants with boundary components $\ell(a) = \ell_1$, $\ell(b) = \ell_2$, and $\ell_3 = \ell(c)$. Let O_Y, $\Gamma(O_Y)$ be as in §3.1. Since $\Gamma(O_Y) = \langle \varphi_\alpha, \varphi_\beta \rangle$ is a non-elementary fuchsian group, it contains elements whose translation length does not belong to $c_0\mathbb{Z}$ [GR], [Dal].

Choose a word of minimal length $\underline{w} = (w_1, \ldots, w_k) \in \mathcal{S}^k$ s.t. $\varphi = \varphi_{w_1} \circ \cdots \circ \varphi_{w_k}$ has translation length outside $c_0\mathbb{Z}$. The minimality of $\underline{w}$ means that

(a) $(w_1, \ldots, w_k)$ is reduced;
(b) $w_1 \neq \overline{w_k}$ (otherwise φ is conjugate to an element of $\Gamma(O_Y)$ with the same translation length and a shorter group representation).

Thus $\underline{w}$ is a symbolic period, and $p_Y(\underline{w}) \notin c_0\mathbb{Z}$.

We arrange for $\underline{w}$ to start with a given $x \in \mathcal{S}$. If $\underline{w}$ contains x, use a cyclic permutation to bring x to the beginning, and note that $p_Y(\underline{w})$ is invariant under cyclic permutations. If $\underline{w}$ contains $\overline{x}$, then $\underline{w} = (\overline{w}_k, \ldots, \overline{w}_1)$ is a symbolic period which contains x and we can proceed as before. Now assume that $\underline{w}$ does not contain x or $\overline{x}$. Then $x\underline{w}^n$ is a symbolic period for all n. We claim that $\exists n$ s.t. $p_Y(x\underline{w}^n) \notin c_0\mathbb{Z}$. This is because

$$\begin{aligned} p_Y(x\underline{w}^n) &= \text{translation length of } \varphi_x \circ \varphi_{\underline{w}}^n \\ &= 2\cosh^{-1}[\tfrac{1}{2}\operatorname{tr}(\varphi_x \circ \varphi_{\underline{w}}^n)] = n p_Y(\underline{w}) + o(1), \text{ and } n \to \infty, \end{aligned}$$

so $p_Y(x\underline{w}^{n+1}) - p_Y(x\underline{w}^n) \xrightarrow[n\to\infty]{} p_Y(\underline{w}) \notin c_0\mathbb{Z}$, which would not have been possible had $p_Y(x\underline{w}^n)$ been an element of $c_0\mathbb{Z}$ for all n. Thus $\exists n$ such that $p_Y(x\underline{w}^n) \notin c_0\mathbb{Z}$.

Now choose $z \in \mathcal{S}$ s.t. (y, z) is rich and reduced. If $\underline{w}$ starts with x and xy is reduced, then $xyzyx\underline{w}^n$ is a rich symbolic period which begins with xy, and the previous argument shows that there is an n so that $p_Y(xyzyx\underline{w}^n) \notin c_0\mathbb{Z}$. This is the symbolic period we were looking for.

We have found a rich symbolic period $\underline{w}$ as required, for a particular pair of pants $Y = Y(\ell_1, \ell_2, \ell_3)$. By lemma 3.5.2, $\underline{w}$ will work for for all $Y(\ell_1', \ell_2', \ell_3')$ with $(\ell_1', \ell_2', \ell_3')$ sufficiently close to (ℓ_1, ℓ_2, ℓ_3). Since the set $\Lambda_L := \{(\ell_1', \ell_2', \ell_3') : L^{-1} \leq \ell_i' \leq L\}$ is compact, we are done. □

4. Proof of the Holonomy Lemma

4.1. **The Busemann Cocycle.** The holonomy lemma deals with the Radon-Nikodym cocycle. It much easier to work with another, more geometric cocycle, defined below.

The *strong stable manifold* and the *weak stable manifold* of $\omega \in T^1(\Gamma \setminus \mathbb{D})$ are defined respectively by

$$\begin{aligned} W^{ss}(\omega) &:= \{\omega' : \mathrm{dist}(g^s\omega, g^s\omega') \xrightarrow[s\to\infty]{} 0\} \\ W^{ws}(\omega) &:= \{\omega' : \exists s_0 \text{ s.t. } \mathrm{dist}(g^s\omega, g^{s_0+s}\omega') \xrightarrow[s\to\infty]{} 0\}. \end{aligned}$$

The *strong unstable manifold* $W^{su}(\omega)$ and the *weak unstable manifold* $W^{wu}(\omega)$ are defined as above, but with $s \to -\infty$. The weak (un)stable manifold of ω only depends on the geodesic of ω.

Define an equivalence relation $\mathfrak{B}$ on the collection of oriented geodesics on $\Gamma\setminus\mathbb{D}$ by $(\gamma, \widetilde{\gamma}) \in \mathfrak{B}$ iff $\gamma, \widetilde{\gamma}$ have the same weak stable manifolds, and the same weak unstable manifolds.

Given two positively directed unit tangent vectors ω_1, ω_2 on an (oriented) geodesic γ, let $d_\gamma(\omega_1, \omega_2) := s$ where $\omega_2 = g^s(\omega_1)$.

We define the *Busemann cocycle* $B : \mathfrak{B} \to \mathbb{R}$ as follows: mark two positively directed unit tangent vectors ω_1, ω_2 on γ; let $\widetilde{\omega}_1$ be the intersection of $\widetilde{\gamma}$ with the unstable horocycle of $\widetilde{\omega}_1$, and let $\widetilde{\omega}_2$ be the intersection of $\widetilde{\gamma}$ with the stable horocycle of ω_2; then

$$B(\gamma, \widetilde{\gamma}) := d_{\widetilde{\gamma}}(\widetilde{\omega}_1, \widetilde{\omega}_2) - d_\gamma(\omega_1, \omega_2).$$

It is easy to verify that this number is independent of the choice of ω_1, ω_2, and that B has the cancellation property $B(\gamma_1, \gamma_2) + B(\gamma_2, \gamma_3) = B(\gamma_1, \gamma_3)$.

It is useful to think about $B(\omega, \widetilde{\omega})$ as a regularization of the meaningless difference 'length$(\widetilde{\gamma})-$length(γ)'.

The following proposition relates the Busemann and Radon-Nikodym cocycles. Let $\gamma[e^{i\theta_1}, e^{i\theta_2}]$ be the Γ–projection of the geodesic on $T^1(\mathbb{D})$ from $e^{i\theta_1}$ to $e^{i\theta_2}$. Recall that $\mathfrak{G}$ is the orbit relation of Γ (c.f. §2.1).

Proposition 4.1.1. *For every $(e^{i\theta}, e^{i\widetilde{\theta}}) \in \mathfrak{G}$ such that $|e^{i\theta} - e^{i\widetilde{\theta}}| < 1$,*

$$R(e^{i\theta}, e^{i\widetilde{\theta}}) = B\big(\gamma, \widetilde{\gamma}\big) \pm 4|e^{i\theta} - e^{i\widetilde{\theta}}|^2 \text{ where } \begin{cases} \gamma := \gamma[-e^{i\theta}, e^{i\theta}] \\ \widetilde{\gamma} := \gamma[-e^{i\theta}, e^{i\widetilde{\theta}}] \end{cases}.$$

Proof. Let $\mathrm{Hor}_{e^{i\theta}}(z)$ denote the stable horocycle passing through $z \in \mathbb{D}$ and tangent to $\partial\mathbb{D}$ at $e^{i\theta}$, equivalently the strong stable manifold of the unit tangent vector based at z and pointing at $e^{i\theta}$.

Recall the definition of *Busemann's function* (see e.g. [Kai]):

$$B_{e^{i\theta}}(z, w) := s, \text{ with } s \text{ s.t. } g^s[W^{ss}(\omega_z(e^{i\theta}))] = W^{ss}(\omega_w(e^{i\theta})), \tag{4.1}$$

where $z, w \in \mathbb{D}, e^{i\theta} \in \partial\mathbb{D}$, and $\omega_{z'}(e^{i\theta'})$ is the unit tangent vector based at z' and pointing at $e^{i\theta'}$. It is known that $\forall\varphi \in \text{Möb}(\mathbb{D})$, $|\varphi'(e^{i\theta})| = \exp[-B_{e^{i\theta}}(\varphi^{-1}(0), 0)]$.

Fix some $(e^{i\theta}, e^{i\widetilde{\theta}}) \in \mathfrak{G}$ such that $|e^{i\theta} - e^{i\widetilde{\theta}}| < 1$, and define the following unit tangent vectors (figure 2):

(1) ω_1, based at 0 and pointing at $e^{i\theta}$;
(2) ω, based at 0 and pointing at $e^{i\widetilde{\theta}}$;

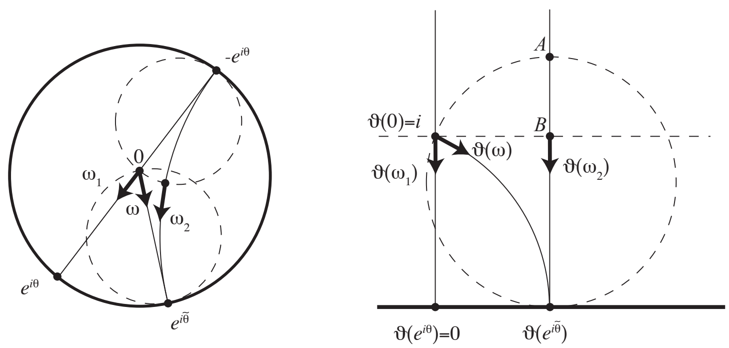

FIGURE 2

(3) ω_2, the intersection of $\widetilde{\gamma} := \gamma[-e^{i\theta}, e^{i\widetilde{\theta}}]$ with the unstable horocycle of ω_1.

By the definition of $\mathfrak{G}$, there exists some $\varphi \in \Gamma$ s.t. $e^{i\widetilde{\theta}} = \varphi(e^{i\theta})$, so

$$\begin{aligned}
R(e^{i\theta}, e^{i\widetilde{\theta}}) &= -\log|\varphi'(e^{i\theta})| = B_{e^{i\theta}}(\varphi^{-1}(0), 0)\\
&= s \text{ s.t. } g^s[W^{ss}(\varphi^{-1}(\omega))] = W^{ss}(\omega_1) \text{ viewed on } \mathbb{D}\\
&= s \text{ s.t. } g^s[W^{ss}(\Gamma\varphi^{-1}(\omega))] = W^{ss}(\Gamma\omega_1) \text{ viewed on } \Gamma\setminus\mathbb{D}\\
&= s \text{ s.t. } g^s[W^{ss}(\Gamma\omega)] = W^{ss}(\Gamma\omega_1) \text{ viewed on } \Gamma\setminus\mathbb{D}.\\
B(\gamma, \widetilde{\gamma}) &= d_{\widetilde{\gamma}}(\Gamma\omega_2, \widetilde{\gamma}\cap W^{ss}(\Gamma\omega_1)) - d_\gamma(\Gamma\omega_1, \Gamma\omega_1),\ \because W^{su}(\Gamma\omega_2) = W^{su}(\Gamma\omega_1)\\
&= d_{\widetilde{\gamma}}(\Gamma\omega_2, \widetilde{\gamma}\cap W^{ss}(\Gamma\omega_1))\\
&= s \text{ s.t. } g^s[W^{ss}(\Gamma\omega_2)] = W^{ss}(\Gamma\omega_1)
\end{aligned}$$

Thus $R(e^{i\theta}, e^{i\theta'}) = \varepsilon + B(\gamma, \widetilde{\gamma})$, where ε is the number s.t. $g^\varepsilon[W^{ss}(\Gamma\omega)] = W^{ss}(\Gamma\omega_2)$, or equivalently $g^\varepsilon[W^{ss}(\omega)] = W^{ss}(\omega_2)$ (viewed on $\mathbb{D}$).

To calculate ε, we work in the upper half plane model. Define for this purpose the isometry $\vartheta : \mathbb{D} \to \mathbb{H}$, $z \mapsto -i\frac{z-e^{i\theta}}{z+e^{i\theta}}$. Then $\vartheta(0) = i$, $\vartheta(e^{i\theta}) = 0$, $\vartheta(-e^{i\theta}) = \infty$, $\vartheta[\gamma] = \{it : t > 0\}$ and $\vartheta[\widetilde{\gamma}] = \{\vartheta(e^{i\widetilde{\theta}}) + it : t > 0\}$. $|\varepsilon|$ is the hyperbolic distance between $A := \mathrm{Hor}_{\vartheta(e^{i\widetilde{\theta}})}(i) \cap \{z : \mathrm{Re}(z) = \vartheta(e^{i\widetilde{\theta}})\}$, and $B := \vartheta(e^{i\widetilde{\theta}}) + i$ (figure 2).

Let $|AB|$ denote the *euclidean* distance between A, B. The (euclidean) triangles $\triangle iBA$, $\triangle\vartheta(e^{i\widetilde{\theta}})Bi$ are similar, so $|AB| : |iB| = |iB| : 1$, whence $|AB| = |\vartheta(e^{i\widetilde{\theta}}) - \vartheta(e^{i\theta})|^2$. Now $|\vartheta'(z)| = 2/|e^{-i\theta}z + 1|^2 < 2$ on the half plane $\{z : \mathrm{Re}(e^{-i\theta}z) > 0\}$, and this set contains $e^{i\theta}, e^{i\widetilde{\theta}}$ ($\because |e^{i\theta} - e^{i\widetilde{\theta}}| < 1$), therefore the segment $[e^{i\theta}, e^{i\widetilde{\theta}}]$. Thus, $|AB| = |\vartheta(e^{i\widetilde{\theta}}) - \vartheta(e^{i\theta})|^2 \le 4|e^{i\widetilde{\theta}} - e^{i\theta}|^2$. Returning to *hyperbolic* distances we see that $|\varepsilon| = \int_1^{1+|AB|}\frac{dy}{y} \le |AB| < 4|e^{i\widetilde{\theta}} - e^{i\theta}|^2$. □

4.2. **Separation of Cases.** Throughout this section $M = \Gamma \setminus \mathbb{D}$ is a weakly tame hyperbolic surface with a pants decomposition as in definition 2, and m is an ergodic invariant Radon measure for the Radon–Nikodym action of Γ on $\partial\mathbb{D} \times \mathbb{R}$.

Recall that $\omega(e^{i\theta}) \in T^1(\mathbb{D})$ denotes the unit tangent vector based at the origin and pointing at $e^{i\theta}$. We describe the m–almost sure behavior of $\Gamma g^s(\omega(e^{i\theta}))$ on $\Gamma \setminus \mathbb{D}$. (The properties we are interested in do not depend on the $\mathbb{R}$–coordinate of points in $\partial\mathbb{D} \times \mathbb{R}$.)

Fix a positive number $\ell_{\max}$, and let $\{Y_j(e^{i\theta})\}_{j=1}^J$ $(0 \le J \le \infty)$ be the ordered sequence of pairs of pants with norm less than $\ell_{\max}$, which $\{\Gamma g^s \omega(e^{i\theta})\}_{s>0}$ crosses. Denote by $a_j(e^{i\theta}), b_j(e^{i\theta}), c_j(e^{i\theta})$ the lengths of the entry, exit, and 'other' boundary component of $Y_j(e^{i\theta})$ (note that the boundary components corresponding to $a_i(e^{i\theta})$, $b_i(e^{i\theta})$ are different because of our definition of 'crossing').

Lemma 4.2.1. *$\exists \ell_{\max} > 0$ such that at least one of the following statements holds:*

(a) *$J = \infty$ a.e., and $\liminf\limits_{j\to\infty} a_j(e^{i\theta})$, $\liminf\limits_{j\to\infty} b_j(e^{i\theta})$, $\liminf\limits_{j\to\infty} c_j(e^{i\theta})$ are positive constants on a set of full measure ;*

(b) *$J = \infty$ a.e., $\liminf\limits_{j\to\infty} a_j(e^{i\theta})$, $\liminf\limits_{j\to\infty} b_j(e^{i\theta})$ are positive constants on a set of full measure, and $\liminf\limits_{j\to\infty} c_j(e^{i\theta}) = 0$ a.e.;*

(c) *$J = \infty$ a.e., and at least one of $\liminf\limits_{j\to\infty} a_j(e^{i\theta})$, $\liminf\limits_{j\to\infty} b_j(e^{i\theta})$ is equal to zero on a set of full measure;*

(d) *there is a connected union of finitely many pants components which traps $\{\Gamma g^s(\omega(e^{i\theta}))\}_{s>0}$ for a.e. $e^{i\theta}$.*

Proof. Suppose that case (d) happens with positive measure for some connected union of pairs of pants $S = Y_1 \cup \cdots \cup Y_k$. This means that

$$\Omega(S) := \{(e^{i\theta}, s) : \Gamma g^s(\omega(e^{i\theta})) \text{ eventually gets trapped in } T^1(S)\}$$

has positive measure. We claim that there is a possibly larger union of pants components S^* for which $\Omega(S^*)$ has full measure.

If $S = M$, then (d) holds everywhere and there is nothing to prove. If $S \neq M$, then ∂S is non-empty, and equals the union of finitely many closed geodesics. Let L_0 be the length of the largest one. Each of the boundary components has a collar neighborhood C_η with $\eta \ge \eta(L_0)$ (proposition 3.2.1). If we take $0 < \delta < \eta(L_0)$, then the δ–neighborhood of S is contained in the set S^* made of S and the pairs of pants attached to ∂S.

Clearly S^* is a finite connected union of pairs of pants, and by construction $S^* \supseteq N_\delta(S)$. We claim that

$$\Omega(S^*) \supseteq \bigcup_{g\in\Gamma} g[\Omega(S)]. \tag{4.2}$$

To see this note that if $\omega(e^{i\theta}) \in \Omega(S)$ and $g \in \Gamma$, then there are constants b, s_0 such that the geodesic rays $\{\Gamma g^s \omega(e^{i\theta})\}_{s>s_0}$, $\{\Gamma g^{s+b}\omega(g \cdot e^{i\theta})\}_{s>s_0}$ δ–shadow each other for all $s > 0$.[7] Thus if the geodesic ray of $\Gamma\omega(e^{i\theta})$ is eventually trapped in S, then the geodesic ray of $\Gamma\omega(g \cdot e^{i\theta})$ will eventually be trapped in $N_\delta(S)$, a subset of S^*. This proves (4.2).

---

[7]Proof: $g^{-1}\omega(g\cdot e^{i\theta})$ determines a geodesic which ends at $e^{i\theta}$, therefore $\exists a, b \in \mathbb{R}$ such that $\omega(g\cdot e^{i\theta}) = g[g^b h^a \omega(e^{i\theta})]$. It follows that $\Gamma g^s \omega(g\cdot\omega(e^{i\theta})) = \Gamma g^{s+b}h^a\omega(e^{i\theta}) = h^{ae^{-(s+b)}}(\Gamma g^{s+b}\omega(e^{i\theta}))$ (recall that the action of Γ commutes with those of the geodesic and horocycle flow).

Since m is ergodic, and $\Omega(S^*)$ contains a Γ–invariant set of positive measure, $\Omega(S^*)$ has full measure. We see that if (d) holds with positive measure, then it holds with full measure.

Alternatively, either (d) holds, or almost no geodesic is trapped in a finite union of pairs of pants. Assume the latter.

Let $Y_j'(e^{i\theta})$ be the list of *all* pants components crossed by $\{\Gamma g^s \omega(e^{i\theta})\}_{s>0}$, and consider the function

$$f(e^{i\theta}) := \liminf_{j\to\infty} \|Y_j'(e^{i\theta})\|.$$

The weak tameness assumption and the almost sure failure of (d) implies that the list of $Y_j'(e^{i\theta})$'s is infinite a.s., and that $f < \infty$ almost everywhere. We claim that f is constant, and then define $\ell_{\max}$ to be 1 plus this constant.

We need some facts on collar neighborhoods. Let $\eta(\cdot)$ be the collar function from proposition 3.2.1. Every boundary component γ_p of a pair of pants Y has a collar neighborhood of size at least $\eta(\|Y\|)$. We say that a geodesic *crosses* the collar neighborhood of γ_p if it enters it and leaves it through the two different boundary components of the collar. Any such curve must cross γ_p, and any geodesic which crosses γ_p must cross the collar of γ_p.[8]

Suppose a geodesic segment γ crosses a boundary component γ_p, then any other geodesic segment γ_1 which $\frac{1}{4}\eta(\|Y\|)$–shadows γ must cross γ_p, whence the collar of γ_p (because if γ crosses γ_p, then γ crosses the $\eta(\|Y\|)$–collar of γ_p, so γ_1 crosses the $\frac{1}{2}\eta(\|Y\|)$–collar of γ_p, whence γ_p itself).

We can now repeat the δ–shadowing argument done above with $\delta = \frac{1}{4}\eta(\|Y\|)$ to deduce that for every ℓ,

$$\{(e^{i\theta}, s) \in \partial\mathbb{D} \times \mathbb{R} : f(e^{i\theta}) < \ell\} \supseteq \bigcup_{g\in\Gamma} g\{(e^{i\theta}, s) \in \partial\mathbb{D} \times \mathbb{R} : f(e^{i\theta}) < \ell\}.$$

The ergodicity of m implies that the level sets of f either have full measure or zero measure, so f must be constant on a set of full measure.

We see that if we set $\ell_{\max}$ to be 1 plus the almost sure value of f, then $J = \infty$ m–almost everywhere. Let $\{Y_j(e^{i\theta})\}_{i=1}^{\infty}$ be the sequence of pairs of pants of norm less than $\ell_{\max}$ which $\{\Gamma g^s\omega(e^{i\theta})\}_{s>0}$ crosses. The argument we used to prove that f is constant on a set of full measure can be used to show that $\liminf_{j\to\infty} a_j(e^{i\theta})$, $\liminf_{j\to\infty} b_j(e^{i\theta})$, $\liminf_{j\to\infty} c_j(e^{i\theta})$ are constant on sets of full measure. Thus at least one of (a)–(c) holds with full measure. □

We prove the holonomy lemma in each of these cases. The reader who is only interested in surfaces which can be partitioned into countably many pairs of pants whose boundary components have lengths bounded away from infinity *and zero*, need not worry about cases (b) and (c).

4.3. **Proof of the holonomy lemma in case (a).** Given $e^{i\theta}$, we are asked to construct some $\varphi_{e^{i\theta}} \in \Gamma$ with good control on $R(e^{i\theta}, \varphi_{e^{i\theta}}(e^{i\theta})) = -\log|(\varphi_{e^{i\theta}})'(e^{i\theta})|$. Proposition 4.1.1 reduces this problem to that of constructing for a given geodesic γ another geodesic $\widetilde{\gamma}$ with good control on $B(\gamma, \widetilde{\gamma})$.

[8] To see this lift γ_p to the positive y–axis in $\mathbb{H}$. The boundary components of the η–collar lift to $\{x + iy : y = x\tan\theta, x > 0\}$ and $\{x + iy : y = -x\tan\theta, x < 0\}$ with θ given by $\mathrm{dist}(i, e^{i\theta}) = \eta$. Any geodesic crosses the the positive y–axis iff it crosses the lifts of the boundary components.

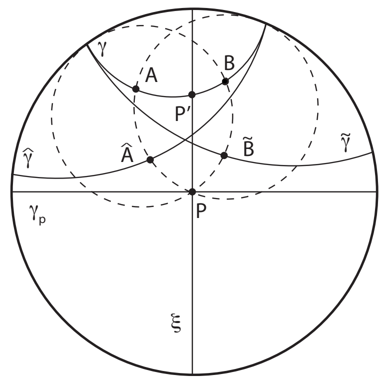

FIGURE 3

- §4.3.1 shows how to construct $\widetilde{\gamma}$, assuming that γ shadows certain closed geodesics a very long time.
- §4.3.2 shows that for almost every $(e^{i\theta}, s)$, $\gamma := \{g^s(\Gamma\omega(e^{i\theta}))\}_{s\in\mathbb{R}}$ shadows such closed geodesics a very long time. This allows us to construct $\varphi_{e^{i\theta}} \in \Gamma$ with good control of $R(e^{i\theta}, \varphi_{e^{i\theta}}(e^{i\theta}))$.
- §4.3.3 completes the proof of the holonomy lemma in case (a), by showing that $(e^{i\theta}, s) \mapsto (\varphi_{e^{i\theta}}(e^{i\theta}), s - \log|\varphi'_{e^{i\theta}}(e^{i\theta})|)$ can be made one-to-one.

4.3.1. *Controlling the values of the Busemann cocycle.* Let γ be a geodesic which crosses a pair of pants Y, and whose cutting sequence in Y begins with $\underline{w}^{2k}$ where $\underline{w} = (\xi_1, \dots, \xi_N)$ is a symbolic period.

There is a point X on $\gamma \cap Y$ such that the cutting sequence up to X ends with $\underline{w}^k$, and the cutting sequence starting from X begins with $\underline{w}^k$. Fix a hyperbolic octagon O_Y which represents Y, and lift γ to $\mathbb{D}$ in such a way that X is in O_Y.

Define $\widetilde{\gamma}$ to be the projection to $\Gamma\backslash\mathbb{D}$ of the geodesic with endpoints

$$\widetilde{\gamma}(-\infty) = \gamma(-\infty)\ ,\ \widetilde{\gamma}(\infty) = \varphi^n[\gamma(\infty)],\ \text{ where } \varphi := \varphi_{\xi_1} \circ \varphi_{\xi_2} \circ \cdots \circ \varphi_{\xi_N}.$$

This has the effect of adding an additional $\underline{w}^n$ to the cutting sequence of γ in Y (see proposition 3.3.1). Note that $(\gamma, \widetilde{\gamma}) \in \mathfrak{B}$, because $\varphi \in \Gamma$.

Proposition 4.3.1. *If* $(k-2)p_Y(\underline{w}) \geq \mathrm{diam}(O_Y) + 5$, *then*

$$\left|B(\gamma, \widetilde{\gamma}) - np_Y(\underline{w})\right| < 100e^{\mathrm{diam}(O_Y)}e^{-(k-2)p_Y(\underline{w})}.$$

Proof. Set $p := p_Y(\underline{w})$. Abusing notation, we identify γ with its lift to $\mathbb{D}$ so that X is in O_Y, and $\widetilde{\gamma}$ with its lift to $\mathbb{D}$ connecting $\gamma(-\infty)$ to $\varphi^n[\gamma(\infty)]$. Let $\widehat{\gamma} := \varphi^{-n}[\widetilde{\gamma}]$ be the geodesic from $\varphi^{-n}[\gamma(-\infty)]$ to $\gamma(\infty)$ (yet another lift of $\widetilde{\gamma}$).

Add to the picture the axis of φ (which we call γ_p), a point P on $\gamma_p \cap O_Y$, and the unique geodesic ξ passing through P which is perpendicular to γ_p (figure 3). Next define

(1) A, $\widehat{A}$: the respective intersection points of γ, $\widehat{\gamma}$ with the horocycle passing through P and $\gamma(\infty)$;
(2) B, $\widetilde{B}$: the respective intersection points of γ, $\widetilde{\gamma}$ with the horocycle passing through P and $\gamma(-\infty)$;
(3) $\widetilde{A} := \varphi^n(\widehat{A})$. Note that $\widehat{A}$, $\widetilde{A}$ project to the same point on $\Gamma \setminus \mathbb{D}$.

Claim 1. γ intersect ξ at a point P', and $\mathrm{dist}(P,P') \leq 6e^{\mathrm{diam}(O_Y)}e^{-(k-2)p}$.

Proof. Map γ_p to the upper half plane $\mathbb{H}$ in such a way that γ_p maps to the y–axis with P landing on i. The geodesic ξ must then map to $\{z \in \mathbb{H} : |z| = 1\}$.

We chose O_Y to contain X, so γ and γ_p cross the same k copies of O_Y in both directions, starting from O_Y. This means that γ passes through through the two (hyperbolic) discs of radius $\mathrm{diam}(O_Y)$ centered at $e^{(k-1)p}i$,$e^{-(k-1)p}i$. These discs are on different sides of ξ, because $(k-1)p > \mathrm{diam}(O_Y)$. Thus γ crosses ξ at some point P'.

We also see that γ has one endpoint in $\{x \in \mathbb{R} : |x| < e^{-(k-1)p+\mathrm{diam}(O_Y)}\}$ and one endpoint in $\{x \in \mathbb{R} : |x| > e^{(k-1)p-\mathrm{diam}(O_Y)}\}$. Write these endpoints as

$$e^{-(M+k)p}x, e^{(N+k)p}y, \text{ where } -(1+\frac{\mathrm{diam}(O_Y)}{p}) < M, N < \infty \text{ and } 1 \leq |x|, |y| < e^p.$$

We may assume w.l.o.g. that $y > 0$ (otherwise apply the isometry $z \mapsto -\overline{z}$).

We can now find P' by setting $P' = e^{i\vartheta}$, and solving

$$\left|e^{i\vartheta} - \left(\frac{e^{-(M+k)p}x + e^{(N+k)p}y}{2}\right)\right| = \left|\frac{e^{-(M+k)p}x - e^{(N+k)p}y}{2}\right|.$$

A direct calculation, using the assumption $(k-2)p \geq \mathrm{diam}(O_Y) + 5$, shows that

$$\begin{aligned} |\cos\vartheta| &= \left|\frac{e^{(N-M)p}xy + 1}{e^{(k+N)p}y + e^{-(k+M)p}x}\right| = \left|\frac{e^{-(M+k)p}x + e^{-(k+N)p}/y}{1 + e^{-(2k+M+N)p}x/y}\right| \\ &\leq \frac{2e^{-(k-2)p+\mathrm{diam}(O_Y)}}{1 - e^{-2[(k-2)p-\mathrm{diam}(O_Y)]}} < 3e^{\mathrm{diam}(O_Y)}e^{-(k-2)p}, \end{aligned}$$

whence $\sin\vartheta = \sqrt{1-\cos^2\vartheta} > \frac{1}{2}$ and

$$\begin{aligned} \sinh\left(\tfrac{1}{2}\,\mathrm{dist}(P,P')\right) &\equiv \frac{|e^{i\vartheta} - i|}{2\sqrt{\mathrm{Im}(e^{i\vartheta})\mathrm{Im}(i)}} = \sqrt{\frac{1-\sin\vartheta}{2\sin\vartheta}} \leq \sqrt{1-\sin\vartheta} \\ &\leq \frac{|\cos\vartheta|}{\sqrt{1+\sin\vartheta}} < |\cos\vartheta| < 3e^{\mathrm{diam}(O_Y)}e^{-(k-2)p}. \end{aligned}$$

Since $t \leq \sinh t$ for all $t \geq 0$, $\mathrm{dist}(P,P') \leq 6e^{\mathrm{diam}(O_Y)}e^{-(k-2)p}$.

Claim 2. $\mathrm{dist}(A,P), \mathrm{dist}(B,P) < 12e^{\mathrm{diam}(O_Y)}e^{-(k-2)p}$.

Proof. Denote the horocycle passing through $z \in \mathbb{D}$ and tangent at $\eta \in \partial\mathbb{D}$ by $\mathrm{Hor}_\eta(z)$, and recall the definition of Busemann's function (4.1). By construction

$$\begin{aligned} \mathrm{dist}(A,P') &= |B_{\gamma(\infty)}(A,P')| && (A, P' \text{ are both on } \gamma) \\ &= |B_{\gamma(\infty)}(P,P')| && (A, P \text{ are both on } \mathrm{Hor}_{\gamma(\infty)}(P)) \\ &\leq \mathrm{dist}(P,P') && (|B_{e^{i\theta}}(z,w)| \leq \mathrm{dist}(z,w)) \end{aligned}$$

and so $\mathrm{dist}(A,P) \leq 2\,\mathrm{dist}(P,P')$. A similar argument works for $\mathrm{dist}(B,P)$, and the claim follows from claim 1.

Claim 3. $\mathrm{dist}(A,B) \leq 24e^{\mathrm{diam}(O_Y)}e^{-(k-2)p}$, $\mathrm{dist}(\widetilde{A},\widetilde{B}) = np \pm 24e^{\mathrm{diam}(O_Y)}e^{-(k-2)p}$.

Proof. The first inequality follows from claim 2. To see the second, note that claims 1 and 2 work for any geodesic γ whose cutting sequence arriving to O_Y ends with

$\underline{w}^k$ and whose cutting sequence from O_Y onwards starts with $\underline{w}^k$. The geodesics $\widetilde{\gamma}$ and $\widehat{\gamma}$ have this property. Thus $\mathrm{dist}(\widehat{A},P),\mathrm{dist}(\widetilde{B},P)\leq 12e^{\mathrm{diam}(O_Y)}e^{-(k-2)p}$. Now

$$\begin{aligned}
\mathrm{dist}(\widetilde{A},\widetilde{B}) &\leq \mathrm{dist}(\widetilde{A},\varphi^n(P))+\mathrm{dist}(\varphi^n(P),P)+\mathrm{dist}(P,\widetilde{B})\\
&= \mathrm{dist}(\widehat{A},P)+np+\mathrm{dist}(P,\widetilde{B})\ (\because \widetilde{A}=\varphi^n(\widehat{A}),\ P \text{ is on the axis of } \varphi)\\
&\leq np+24e^{\mathrm{diam}(O_Y)}e^{-(k-2)p};\\
\mathrm{dist}(\widetilde{A},\widetilde{B}) &\geq \mathrm{dist}(\varphi^n(P),P)-\mathrm{dist}(P,\widetilde{B})-\mathrm{dist}(\widetilde{A},\varphi^n(P))\\
&= np-\mathrm{dist}(P,\widetilde{B})-\mathrm{dist}(\widehat{A},P)\\
&\geq np-24e^{\mathrm{diam}(O_Y)}e^{-(k-2)p}.
\end{aligned}$$

The claim follows.

We can now prove the proposition. Let ω_A,ω_B denote the (positively directed) unit tangent vectors to the projection of γ based at the projections of A,B. Let $\omega_{\widetilde{A}}$, $\omega_{\widetilde{B}}$ denote the (positively directed) unit tangent vectors to the projection of $\widetilde{\gamma}$ based at the projections of $\widetilde{A},\widetilde{B}$. Then

$$\begin{aligned}
&\lim_{s\to\infty}\mathrm{dist}(g^s\omega_A,g^s\omega_{\widetilde{A}})=0 &&\because \omega_A,\omega_{\widetilde{A}}\in\Gamma[\mathrm{Hor}_{\gamma(\infty)}(P)],\\
&\lim_{s\to\infty}\mathrm{dist}(g^{-s}\omega_B,g^{-s}\omega_{\widetilde{B}})=0 &&\because \omega_B,\omega_{\widetilde{B}}\in\Gamma[\mathrm{Hor}_{\gamma(-\infty)}(P)].
\end{aligned}$$

It follows that $B(\gamma,\widetilde{\gamma})=\mathrm{dist}(\widetilde{A},\widetilde{B})-\mathrm{dist}(A,B)=np\pm 48e^{\mathrm{diam}(O_Y)}e^{-(k-2)p}$. $\square$

4.3.2. *The support of m in case (a).* Assume we are in case (a) of lemma 4.2.1.

Proposition 4.3.2. *$\exists L>0$ s.t. $\forall c_0>0\ \exists M(c_0),\delta(c_0),\varepsilon(c_0)>0$ s.t. $\forall n\in\mathbb{N}$ and for a.e. $e^{i\theta}$, $\{\Gamma g^s(\omega(e^{i\theta}))\}_{s>0}$ crosses infinitely many times pairs of pants Y s.t.:*

(a) *the lengths of the boundary components of Y are all in $[L^{-1},L]$,*
(b) *the cutting sequence in Y contains the word $\underline{w}^{2n}$ with $\underline{w}$ a rich symbolic period, and*
(c) *$\delta(c_0)<p_Y(\underline{w})\leq M(c_0)$, $\mathrm{dist}(p_Y(\underline{w}),c_0\mathbb{Z})>\varepsilon(c_0)$.*

Proof. In case (a) there is an $L>0$ such that for m–almost every $(e^{i\theta},s)$, the geodesic ray $\{\Gamma g^s(\omega(e^{i\theta}))\}_{s>0}$ crosses infinitely many times pairs of pants Y whose boundary components have lengths in $[L^{-1},L]$.

Recall the definition of $\mathcal{W}(L,c_0)$ from proposition 3.5.1, and define (relative to a fixed weakly tame pants decomposition):

$$\begin{aligned}
\mathcal{Y}&:=\{\text{all pants components whose boundary component lengths are in } [L^{-1},L]\},\\
D&:=\sup\{\mathrm{diam}(O_Y):Y\in\mathcal{Y}\},\ (O_Y:=\text{octagon representing } Y)\\
M_0&:=\sup\{p_Y(\underline{w}):\underline{w}\in\mathcal{W}(L,c_0),Y\in\mathcal{Y}\},\\
\delta_0&:=\inf\{p_Y(\underline{w}):\underline{w}\in\mathcal{W}(L,c_0),Y\in\mathcal{Y}\},\\
\varepsilon_0&:=\tfrac{1}{2}\sup\{\varepsilon:\text{for every pants space } Y\in\mathcal{Y} \text{ and } (x,y)\in\mathcal{S}^2 \text{ reduced},\\
&\qquad \exists \text{ rich } \underline{w}\in\mathcal{W}(L,c_0) \text{ s.t. } \mathrm{dist}(p_Y(\underline{w}),c_0\mathbb{Z})>\varepsilon\ ,\ (w_1,w_2)=(x,y)\}.
\end{aligned}$$

D is finite because of the hexagon formula; M_0,δ_0 and ε_0 are finite and positive, because of lemma 3.5.2 and the finiteness of $\mathcal{W}(L,c_0)$.

Recall the definition of $\eta(\cdot)$ and $\varepsilon_{sh}(\cdot)$ from propositions 3.2.1 and 3.4.1. We can make $\varepsilon_{sh}(\cdot)$ arbitrarily small. Make it so small that

$$\varepsilon_{sh}(\cdot) < \frac{1}{4}\eta(\cdot), \text{ and } 0 < x < \max \varepsilon_{sh}(\cdot) \Rightarrow \frac{2\tanh\frac{x}{2}}{\sqrt{1-\tanh^2\frac{x}{2}}} < 2x. \tag{4.3}$$

Given $e^{i\theta}$, let $\{Y_k(e^{i\theta})\}_{k=1}^{\infty}$ denote the ordered list of pairs of pants in $\mathcal{Y}$ which $\{\Gamma g^s(\omega(e^{i\theta}))\}_{s>0}$ crosses. In case (a), this list is well-defined and infinite almost everywhere. The sequence of $Y_i(e^{i\theta})$'s lifts to a sequence of hyperbolic octagons crossed by $\{g^s\omega(e^{i\theta})\}_{s>0}$. Let $O_i(e^{i\theta}) := O_{Y_i(e^{i\theta})}$ be the lift of $Y_i(e^{i\theta})$ which contains the lift of the point of entry to $Y_i(e^{i\theta})$.

Label the the sides of $O_i(e^{i\theta})$ in such a way that $g^s(\omega(e^{i\theta}))$ enters $Y_i(e^{i\theta})$ through boundary component a and leaves it through boundary component c. Let $\varphi_x^{O_i(e^{i\theta})}$, $x \in \mathcal{S}$, be the side pairings. These are elements of Γ, because they match the sides $x, \overline{x}$ of $O_i(e^{i\theta})$, and $x, \overline{x}$ project to the same directed geodesic segment on $\Gamma\backslash\mathbb{D}$.

Let $\Omega(n)$ denote the collection of $(e^{i\theta}, s) \in \partial\mathbb{D}\times\mathbb{R}$ such that there is *no* k for which $\{\Gamma g^s\omega(e^{i\theta})\}_{s>0}$ $\varepsilon_{sh}(L)$–shadows in $Y_k(e^{i\theta})$ a closed geodesic with cutting sequence $\underline{w} \in \mathcal{W}(L, c_0)$ s.t. $\operatorname{dist}(p_{Y_k(e^{i\theta})}(\underline{w}), c_0\mathbb{Z}) \geq \varepsilon_0$, during a time period longer than $np_{Y_k(e^{i\theta})}(\underline{w})$.

If we can show that $m[\bigcup_{n\in\mathbb{N}}\Omega(n)] = 0$ then we are done, because of the connection between shadowing and cutting sequences given in proposition 3.4.1.

Assume by way of contradiction that $m[\Omega(n)] \neq 0$ for some n. W.l.o.g., there exist a pair of pants $Y \subset \Gamma\setminus\mathbb{D}$, and a compact interval I, such that

$$\Omega_Y(n, I) := \{(e^{i\theta}, s) \in \Omega(n) : \omega(e^{i\theta}) \in \operatorname{int}[T^1(Y)], s \in I\}$$

has positive measure (to arrange this pass to a conjugate copy of Γ so that the Γ–orbit of the origin in $\mathbb{D}$ is identified with a point outside the union of the boundaries of all pants components). Since m is Radon, $0 < m[\Omega_Y(n)] < \infty$.

Define $\kappa_i : \Omega_Y(n, I) \to \partial\mathbb{D}\times\mathbb{R}$ for $i \in \mathbb{N}$ as follows. Fix some $n_0 \gg n$, to be determined later. $\{\Gamma g^s(\omega(e^{i\theta}))\}_{s>0}$ enters $Y_i(e^{i\theta})$ through side a and leaves it through side c, so the endpoints of $\{g^s(\omega(e^{i\theta}))\}_{s\in\mathbb{R}}$ satisfy

$$\begin{aligned} -e^{i\theta} &\in (\varphi_\alpha^{O_i(e^{i\theta})})^k[Q(a)]; \\ e^{i\theta} &\in (\varphi_\alpha^{O_i(e^{i\theta})})^\ell(\varphi_{u_1}^{O_i(e^{i\theta})})\cdots(\varphi_{u_N}^{O_i(e^{i\theta})})(\varphi_\beta^{O_i(e^{i\theta})})^m[Q(c)], \end{aligned}$$

where $|\ell|, |m|$ are maximal and $(u_1, \ldots, u_N) \in \mathcal{S}^N$ is a reduced word such that $u_1 \neq \alpha, \overline{\alpha}$, $u_N \neq \beta, \overline{\beta}$. Choose $\underline{w} = \underline{w}(\underline{u}) = (w_1, \ldots, w_k) \in \mathcal{W}(L, c_0)$ in such a way that $w_1 = u_1$, $(w_1, w_2) \neq (u_1, u_2)$, and $\operatorname{dist}(p_{Y_i(e^{i\theta})}(\underline{w}), c_0\mathbb{Z}) \geq \varepsilon_0$. Such a word exists, because of the definition of ε_0. Set

$$\begin{aligned} \varphi_{e^{i\theta}} &:= (\varphi_\alpha^{O_i(e^{i\theta})})^\ell \circ [\varphi_{w_1}^{O_i(e^{i\theta})} \circ \cdots \circ \varphi_{w_k}^{O_i(e^{i\theta})}]^{n_0} \circ (\varphi_\alpha^{O_i(e^{i\theta})})^{-\ell}, \\ \kappa_i(e^{i\theta}, s) &:= (\varphi_{e^{i\theta}}(e^{i\theta}), s - \log|(\varphi_{e^{i\theta}})'(e^{i\theta})|). \end{aligned}$$

We show below that for a suitable choice of n_0, j_0

- κ_i are measure preserving for all i large enough,
- $\bigcup_{i>j_0}\kappa_i[\Omega_Y(n, I)]$ is precompact,
- $\{\kappa_i[\Omega_Y(n, I)]\}$ contains an infinite family of pairwise disjoint sets.

The union of this family is a precompact set with infinite measure, in contradiction to the Radon property of m. This contradiction shows that $m[\Omega(n)] = 0$ is impossible. Since n was arbitrary, this proves the proposition.

Step 1. Define $e^{i\widetilde{\theta}}$ by $\kappa_i(e^{i\theta},s)=(e^{i\widetilde{\theta}},*)$, and let $\widetilde{\gamma}^+$ denote the geodesic ray $\{g^s(\omega(e^{i\widetilde{\theta}}))\}_{s>0}$. There is a constant $t_0=t_0(L)$ such that if n_0 and i_0 are large enough, then for all $i>i_0$

(a) $Y_i(e^{i\theta})$ is the first $Y\in\mathcal{Y}$ where $\Gamma\widetilde{\gamma}^+$ $\frac{1}{2}\varepsilon_{sh}(L)$–shadows a closed geodesic γ_p with cutting sequence in $\mathcal{W}(L,c_0)$ and length $\ell(\gamma_p)$ s.t. $\mathrm{dist}(\ell(\gamma_p),c_0\mathbb{Z})>\varepsilon_0$, for more than $n\ell(\gamma_p)+t_0$ units of times.

(b) There is a constant $N(L)$, independent of $e^{i\theta}$, such that $Y_i(e^{i\theta})=Y_j(e^{i\widetilde{\theta}})$ where $|j-i|\le N(L)$.

Proof. Consider the geodesics γ from $-e^{i\theta}$ to $e^{i\theta}$; $\widehat{\gamma}$ from $-e^{i\theta}$ to $e^{i\widetilde{\theta}}$; and $\widetilde{\gamma}$ from $-e^{i\widetilde{\theta}}$ to $e^{i\widetilde{\theta}}$. The endpoints of $\gamma,\widehat{\gamma}$ are such that the $(\varphi_\alpha^{O_i(e^{i\theta})})^{-\ell}$–images of $\gamma,\widehat{\gamma}$ enter $O_i(e^{i\theta})$ through two points $(\varphi_\alpha^{O_i(e^{i\theta})})^{-\ell}(P),(\varphi_\alpha^{O_i(e^{i\theta})})^{-\ell}(Q)$ on sides $\overline{\alpha},a,\alpha$ of $O_i(e^{i\theta})$ (proposition 3.3.1). Since $\widetilde{\gamma}^+:=\{g^s\omega(e^{i\widetilde{\theta}})\}_{s>0}$ is between γ and $\widehat{\gamma}$, its $(\varphi_\alpha^{O_i(e^{i\theta})})^{-\ell}$–image must also enter $O_i(e^{i\theta})$ through some point $(\varphi_\alpha^{O_i(e^{i\theta})})^{-\ell}(R)$ on one of the sides $\overline{\alpha},a,\alpha$. The distance between these points cannot exceed the diameter of $O_i(e^{i\theta})$, so

$$\mathrm{dist}(P,Q),\mathrm{dist}(P,R)\le D. \tag{4.4}$$

If we set $\widehat{P}$ to be the point on $\widehat{\gamma}\cap\mathrm{Hor}_{-e^{i\theta}}(P)$, then the length of the horocyclic arc of $\mathrm{Hor}_{-e^{i\theta}}(P)$ connecting P to $\widehat{P}$ is bounded by $2/(1-\tanh D)$.[9] This means that the unit tangent vector

$$\widehat{\omega}:=g^{-\mathrm{dist}(o,P)}[\text{ tangent vector to }\widehat{\gamma}\text{ at }\widehat{P}]$$

satisfies

$$\mathrm{dist}(g^s\omega(e^{i\theta}),g^s\widehat{\omega})\le\frac{1}{10}\varepsilon_{sh}(L)\text{ for all }s<\mathrm{dist}(o,P)-d^*, \tag{4.5}$$

where $d^*:=\ln\big[20/[\varepsilon_{sh}(L)(1-\tanh D)]\big]$ and o is the origin.

$\Gamma g^s\omega(e^{i\theta})$ crosses $i-1$ elements of $\mathcal{Y}$ during the time interval $[0,\mathrm{dist}(o,P)]$, because at the end of this time interval it is beginning its crossing of $Y_i(e^{i\theta})$. It takes at least $\eta(L)$ units of time to cross a pair of pants in $\mathcal{Y}$, because of the time it takes to cross the collars of Y. Thus $\mathrm{dist}(o,P)>(i-1)\eta(L)$.

If we choose

$$i_0=i_0(L)>\frac{d^*}{\eta(L)}+1,$$

then for every $i>i_0(L)$, $\mathrm{dist}(o,P)>d^*$, whence by (4.5)

$$\mathrm{dist}(\omega(e^{i\theta}),\widehat{\omega})<\frac{1}{10}\varepsilon_{sh}(L).$$

Now consider the point $\check{P}$ at the intersection of $\mathrm{Hor}_{e^{i\widetilde{\theta}}}(o)$ and $\widehat{\gamma}$, and let $\check{\omega}$ be the unit tangent vector to $\widehat{\gamma}$ at $\check{P}$. Since the distance between the base points of

[9] Draw P,Q in $\mathbb{H}$ so that $-e^{i\theta}$ is at infinity, and P is at i. The length of the horocyclic arc connecting $P,\widehat{P}$ is $A=|\mathrm{Re}(Q)|=|\mathrm{Re}(\widehat{P})|$. By [K], theorem 1.2.6, $\tanh[\frac{1}{2}\mathrm{dist}(P,\widehat{P})]=\frac{|P-\widehat{P}|}{|P-\overline{\widehat{P}}|}=\frac{|\mathrm{Re}(\widehat{P})|}{|\mathrm{Re}(\widehat{P})+2i|}=A/\sqrt{A^2+4}$, so $A=2\tanh\frac{1}{2}\mathrm{dist}(P,\widehat{P})/\sqrt{1-\tanh^2\frac{1}{2}\mathrm{dist}(P,\widehat{P})}$. Since $\mathrm{dist}(P,\widehat{P})\le\mathrm{dist}(P,Q)+\mathrm{dist}(Q,\widehat{P})\le 2\,\mathrm{dist}(P,Q)\le 2D$ and $\tanh D\in(0,1)$, $A\le 2/(1-\tanh D)$.

$\widehat{\omega}$ and $\omega(e^{i\widetilde{\theta}})$ is less than $\frac{1}{10}\varepsilon_{sh}(L)$, the length of the horocyclic arc of $\mathrm{Hor}_{e^{i\widetilde{\theta}}}(o)$ connecting the base points of $\breve{\omega}$ and $\omega(e^{i\widetilde{\theta}})$ is less than

$$\frac{2\tanh\frac{1}{20}\varepsilon_{sh}(L)}{\sqrt{1-\tanh^2\frac{1}{20}\varepsilon_{sh}(L)}} < \frac{1}{5}\varepsilon_{sh}(L)$$

((4.3) and footnote 9). Therefore, $\mathrm{dist}(g^s\omega(e^{i\widetilde{\theta}}), g^s\breve{\omega}) \le \frac{1}{5}\varepsilon_{sh}(L)$ for all $s > 0$.

$\widehat{\omega}, \breve{\omega}$ are tangent to the same geodesic, and the distance between their base points is $|B_{e^{i\widetilde{\theta}}}(\text{base point of } \widehat{\omega}, o)| \le \mathrm{dist}(\widehat{\omega}, \omega(e^{i\theta})) < \frac{1}{10}\varepsilon_{sh}(L)$, therefore

$$\mathrm{dist}(g^s\omega(e^{i\widetilde{\theta}}), g^s\widehat{\omega}) \le \frac{3}{10}\varepsilon_{sh}(L) \text{ for all } s > 0. \tag{4.6}$$

Combining (4.4), (4.5) and (4.6), we see that if $i > i_0$, then

$$\mathrm{dist}(g^s\omega(e^{i\widetilde{\theta}}), g^s\omega(e^{i\theta})) < \frac{1}{2}\varepsilon_{sh}(L) \text{ for all } 0 < s < \mathrm{dist}(o,R) - [d^* + D]. \tag{4.7}$$

By assumption $\omega \in \Omega_Y(n, I)$, so $\{\Gamma g^s(\omega(e^{i\theta}))\}_{s>0}$ does *not* cross a $Y \in \mathcal{Y}$ where it $\varepsilon_{sh}(L)$–shadows a closed geodesic γ_p with cutting sequence in $\mathcal{W}(L, c_0)$ and length ε_0–away from $c_0\mathbb{Z}$, for more than $n\ell(\gamma_p)$ units of time. Thus by (4.7), $g^s(\omega(e^{i\widetilde{\theta}}))$ cannot cross a $Y \in \mathcal{Y}$ sometime in the interval $0 < s < \mathrm{dist}(o,R) - [d^* + D]$, where it $\frac{1}{2}\varepsilon_{sh}(L)$–shadows a closed geodesic as above.

Recall that at time $s = \mathrm{dist}(o,R)$, $\Gamma g^s\omega(e^{i\widetilde{\theta}})$ is in, or entering $Y_i(e^{i\theta})$. Thus $\widetilde{\gamma}^+ := \{g^s(\omega(e^{i\widetilde{\theta}}))\}_{s>0}$ does not cross a $Y \in \mathcal{Y}$ where it $\frac{1}{2}\varepsilon_{sh}(L)$–shadows a closed geodesic γ_p with cutting sequence in $\mathcal{W}(L, c_0)$ and length ε_0–away from $c_0\mathbb{Z}$, for more than $n\ell(\gamma_p) + d^* + D$ units of time, before it enters $Y_i(e^{i\theta})$.

We now show that if n_0 is large enough, then $\{g^s(\omega(e^{i\widetilde{\theta}}))\}_{s>0}$ *does* $\frac{1}{2}\varepsilon_{sh}(L)$–shadow in $Y_i(e^{i\theta})$ a closed geodesic γ_p with cutting sequence in $\mathcal{W}(L, c_0)$ and length ε_0–away from $c_0\mathbb{Z}$, for more than $n\ell(\gamma_p) + d^* + D$ units of time. Take for this purpose

$$n_0 := n_0(n, L) > n + 16 + \frac{2t_{sh}(L) + \ell_{sh}(L) + d^* + D}{\delta_0} \quad \text{(c.f. proposition 3.4.1).}$$

By proposition 3.3.1, $\widehat{\gamma}$ crosses $Y_i(e^{i\theta})$ and its cutting sequence there contains $\underline{w}^{n_0}$. Let γ_p be the closed geodesic with symbolic period $\underline{w}$. By the choice of n_0,

$$(n_0 - 3)\ell(\gamma_p) \ge (n_0 - 3)\delta_0 > \ell_{sh}(L),$$

so proposition 3.4.1(b) says that $\widehat{\gamma}^+$ $\frac{1}{2}\varepsilon_{sh}(L)$–shadows γ_p for more than

$$(n_0 - 3)\ell(\gamma_p) - t_{sh}(L) > (n + 13)\ell(\gamma_p) + t_{sh}(L) + \ell_{sh}(L) + d^* + D$$

units of time. By (4.6), $\widetilde{\gamma}^+$ $\varepsilon_{sh}(L)$–shadows γ_p for this duration, so by proposition 3.4.1(a) the cutting sequence of $\widetilde{\gamma}^+$ contains the word $\underline{w}^m$ with

$$m > n + 3 + \frac{t_{sh}(L) + \ell_{sh}(L) + d^* + D}{\ell(\gamma_p)}.$$

By proposition 3.4.1(b), $\widetilde{\gamma}^+$ $\frac{1}{2}\varepsilon_{sh}(L)$–shadows γ_p for $T > n\ell(\gamma_p) + d^* + D$ units of time.

We conclude that $Y_i(e^{i\theta})$ is the first pair of pants in $\mathcal{Y}$ which $\widetilde{\gamma}^+$ crosses, where it shadows a closed geodesic γ_p with cutting sequence in $\mathcal{W}(L, c_0)$ for more than $n\ell(\gamma_p) + d^* + D$ units of time. This is part (a) of step 1, with $t_0 := d^* + D$.

To see part (b), recall from (4.3) that $\varepsilon_{sh}(L) < \frac{1}{4}\eta(L)$ where $\eta(L)$ is a lower bound for the size of the collar neighborhood of a boundary component of length less than L. This means that if γ_1, γ_2 $\varepsilon_{sh}(L)$–shadow each other, then any time γ_1 crosses a pants component with norm less than L, γ_2 crosses it as well. This means that $\Gamma g^s(\omega(e^{i\theta}))$ and $\Gamma g^s(\omega(e^{i\widetilde{\theta}}))$ cross the same elements of $\mathcal{Y}$ during the time interval $0 < s < \mathrm{dist}(o, R) - [d^* + D]$. In a time interval of length $d^* + D$, neither geodesic can cross more than

$$N(L) := \frac{d^* + D}{\eta(L)}$$

elements of $\mathcal{Y}$. It follows that $Y_i(e^{i\theta}) = Y_j(e^{i\widetilde{\theta}})$ where $|i - j| \leq 2N(L)$.

Step 2. κ_i is injective, whence measure preserving.

Proof. It is a general fact that any measure which is invariant under a countable group action, is also invariant under any bi-measurable invertible map $\kappa : A \to B$ such that for every x, the orbit of x is equal to the orbit of $\kappa(x)$ (here A, B are measurable subsets). Thus it is enough to show that $\kappa_i : \Omega_Y(n, I) \to \kappa_i[\Omega_Y(n, I)]$ is invertible.

Define $e^{i\widetilde{\theta}}$ by $\kappa_i(e^{i\theta}, s) = (e^{i\widetilde{\theta}}, *)$. We show how to reconstruct $e^{i\theta}$ from $e^{i\widetilde{\theta}}$.

Here is how to read $O_i(e^{i\theta})$ from $e^{i\widetilde{\theta}}$: Follow the geodesic ray $\widetilde{\gamma}^+ := \{g^s(\omega(e^{i\widetilde{\theta}})\}_{s>0}$ until the first time t its projection to $\Gamma\backslash\mathbb{D}$ crosses a pair of pants $Y \in \mathcal{Y}$ where it shadows a closed geodesic γ_p with cutting sequence $\underline{w} \in \mathcal{W}(L, c_0)$ and length ε_0–away from $c_0\mathbb{Z}$, for more than $n\ell(\gamma_p) + t_0$ units of time. By step 1(a), $Y = Y_i(e^{i\theta})$, and its unique lift to $\mathbb{D}$ which contains $g^t(\omega(e^{i\widetilde{\theta}}))$ is $O_i(e^{i\theta})$.

Note that although we know the octagon $O_i(e^{i\theta})$, we do not know the particular choice of labels of its sides (which is α and which is $\overline{\alpha}$?). But this does not matter: for any labeling $\mathcal{S}'$, there must be labels $x', w'_1, \ldots, w'_k, u'_1, \ldots, u'_N, y', z'$ such that

$$e^{i\widetilde{\theta}} \in (\varphi_{x'}^{O_i(e^{i\theta})})^{\ell}[(\varphi_{w'_1}^{O_i(e^{i\theta})}) \cdots (\varphi_{w'_k}^{O_i(e^{i\theta})})]^{n_0}(\varphi_{u'_1}^{O_i(e^{i\theta})}) \cdots (\varphi_{u'_N}^{O_i(e^{i\theta})})(\varphi_{y}^{O_i(e^{i\theta})})^m[Q(z')]$$

with $|\ell|, |m|$ maximal, and $(w'_1, w'_2) \neq (u'_1, u'_2)$. This determines $(w'_1, \ldots, w'_k)$, and we must have $(\varphi_{w'_1}^{O_i(e^{i\theta})}) \cdots (\varphi_{w'_k}^{O_i(e^{i\theta})}) = (\varphi_{w_1}^{O_i(e^{i\theta})}) \cdots (\varphi_{w_k}^{O_i(e^{i\theta})})$ in the original labeling of $O_i(e^{i\theta})$. Thus

$$e^{i\theta} = (\varphi_{x'}^{O_i(e^{i\theta})})^{\ell}[(\varphi_{w'_1}^{O_i(e^{i\theta})}) \cdots (\varphi_{w'_k}^{O_i(e^{i\theta})})]^{-n_0}(\varphi_{x'}^{O_i(e^{i\theta})})^{-\ell}(e^{i\widetilde{\theta}}).$$

We reconstructed $e^{i\theta}$ from $e^{i\widetilde{\theta}}$. This is enough to invert κ_i.

Step 3. There exists $j_0 = j_0(L)$ such that $\bigcup_{i>j_0} \kappa_i[\Omega_Y(n, I)]$ is precompact.

Proof. Fix i, and define $e^{i\widetilde{\theta}}$ as before by $(e^{i\widetilde{\theta}}, *) = \kappa_i(e^{i\theta}, *)$. We find a bound, independent of θ and i, for $-\log|(\varphi_{e^{i\theta}})'(e^{i\theta})| = R(e^{i\theta}, e^{i\widetilde{\theta}})$.

Equations (4.4) and (4.7), and the fact that it takes at least $\eta(L)$ units of time to cross a pair of pants in $\mathcal{Y}$, show that $\mathrm{dist}(g^s\omega(e^{i\theta}), g^s\omega(e^{i\widetilde{\theta}})) < \frac{1}{2}\varepsilon_{sh}(L)$ for some

$$s > \mathrm{dist}(o, P) - [d^* + 2D] - 1 > (i-1)\eta(L) - [d^* + 2D + 1].$$

If two geodesics emanating from the origin stay close after a long time, then their limit points in $\partial\mathbb{D}$ must be close. Thus $\exists j_0 = j_0(L)$ so large that

$$i > j_0 \Longrightarrow |e^{i\theta} - e^{i\widetilde{\theta}}| < 1.$$

Indeed we can make $|e^{i\theta} - e^{i\widetilde{\theta}}|$ arbitrarily small by choosing j_0 sufficiently large.

By proposition 4.1.1, $|R(e^{i\theta}, e^{i\widetilde{\theta}})| \leq |B(\Gamma\gamma, \Gamma\widehat{\gamma})| + 4$, where γ is the $\mathbb{D}$-geodesic from $-e^{i\theta}$ to $e^{i\theta}$, and $\widehat{\gamma}$ is the $\mathbb{D}$–geodesic from $-e^{i\theta}$ to $e^{i\widetilde{\theta}}$. We estimate $B(\Gamma\gamma, \Gamma\widehat{\gamma})$. By proposition 3.3.1, there is an ℓ such that

(a) $\gamma^* := (\varphi_\alpha^{O_i(e^{i\theta})})^{-\ell}[\gamma]$ and $\widehat{\gamma}^* := (\varphi_\alpha^{O_i(e^{i\theta})})^{-\ell}[\widehat{\gamma}]$ enter $O_i(e^{i\theta})$ through points $A, \widehat{A} \in \partial O_i(e^{i\theta})$, respectively. Let $\omega_A, \omega_{\widehat{A}}$ be the unit tangent vectors to γ^*, $\widehat{\gamma}^*$ at $A, \widehat{A}$. Note that $\gamma^*(-\infty) = \widehat{\gamma}^*(-\infty)$.

(b) γ^* and $\widehat{\gamma}^{\#}{:=}(\varphi_\alpha^{O_i(e^{i\theta})})^{\ell}[(\varphi_{\overline{w}_k}^{O_i(e^{i\theta})})\cdots(\varphi_{\overline{w}_1}^{O_i(e^{i\theta})})]^{n_0}(\varphi_\alpha^{O_i(e^{i\theta})})^{-\ell}[\widehat{\gamma}]$ leave $O_i(e^{i\theta})$ through $B, \widehat{B} \in \partial O_i(e^{i\theta})$, respectively. Let $\omega_B, \omega_{\widehat{B}}$ be the unit tangent vectors to γ^*, $\widehat{\gamma}^{\#}$ at $B, \widehat{B}$. Note that $\gamma^*(\infty) = \widehat{\gamma}^{\#}(\infty)$.

(c) The geodesic segment $(\Gamma\widehat{A}, \Gamma\widehat{B})$ (on $\Gamma\widehat{\gamma}$) has cutting sequence $(w_1, \ldots, w_k)^{n_0}$ (see the proof of proposition 3.3.1), so it D–shadows n_0 windings of the closed geodesic γ_p with symbolic period $\underline{w}$.

By construction $g^{B_{\gamma^*(-\infty)}(A,\widehat{A})}\omega_A, \omega_{\widehat{A}}$ are backward asymptotic w.r.t. the geodesic flow, and $g^{B_{\gamma^*(\infty)}(B,\widehat{B})}\omega_B, \omega_{\widehat{B}}$ are forward asymptotic w.r.t. the geodesic flow. Therefore,

$$\begin{aligned}|B(\Gamma\gamma, \Gamma\widetilde{\gamma})| &\leq |B_{\gamma^*(-\infty)}(A, \widehat{A})| + \big|\mathrm{dist}(\widehat{A}, \widehat{B}) - \mathrm{dist}(A, B)\big| + |B_{\gamma^*(\infty)}(B, \widehat{B})| \\ &\leq \mathrm{dist}(A, \widehat{A}) + [\mathrm{dist}(\widehat{A}, \widehat{B}) + \mathrm{dist}(A, B)] + \mathrm{dist}(B, \widehat{B}) \\ &\leq D + [(D + n_0\ell(\gamma_p) + D) + D] + D \leq M_0 n_0 + 5D.\end{aligned}$$

Thus $\big|-\log|(\varphi_{e^{i\theta}})'(e^{i\theta})|\big| = |R(e^{i\theta}, e^{i\widetilde{\theta}})| \leq M_0 n_0 + 5D + 4 =: K_0$.

It follows that $\bigcup_{i>j_0} \kappa_i[\Omega_Y(n, I)] \subset \partial\mathbb{D}\times[-K_0-|I|, K_0+|I|]$, so it is pre-compact.

Step 4. $\{\kappa_i[\Omega_Y(n, I)]\}_{i\geq i_0}$ contains an infinite family of pairwise disjoint sets.

Proof. Suppose $(e^{i\widetilde{\theta}}, s) \in \kappa_i[\Omega_Y(n, I)]$ and $i > i_0$, then the first $Y \in \mathcal{Y}$ where $\{g^s(\omega(e^{i\widetilde{\theta}}))\}_{s>0}$ $\frac{1}{2}\varepsilon_{sh}(L)$ shadows a closed geodesic γ_p with cutting sequence in $\mathcal{W}(L, c_0)$ and length ε_0–away from $c_0\mathbb{Z}$ for more than $n\ell(\gamma_p) + t_0$ units of time is $Y_j(e^{i\widetilde{\theta}})$, where $j \in [i - N(L), i + N(L)]$. This means that $\{\kappa_{4N(L)i}[\Omega_Y(n, I)]\}_{i>i_0}$ are pairwise disjoint.

We have just found an infinite collection of pairwise disjoint sets, with the same measure, whose union is precompact. But this is impossible, because m is a Radon measure. The proposition follows with $\delta(c_0){:=}\delta_0, M(c_0){:=}M_0$ and $\varepsilon(c_0){:=}\varepsilon_0$. □

4.3.3. *Proof of the holonomy lemma in case (a).* Fix ε, c_0 positive, and let L, $M(c_0)$, $\delta(c_0)$, and $\varepsilon(c_0)$ be as in proposition 4.3.2. Define $\mathcal{Y}$ to be the collection of pants components whose boundary components have lengths in $[L^{-1}, L]$, and let D denote the supremum of the diameters of the hyperbolic octagons representing elements of $\mathcal{Y}$. Fix some $N_0, k_0, s_0 \in \mathbb{N}$, to be determined later.

Denote the ordered sequence of elements of $\mathcal{Y}$ which $\{\Gamma g^s\omega(e^{i\theta})\}_{s>0}$ crosses by $\{Y_i(e^{i\theta})\}_{i\geq 1}$. Let $j = j(e^{i\theta}, k_0, s_0)$ be the first j s.t. $\{\Gamma g^{s+s_0}\omega(e^{i\theta})\}_{s>0}$ crosses $Y_j(e^{i\theta})$ and $\varepsilon_{sh}(L)$–shadows there a rich closed geodesic γ_p such that

$$\delta_0(c_0) \leq \ell(\gamma_p) \leq M(c_0) \text{ and } \mathrm{dist}(\ell(\gamma_p), c_0\mathbb{Z}) > \varepsilon(c_0),$$

for more than $2k_0\ell(\gamma_p)$ units of time.

If there is more than one such geodesic, then we always assume that γ_p is the one which is shadowed for the longest time. If there is more than one closed geodesic achieving the maximum, choose the one whose cutting sequence comes first in the lexicographic order. Let $\gamma_p = \gamma_p(e^{i\theta})$ be the unique resulting closed geodesic.

Let $\underline{w} = \underline{w}(e^{i\theta})$ be the symbolic period of $\gamma_p(e^{i\theta})$ (choose the cyclic permutation which appears first in the lexicographic order). By proposition 3.4.1, the cutting sequence of $\{\Gamma g^s\omega(e^{i\theta})\}$ in $Y_j(e^{i\theta})$ contains $\underline{w}^{2k_0'}$ where $k_0' > k_0 - 5$. There is a point X on $\{\Gamma g^s\omega(e^{i\theta})\}_{s>s_0} \cap Y_j(e^{i\theta})$ such that the cutting sequence (in Y_j) up to X ends with $\underline{w}^{k_0-5}$, and the cutting sequence starting at X begins with $\underline{w}^{k_0-5}$. Define s_X by $\Gamma g^{s_X}\omega(e^{i\theta}) = X$.[10]

Lift $Y_j(e^{i\theta})$ into an octagon $O_j := O_{j(e^{i\theta},k_0,s_0)}(e^{i\theta}) \subset \mathbb{D}$ in such a way that O_j contains X. Label the sides of O_j as in figure 1. Let $\varphi_s^{O_j} \in \Gamma$, $s \in \mathcal{S}$, be the side pairing transformations of O_j, and define $\varphi_{e^{i\theta}} := \varphi_{\xi_1}^{O_j} \circ \cdots \circ \varphi_{\xi_N}^{O_j}$ where $\underline{w} = (\xi_1, \ldots, \xi_N)$ is the cutting sequence of γ_p.

Fix some bounded function $\nu : [\delta(c_0), M(c_0)] \to \mathbb{N}$, and define

$$\kappa(e^{i\theta}) \quad := \quad (\varphi_{e^{i\theta}})^{N_0+\nu(\ell(\gamma_p(e^{i\theta})))}(e^{i\theta}).$$

We show how to choose s_0, N_0, k_0 and $\nu(\cdot)$ to obtain the Γ–holonomies of the holonomy lemma.

Set $e^{i\widetilde{\theta}} := \kappa(e^{i\theta})$, and consider the geodesics γ from $-e^{i\theta}$ to $e^{i\theta}$ and $\widehat{\gamma}$ from $-e^{i\theta}$ to $e^{i\widetilde{\theta}}$. Repeating the argument done in the previous section, we obtain an analogue of (4.7): there are constants $s_2 = s_2(L)$ and $d^* = d(L)$ such that if $s_0 > s_2$, then

$$\mathrm{dist}(g^s\omega(e^{i\widetilde{\theta}}), g^s\omega(e^{i\theta})) < \tfrac{1}{2}\varepsilon_{sh}(L) \text{ for all } 0 < s < \widetilde{s}_X - [d^* + D], \tag{4.8}$$

where $\widetilde{s}_X$ is the time $g^{\widetilde{s}_X}\omega(e^{i\widetilde{\theta}})$ enters $O_j(e^{i\theta})$. As before $|\widetilde{s}_X - s_X| < D$.

Evidently, $\widetilde{s}_X - [d^* + D] > s_X - 2D - d^* > s_0 - 2D - d^*$, so we can choose $s_0 = s_0(L)$ so large that (4.8) forces

$$|\kappa(e^{i\theta}) - e^{i\theta}| = |e^{i\widetilde{\theta}} - e^{i\theta}| < \min\{1, \sqrt{\varepsilon(c_0)/10}, \varepsilon\},$$

regardless of the choice of k_0, N_0, and $\nu(\cdot)$. This gives us the choice of s_0.

Next we choose $k_0 = k_0(L)$ to be so large that $((k_0 - 5) - 2)\delta(c_0) > D + 5$ and $100e^{D-((k_0-5)-2)\delta(c_0)} < \frac{1}{10}\varepsilon(c_0)$ (see proposition 4.3.1 for motivation). This choice guarantees that no matter which $N_0, \nu(\cdot)$ we choose,

$$\begin{aligned} R(e^{i\theta}, \kappa(e^{i\theta})) &= B(\Gamma\gamma, \Gamma\widehat{\gamma}) \pm \tfrac{1}{10}\varepsilon(c_0) && \text{(Proposition 4.1.1)} \\ &= [N_0 + \nu(\ell(\gamma_0))]\ell(\gamma_p) \pm \tfrac{1}{5}\varepsilon(c_0). && \text{(Proposition 4.3.1)} \end{aligned}$$

Next we choose N_0 so that for every choice of $\nu(\cdot)$, κ is invertible. By construction, the cutting sequence of $\Gamma\widehat{\gamma}$ in $Y_j(e^{i\theta})$ contains the word $\underline{w}^{N_0}$. Recall the definitions of $\ell_{sh}(\cdot)$ and $t_{sh}(\cdot)$ from proposition 3.4.1, and choose N_0 so that $(N_0 - 3)\delta(c_0) > \ell_{sh}(L)$ and $(N_0 - 3)\ell(\gamma_p) - t_{sh}(L) > 100(D + d^*) + 2k_0M(c_0)$. This has the effect of guaranteeing that $\Gamma\widehat{\gamma}$ $\frac{1}{2}\varepsilon_{sh}(L)$–shadows γ_p in $Y_j(e^{i\theta})$ for more than $100(D + d^*) + 2k_0M(c_0)$ units of time.

$\{\Gamma g^s\omega(e^{i\widetilde{\theta}})\}_{s>s_0}$ cannot $\frac{1}{2}\varepsilon_{sh}(L)$–shadow a rich closed geodesic γ_p' with length $\delta(c_0) \le \ell(\gamma_p') \le M(c_0)$, $\mathrm{dist}(\ell(\gamma_p'), c_0\mathbb{Z}) > \varepsilon(c_0)$ for such a long period of time at some $Y \in \mathcal{Y}$ *before* arriving to $Y_j(e^{i\theta})$: Had it done so, then by (4.8) it would have

[10] s_X is well defined provided X is not a point where $\{\Gamma g^s\omega(e^{i\theta})\}_{s>s_0}$ crosses itself. We can always assume this to be case, because we are free to move X a little.

had to $\frac{1}{2}\varepsilon_{sh}(L)$–shadow γ_p' at least $2k_0\ell(\gamma_p')$ units of time when it is also $\frac{1}{2}\varepsilon_{sh}(L)$–shadowing $\{\Gamma g^s\omega(e^{i\theta})\}_{s>s_0}$; This would force $\{\Gamma g^s\omega(e^{i\theta})\}_{s>s_0}$ to $\varepsilon_{sh}(L)$–shadow γ_p' for a time a period longer than $2k_0\ell(\gamma_p')$ before arriving to $Y_j(e^{i\theta})$ – a contradiction to the minimality of j.

Thus if N_0 is chosen as above, then we can always 'read' $Y_j(e^{i\theta})$ from $e^{i\widetilde{\theta}}$. Since we can read $Y_j(e^{i\theta})$ from $e^{i\widetilde{\theta}}$, we can read $\gamma_p(e^{i\theta})$ from $e^{i\widetilde{\theta}}$: It is the *unique* rich closed geodesic γ_p with $\ell(\gamma_p)\in[\delta(c_0),M(c_0)]$, $\operatorname{dist}(\ell(\gamma_p),c_0\mathbb{Z})>\delta(c_0)$ which $\{g^s\omega(e^{i\widetilde{\theta}})\}_{s>s_0}$ shadows for the *longest* time in $Y_j(e^{i\theta})$. Having read γ_p, we can now determine $\nu(\ell(\gamma_p))$, $\underline{w}(e^{i\theta})$ and $\varphi_{e^{i\theta}}$, and invert κ.

To summarize, for every function $\nu:[\delta(c_0),M(c_0)]\to\mathbb{N}$, we have constructed a Γ–holonomy κ, whose domain of definition A satisfies $m[A^c\times\mathbb{R}]=0$, which satisfies $|\kappa(e^{i\theta})-e^{i\theta}|<\varepsilon$, and

$$\left|R(e^{i\theta},\kappa(e^{i\theta}))-[N_0+\nu(\ell(\gamma_p))]\ell(\gamma_p)\right|<\frac{1}{5}\varepsilon(c_0).$$

For part (a) of the holonomy lemma, choose (for a given $n\in\mathbb{N}$), $\nu(x):=\lfloor n/x\rfloor$, $\alpha=1$, and $M_0=(N_0+1)M(c_0)+\frac{1}{5}\varepsilon(c_0)$.

For part (b) of the holonomy lemma, construct $\nu':[\delta(c_0),M(c_0)]\to\mathbb{N}$ bounded measurable s.t. $\operatorname{dist}([N_0+\nu(x)]x,c_0\mathbb{Z})<\frac{1}{5}\varepsilon(c_0)$ for all $x\in[\delta(c_0),M(c_0)]$).[11] Set

$$\nu(x):=\nu'(x)+1.$$

Since $\operatorname{dist}(\ell(\gamma_p),c_0\mathbb{Z})>\varepsilon(c_0)$, $\operatorname{dist}([N_0+\nu(\ell(\gamma_p))]\ell(\gamma_p),c_0\mathbb{Z})>\frac{4}{5}\varepsilon(c_0)$, whence $\operatorname{dist}(R(e^{i\theta},\kappa(e^{i\theta})),c_0\mathbb{Z})>\frac{3}{5}\varepsilon(c_0)$. Since $\nu'(\cdot)$ is bounded, $|R(e^{i\theta},\kappa(e^{i\theta}))|\le[N_0+\sup\nu'+1]M(c_0)+\frac{1}{5}\varepsilon(c_0)$. We get a holonomy as in part (b) of the holonomy lemma, except for the naming of constants. □

4.4. **Proof of the holonomy lemma in case (b).** In case (b) almost every geodesic crosses a sequence of pairs of pants through boundary components with length bounded away from zero and infinity, and where the other component has length tending to zero. The diameter of such pairs of pants tends to infinity, so the argument we used in case (a) breaks down.

To deal with this we show that, almost surely, the geodesic will spend very long time in the collar of the 'other' boundary component. Then we will utilize the special geometric properties of collars of very short closed geodesics.

4.4.1. *Controlling the Busemann cocycle in case (b).* Suppose γ is a geodesic which crosses a pair of pants Y entering through side a, and leaving through b. Abusing notation we write $c=\ell(c)$, and let $s=s(c)=\tan\theta$ where θ solves $\operatorname{dist}(e^{i\theta},i)=\eta(c)$ (c.f. Proposition 3.4.2). We have $s\sim\frac{c}{2}$ as $c\to 0^+$.

Choose a positive constant $c_{\max}<1/100$ so small that if $0<c<c_{\max}$, then

(1) $\frac{2s}{c}\in[\frac{1}{2},2]$,
(2) $\cos\theta\ge\frac{1}{2}$,
(3) $(1-e^{-mc})^2=2^{\pm1}(mc)^2$ for all $8\le m\le 21$.

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[11] If p_n/q_n are the principal convergents of $\alpha=p/c_0$, then $\operatorname{dist}(q_np,c_0\mathbb{Z})<\frac{c_0}{q_n}$ so $N(p):=q_n-N_0$ works for p for n large enough. The same $N(p)$ works for all p' sufficiently close to p. A compactness argument finishes the construction.

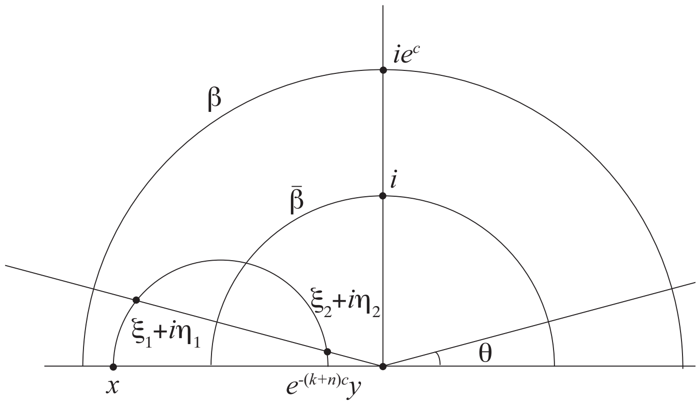

FIGURE 4. Length of a geodesic passing through a collar

Suppose the cutting sequence of γ in Y contains the word β^k, with k maximal (β labels a seam ending at c). Let O_Y be an octagon representing Y, and lift γ to $\mathbb{D}$ in such a way that it enters O_Y through a point X so that the cutting sequence of γ from X onwards starts with β^k.

Define $\widetilde{\gamma}$ to be the projection to $\Gamma \setminus \mathbb{D}$ of the geodesic with endpoints $\widetilde{\gamma}(-\infty) = \gamma(-\infty)$ and $\widetilde{\gamma}(\infty) = \varphi_\beta^n[\gamma(\infty)]$.

Proposition 4.4.1. *If $0 \le c < c_{\max}$, $k > 10$, then there exists $\varepsilon \in [-2, 2]$ such that $B(\gamma, \widetilde{\gamma}) = [f_{c,\varepsilon}(k+n) - f_{c,\varepsilon}(k)] \pm 100[c + \frac{1}{k}]$, where*

$$f_{c,\varepsilon}(t) := \begin{cases} 2\sinh^{-1}\sqrt{\frac{1}{s(c)^2}\sinh^2\frac{(t+\varepsilon)c}{2} - 1} & c \neq 0 \\ 2\sinh^{-1}\sqrt{(t+\varepsilon)^2 - 1} & c = 0. \end{cases}$$

Proof. We do the case $c \neq 0$, and leave the (much easier) case $c = 0$ to the reader.

Step 1. $\gamma, \widetilde{\gamma}$ enter the collar of c.

Proof. Applying a suitable isometry we draw $\gamma, \widetilde{\gamma}$ and O_Y in the upper half plane model $\mathbb{H}$ in such a way that c is the segment $[i, e^c i]$, β is $\{z \in \mathbb{H} : |z| = e^c, \mathrm{Re}(z) < 0\}$, $\overline{\beta}$ is $\{z \in \mathbb{H} : |z| = 1, \mathrm{Re}(z) < 0\}$, and O_Y is to the left of $[i, e^c i]$ (figure 4).

In this model φ_β is represented by the isometry $z \mapsto e^{-c}z$, γ is a geodesic with beginning point x and endpoint $e^{-kc}y$ where $x, y \in (-e^c, -e^{-c})$, and $\widetilde{\gamma}$ is the geodesic with beginning x and end $e^{-(k+n)c}y$. The collar of c is $\{z \in \mathbb{H} : |\mathrm{Im}(z)| \ge |s\mathrm{Re}(z)|\}$ where $s = s(c) = \tan\theta$. By the choice of $c_{\max}$, $s = 2^{\pm 1}\frac{c}{2} \in [\frac{c}{4}, c]$.

A geodesic with endpoints $x, e^{-\mu y}$ $(x, y \in (-e^c, -e^{-c}))$ enters the collar iff the following system of equations has two solutions $\xi_j + i\eta_j$, $\xi_j, \eta_j \in \mathbb{R}$, $(j = 1, 2)$:

$$(\xi - \tfrac{x + e^{-\mu c}y}{2})^2 + \eta^2 = (\tfrac{x - e^{-\mu c}y}{2})^2 \ , \ \eta = -s\xi, \tag{4.9}$$

This resolves to $(1+s^2)\xi^2 - (x+e^{-\mu c}y)\xi + e^{-\mu c}xy = 0$. The discriminant is

$$\Delta \;=\; x^2\left[(1-e^{-(\mu+\varepsilon)c})^2 - 4s^2 e^{-(\mu+\varepsilon)c}\right], \text{ where } \varepsilon := \frac{1}{c}\ln\frac{x}{y} \in [-2,2].$$

If $\mu > 10$ and $0 < c < c_{\max}$, then $\Delta \geq x^2[(1-e^{-8c})^2 - 4c^2] > 0$, the system has two real solutions, and γ enters the collar. Since by assumption $k > 10$, $\mu > 10$ for $\gamma, \widetilde{\gamma}$ and we are done.

Step 2. Comparison of the distance travelled by $\gamma, \widetilde{\gamma}$ inside the collar.

Solving (4.9) we see that points of entry and exit of $\gamma, \widetilde{\gamma}$ to the collar are

$$z_j := \xi_j + i\eta_j = \frac{1-is}{2(1+s^2)}\left[(x+e^{-\mu c}y) + (-1)^j\sqrt{(x+e^{-\mu c}y)^2 - 4(1+s^2)xye^{-\mu c}}\right]$$

with $\mu = k$ for γ, and $\mu = k+n$ for $\widetilde{\gamma}$. We see that

$$\sinh\left(\tfrac{1}{2}\operatorname{dist}(z_1,z_2)\right) \;\equiv\; \frac{|z_1-z_2|}{2\sqrt{\operatorname{Im}(z_1)\operatorname{Im}(z_2)}} = \sqrt{\frac{1}{s^2}\sinh^2\frac{(\mu+\varepsilon)c}{2} - 1},$$

so the difference between the distances covered by $\gamma, \widetilde{\gamma}$ in the collar is

$$D_0 := f_{c,\varepsilon}(k+n+\varepsilon) - f_{c,\varepsilon}(k+\varepsilon).$$

Step 3. Estimation of the distance between the entry points $P, \widetilde{P}$ of $\gamma, \widetilde{\gamma}$ to the collar, and between the exit points $Q, \widetilde{Q}$ of $\varphi_\beta^{-k}[\gamma]$, $\varphi_\beta^{-(k+n)}[\widetilde{\gamma}]$ from the collar.

Draw $\gamma, \widetilde{\gamma}$ as in figure 4. The entry point of $\widetilde{\gamma}$ to the collar is between the entry point of γ (which we denote by $z = \xi + i\eta$), and the entry point of the geodesic $\{z \in \mathbb{H} : \operatorname{Re}(z) = x\}$ (which is $w = x - isx$). It follows that $\operatorname{dist}(P, \widetilde{P})$ is bounded by the hyperbolic length of the (euclidean) segment $[z,w]$:

$$\operatorname{dist}(P,\widetilde{P}) \leq \ell[z,w] = \left|\frac{1}{\sin\theta}\ln\frac{\operatorname{Im}(z)}{\operatorname{Im}(w)}\right|. \tag{4.10}$$

It is clear that $\operatorname{Im}(z)/\operatorname{Im}(w) < 1$. We estimate it from below. The geodesic γ is a semi-circle of radius

$$\begin{aligned} r \;&=\; \frac{1}{2}(e^{-kc}y - x) = \frac{1}{2}|x|(1-e^{-(k+\varepsilon)c}), \text{ where } |\varepsilon| < 2 \text{ is given by } x = e^{\varepsilon c}y \\ &=\; \frac{1}{2}c|x| \cdot \frac{1-e^{-(k+\varepsilon)c}}{c} > \frac{1}{4}c|x|\min\left\{(k+\varepsilon), \frac{1}{c}\right\}. \end{aligned}$$

(If $(k+\varepsilon)c \geq 1$, this is obvious. If $(k+\varepsilon)c < 1$, use $(1-e^{-x})/x > 1 - e^{-1}$ for $0 < x < 1$.) If C is the center of this circle, and $\alpha := \measuredangle zCx$ (measured in radians), then $\alpha = \arcsin\frac{\eta}{r} \leq \arcsin\frac{s|x|}{r} \leq \frac{\pi}{2}\cdot\frac{s|x|}{r}$ ($\because$ arcsin is convex on $[0,\frac{\pi}{2}]$), so

$$|x-\xi| = r(1-\cos\alpha) = 2r\sin^2\frac{\alpha}{2} \leq \frac{1}{2}r\alpha^2 \leq \frac{\pi^2 s^2 x^2}{8r},$$

whence $\operatorname{Im}(z) = \eta = s|x| - s|x-\xi| \geq s|x| - \frac{\pi^2 s^3 x^2}{8r} = \operatorname{Im}(w)\left(1 - \frac{\pi^2 s^2|x|}{8r}\right)$.

Recalling that $0 < c < c_{\max}$, we see that

$$\begin{aligned} \ell[z,w] \;&\leq\; \frac{1}{\sin\theta}\left|\ln\left(1 - \frac{\pi^2 s^2|x|}{8r}\right)\right| \\ &<\; \frac{2}{s}\left|\ln\left(1 - \frac{5s}{\min\{(k+\varepsilon), 1/c\}}\right)\right| \quad \left(\because\ \sin\theta = s\cos\theta \geq \frac{s}{2},\ \text{and } \frac{s}{c} \leq 1\right). \end{aligned}$$

Since $k>10$, $|\varepsilon|<2$, $s<c$, and $c<\frac{1}{100}$, $\frac{5s}{\min\{(k+\varepsilon),1/c\}}<\frac{1}{2}$. Since $|\ln(1-t)|<2t$ for all $0<t<\frac{1}{2}$, we see that the distance between the entry points is at most

$$\ell[z,w]<\frac{20}{\min\{(k+\varepsilon),1/c\}}<20\left(\frac{1}{k-2}+c\right)<25(\frac{1}{k}+c).$$

The estimate for the exit points is obtained in a similar way, and is omitted.

Step 4. Calculation of $B(\gamma,\widehat{\gamma})$.

Let $P,\widetilde{P}$ and $Q,\widetilde{Q}$ be as above, then

$$\begin{aligned} B(\gamma,\widetilde{\gamma}) &= B_{\gamma(-\infty)}(P,\widetilde{P})+[\mathrm{dist}(P,\varphi_\beta^k Q)-\mathrm{dist}(\widetilde{P},\varphi_\beta^{k+n}\widetilde{Q})]+B_{\varphi_\beta^{-k}\gamma(\infty)}(Q,\widetilde{Q}) \\ &= D_0\pm[\mathrm{dist}(P,\widetilde{P})+\mathrm{dist}(Q,\widetilde{Q})]=D_0\pm 50\left(c+\frac{1}{k}\right). \end{aligned}$$

The proposition follows in the case $c\neq 0$, because of step 2. □

Lemma 4.4.1. *Suppose $\varepsilon\in[-2,2]$ and $c\geq 0$. Then*

(a) *$f_{c,\varepsilon}(\cdot)$ is strictly increasing, and $f_{c,\varepsilon}(t)\xrightarrow[t\to\infty]{}\infty$, and $f'_{c,\varepsilon}(\cdot)$ is decreasing,*
(b) $\sup\{f'_{c,\varepsilon}(t):0<c<1,t\geq t_0,|\varepsilon|\leq 2\}\xrightarrow[t_0\to\infty]{}0$.

4.4.2. *The support of m in case (b).* In case (b), there are positive constants $\ell_{\min}<\ell_{\max}$ such that for a.e. $e^{i\theta}$, $\{\Gamma g^s\omega(e^{i\theta})\}_{s>0}$ crosses infinitely many pants with norm less than $\ell_{\max}$, entering and leaving through boundary components with lengths longer than $\ell_{\min}$. Without loss of generality, $\ell_{\min}<\min\{c_{\max},c_0^*\}$, where $c_{\max}$ is as in the previous section and c_0^* is as in proposition 3.4.2 .

Let $\mathcal{Y}$ be the collection of all pairs of pants which have two boundary components with lengths in $[\ell_{\min},\ell_{\max}]$ ('long'), and one boundary component with length in $[0,\ell_{\min})$ ('short'). Let

(a) $\{Y_j(e^{i\theta})\}_{j\geq 1}$ be the sequence of pairs of pants in $\mathcal{Y}$ which $\{\Gamma\omega(e^{i\theta})\}_{s>0}$ crosses, entering and leaving through the long boundary components;
(b) $c_j(e^{i\theta})$:=the length of the short boundary component of $Y_j(e^{i\theta})$ (possibly equal to zero);
(c) $k_j(e^{i\theta})$:=the maximal length of the cutting sequence of $\{\Gamma\omega(e^{i\theta})\}_{s>0}$ in the collar of the short boundary component (if $\{\Gamma\omega(e^{i\theta})\}_{s>0}$ does not enter that collar, set $k_j(e^{i\theta}):=0$).

Proposition 4.4.2. *In case* (b), $\limsup_{j\to\infty}k_j(e^{i\theta})=\infty$ *almost everywhere.*

Proof. We show that for every $n_0\in\mathbb{N}$ and every compact interval $I\subset\mathbb{R}$,

$$\Omega(n_0,I):=\{(e^{i\theta},s)\in\partial\mathbb{D}\times I:\nexists j>0\text{ s.t. }k_j(e^{i\theta})>n_0\}$$

has measure zero.

The strategy is to assume by way of contradiction that $\Omega(n_0,I)$ has positive measure for some n_0,I, and then derive a contradiction using the same method we used in case (a). But the implementation is different, because in case (b) $\mathrm{diam}[Y_j(e^{i\theta})]\xrightarrow[j\to\infty]{}\infty$.

Fix $(e^{i\theta},s)\in\Omega(n_0,I)$ such that $\{Y_j(e^{i\theta})\}$ is an infinite sequence (almost all $e^{i\theta}$ have this property). Label the boundary components and seams of $Y_j(e^{i\theta})$ in accordance to figure 1, in such a way that the entry boundary component is a, and the exit boundary component is b_2, then

$$\begin{aligned}\ell(a),\ell(b_1),\ell(b_2) &\in [\ell_{\min},\ell_{\max}],\\ \ell(c) &\le \ell_{\min},\\ \ell(\alpha),\ell(\overline{\alpha}) &\le \cosh^{-1}\left(\frac{\cosh\frac{\ell_{\max}}{2}+\cosh^2\frac{\ell_{\max}}{2}}{\sinh^2\frac{\ell_{\min}}{2}}\right), \ (\because \text{hexagon formula (3.1)})\\ \ell(\beta),\ell(\overline{\beta}) &\ge \eta(c_i(e^{i\theta})), \text{ and could tend to infinity on a subsequence.}\end{aligned}$$

Let $\underline{w}$ denote the cutting sequence of $\{\Gamma g^s\omega(e^{i\theta})\}_{s>0}$ in $Y_j(e^{i\theta})$. We can always write $\underline{w}=\underline{w}'\beta^k$ or $\underline{w}=\underline{w}'\overline{\beta}^k$ with k maximal (possibly zero) and $\underline{w}'$ a reduced word (possible empty). Let η be the last symbol of $\underline{w}'$ when this word is not empty, and $\eta=a$ otherwise.

For every j, let $O_j(e^{i\theta})$ be the a lift of $Y_j(e^{i\theta})$ to an octagon in $\mathbb{D}$ done in such a way that $O_j(e^{i\theta})$ contains the segment of $\{g^s(e^{i\theta})\}$ from side η to $\beta^k b_2$ or $\overline{\beta}^k b_2$. Let $\varphi_x^{O_j(e^{i\theta})}\in\Gamma$ $(x\in\mathcal{S})$ denote the side pairing transformations of $O_j(e^{i\theta})$.

Fix N, j_0 (to be determined later), and define for an arbitrary fixed $j>j_0$

$$\begin{aligned}\varphi_{e^{i\theta}} &:= \begin{cases}(\varphi_\beta^{O_j(e^{i\theta})})^N & \underline{w} \text{ does not end with } \overline{\beta}\\ (\varphi_{\overline{\beta}}^{O_j(e^{i\theta})})^N & \underline{w} \text{ ends with } \overline{\beta}\end{cases}\\ \kappa_j(e^{i\theta},s) &:= \big(\varphi_{e^{i\theta}}(e^{i\theta}), s-\log|\varphi'_{e^{i\theta}}(e^{i\theta})|\big).\end{aligned}$$

Step 1. Define $e^{i\widetilde{\theta}}$ by $\kappa_j(e^{i\theta},s)=(e^{i\widetilde{\theta}},*)$, and let $\widetilde{\gamma}^+$ denote the geodesic ray $\{g^s\omega(e^{i\widetilde{\theta}})\}_{s>0}$. If j_0,N are large enough (how large depends on $n_0,\ell_{\min},\ell_{\max}$ but not on $e^{i\theta}$), then for all $j>j_0$

(a) $Y_j(e^{i\theta})$ is the first $Y\in\mathcal{Y}$ crossed by $\Gamma\widetilde{\gamma}^+$ through its long boundary components, where $\Gamma\widetilde{\gamma}^+$ enters the collar of the short boundary component, and has cutting sequence of length more than $N-25$ in that collar;

(b) there is a constant $N(\ell_{\max},\ell_{\min},n_0)$ such that $Y_j(e^{i\theta})=Y_{j'}(e^{i\widetilde{\theta}})$ with $|j'-j|<N(\ell_{\max},\ell_{\min},n_0)$.

Proof. We do the proof under the assumption that $c_j(e^{i\theta})\neq 0$. The modifications needed for the case $c_j(e^{i\theta})=0$ are obvious, and are left to the reader.

Let γ be the geodesic from $-e^{i\theta}$ to $e^{i\theta}$; $\widehat{\gamma}$ the geodesic from $-e^{i\theta}$ to $e^{i\widetilde{\theta}}$; and $\widetilde{\gamma}$ the geodesic from $-e^{i\widetilde{\theta}}$ to $e^{i\widetilde{\theta}}$.

Claim 1. If $N>10$, then $\Gamma\widehat{\gamma}$ enters the collar of the short boundary component, and its cutting sequence there has length $(k_j(e^{i\theta})+N)\pm 15$.

Proof. We do the case when $\underline{w}$ does not end with $\overline{\beta}$ (the other case is exactly the same). Applying a suitable isometry we draw $\widehat{\gamma},\gamma$ and $O_j(e^{i\theta})$ in the upper half plane $\mathbb{H}$ in such a way that its side c is the vertical segment $[i,e^c i]$ and $O_j(e^{i\theta})$ is to the left of $[i,e^c i]$, where we have have set $c=c_j(e^{i\theta})$ (figure 4).

Again $\varphi_\beta^{O_j(e^{i\theta})}$ is represented by the isometry $z\mapsto e^{-c}z$, β is $\{z\in\mathbb{H}:|z|=e^c,\mathrm{Re}(z)<0\}$, $\overline{\beta}$ is $\{z\in\mathbb{H}:|z|=1,\mathrm{Re}(z)<0\}$, γ is a geodesic with beginning point x and endpoint $e^{-kc}y$ where $x,y\in[-e^c,-1]$, and $\widehat{\gamma}$ is the geodesic with beginning x and end $e^{-(k+N)c}y$ (we shall comment later on the relation between k

and $k_j(e^{i\theta})$). The collar of c is $\{z \in \mathbb{H} : |\mathrm{Im}(z)| \geq |s\mathrm{Re}(z)|\}$ where $s = \tan\theta$ and θ is the root of $\mathrm{dist}(i, e^{i\theta}) = \eta(c)$. Since $\ell_{\min} < c_{\max}$, $s = 2^{\pm 1}\frac{c}{2}$.

The proof of step 1 of proposition 4.4.1 shows that if $k + N > 10$ (which is the case when $N > 10$), then $\widehat{\gamma}$ enters the collar of c. Equation (3.3) and the assumption $c < \ell_{\min} \leq \min\{1, c_0^*\}$ show that the cutting sequence of $\Gamma\widehat{\gamma}$ in the collar has length $(k + N) \pm 5$.

We compare k to $k_j(e^{i\theta})$. If $\Gamma\gamma$ enters the collar, then (3.3) says that $k_j(e^{i\theta}) = k \pm 4$. If $\Gamma\gamma$ does not enter the collar then necessarily $k \leq 10$, (see the proof of step 1 in proposition 4.4.1), and $k_j(e^{i\theta}) \equiv 0 = k \pm 10$. In both cases, $k_j(e^{i\theta}) = k \pm 10$, and we deduce that the length of cutting sequence of $\Gamma\widehat{\gamma}$ in $Y_j(e^{i\theta})$ is $(N + k_j(e^{i\theta})) \pm 15$.

Claim 2. The length of the pieces of $\Gamma\widehat{\gamma}$ inside the collar of c is bounded from above by some constant which only depends on $\ell_{\min}, \ell_{\max}, n_0$, and N.

Proof. This length was calculated in step 2 of the proof of proposition 4.4.1 to be

$$D_0 = 2\sinh^{-1}\sqrt{\frac{1}{s^2}\sinh^2\frac{(\mu + \varepsilon)c}{2} - 1}.$$

where $\mu = N + k$ and $|\varepsilon| \leq 2$. We saw above that $k \leq k_j(e^{i\theta}) + 10$, and by assumption $(e^{i\theta}, *) \in \Omega(n_0, I)$, so $k_j(e^{i\theta}) \leq n_0$. Thus $\mu + \varepsilon < N + n_0 + 12$. Recalling that $s \geq \frac{1}{4}c$, we see that $D_0 \leq 2\sinh^{-1}\left(\frac{4}{c}\sinh\frac{(N+n_0+12)c}{2}\right)$. This expression has a continuous extension at $c = 0$, and is therefore bounded on the set $\{c : 0 \leq c \leq \ell_{\min}\}$, with a bound which only depends on $\ell_{\min}, \ell_{\max}, N$ and n_0.

Claim 3. There are $k_1 \leq n_0 + 26$, $|k_2 - N| \leq n_0 + 15$ such that $\widehat{\gamma}$ enters the collar of c at $(\varphi_\beta^{O_j(e^{i\theta})})^{k_1}[O_j(e^{i\theta})]$, and leaves it at $(\varphi_\beta^{O_j(e^{i\theta})})^{k_2}[O_j(e^{i\theta})]$, and if we define

- $\widehat{P}$ to be the point of entry of $\widehat{\gamma}$ to the collar of c in $(\varphi_\beta^{O_j(e^{i\theta})})^{k_1}[O_j(e^{i\theta})]$,
- $\widehat{Q}$ to be the point of exit of $\widehat{\gamma}$ from the collar of c in $(\varphi_\beta^{O_j(e^{i\theta})})^{k_2}[O_j(e^{i\theta})]$,
- P the point of entry of γ to $O_j(e^{i\theta})$,
- Q the point of exit of $\varphi_{e^{i\theta}}[\gamma]$ from $\varphi_{e^{i\theta}}[(\varphi_\beta^{O_j(e^{i\theta})})^k O_j(e^{i\theta})]$ (we claim that this point exists),

then $\mathrm{dist}(P, \widehat{P})$, $\mathrm{dist}(Q, \widehat{Q})$, and $\mathrm{dist}(P, \varphi_{e^{i\theta}}^{-1}(Q))$ are bounded from above by a constant which only depends on $\ell_{\min}, \ell_{\max}$, and n_0.

Proof. Proposition 3.3.1 shows that $\gamma, \widehat{\gamma}$ cross $O_j(e^{i\theta})$, then $\varphi_\beta^{O_j(e^{i\theta})}[O_j(e^{i\theta})]$, then $(\varphi_\beta^{O_j(e^{i\theta})})^2[O_j(e^{i\theta})]$, and so on up to $(\varphi_\beta^{O_j(e^{i\theta})})^k[O_j(e^{i\theta})]$ in the case of γ, and $(\varphi_\beta^{O_j(e^{i\theta})})^{k+N}[O_j(e^{i\theta})]$ in the case of $\widehat{\gamma}$. In particular, P, Q exist.

In the upper half plane picture described in the proof of claim 1, $O_j(e^{i\theta})$ lies in the half annulus $\{z : 1 \leq |z| \leq e^c\}$, $\varphi_\beta^{O_j(e^{i\theta})}$ is $z \mapsto e^{-c}z$, and $\widehat{\gamma}$ is a geodesic with endpoints in $(-e^c, -e^{-(k+N+1)c})$. Thus $\widehat{\gamma}$ cannot intersect $(\varphi_\beta^{O_j(e^{i\theta})})^i(O_j(e^{i\theta}))$ when $i > k + N + 1$. Recalling that $k \leq k_j(e^{i\theta}) + 10 \leq n_0 + 10$, we see that

$$k_2 \leq k + N + 1 \leq N + n_0 + 11.$$

On the other hand, $k_2 - k_1$ is at least as large as the length of the cutting sequence of $\Gamma\widehat{g}$ at the collar, which is at least $N - 15$ (claim 1), so $k_2 \geq N - 15$ and $k_1 < k_2 + 15 - N \leq n_0 + 26$. This proves the existence of $k_1, k_2, \widehat{P}, \widehat{Q}$.

We estimate $\text{dist}(P, \widehat{P})$. Let $\widehat{P}_i$ be the entry points of $\widehat{\gamma}$ to $(\varphi_\beta^{O_j(e^{i\theta})})^i[O_j(e^{i\theta})]$. By the definition of k_1, $\widehat{P}_i$ are outside the interior of collar of c for $i = 0, \dots, k_1$, and $P = \widehat{P}_0$, so

$$\begin{aligned}
\text{dist}(P, \widehat{P}) &\leq \sum_{i=1}^{k_1} \text{dist}(\widehat{P}_{i-1}, \widehat{P}_i) + \text{dist}(\widehat{P}_{k_1}, \widehat{P}) \\
&\leq (k_1+1)\text{diam}\big(O_j(e^{i\theta}) \setminus \text{collar of } c\big) \\
&= (k_1+1)\text{diam}\big(\text{core of } O_j(e^{i\theta}) \cup \text{collars of } a, b_1, b_2\big) \\
&\leq (n_0+27)[M_{\text{core}}(\ell_{\max}) + 2\eta(\tfrac{\ell_{\min}}{2})] \quad (\because \tfrac{1}{2}\ell_{\min} \leq \ell(a), \ell(b_i) \leq \ell_{\max})
\end{aligned}$$

($M_{core}(\cdot)$ is defined in lemma 3.2.2). In the same way, one shows that

$$\begin{aligned}
\text{dist}(Q, \widehat{Q}) &\leq [(k+N) - k_2 + 1][M_{\text{core}}(\ell_{\max}) + 2\eta(\ell_{\min}/2)] \\
&\leq (2n_0 + 26)[M_{\text{core}}(\ell_{\max}) + 2\eta(\ell_{\min}/2)].
\end{aligned}$$

Set $\widetilde{Q} := \varphi_{e^{i\theta}}^{-1}(Q)$. We estimate $\text{dist}(P, \widetilde{Q})$. Let A be the geodesic segment from P to $\widetilde{Q}$. This is the part of $\{g^s\omega(e^{i\theta})\}_{s>0}$ with cutting sequence $\eta\beta^k$ or $\eta\overline{\beta}^k$. It can be divided into three pieces: A_1, from P to the the collar of c; A_2 in the collar of c; and A_3 from the collar of c to $\widetilde{Q}$ (if A does not intersect the interior of the collar of c, set $A_1 := A$ and $A_2, A_3 := \varnothing$).

A_1 and A_3 have cutting sequences of length at most $k+1$, and they do not intersect the interior of the collar of c, therefore their lengths are no more that

$$2(k+1)[M_{\text{core}}(\ell_{\max}) + 2\eta(\ell_{\min}/2)] \leq (2n_0+22)[M_{\text{core}}(\ell_{\max}) + 2\eta(\ell_{\min}/2)].$$

A_2 is the sojourn of $\Gamma\gamma$ in the collar of c. Since the length of the cutting sequence of γ in that collar is bounded by n_0, the length of A_2 is bounded by some constant which only depends on $\ell_{\min}$ and n_0. We conclude that $\text{dist}(P, \widetilde{Q})$ is bounded by some constant which only depends on $\ell_{\min}, \ell_{\max}$, and n_0.

Claim 4. Proof of step 1, part (a).

Proof. Let $C = C(\ell_{\min}, \ell_{\max}, n_0)$ denote the upper bound for $\text{dist}(P, \widehat{P})$ found in the previous claim.

Let $\breve{P}$ be the intersection of $\widehat{\gamma}$ with $\text{Hor}_{-e^{i\theta}}(o)$, and let $\mathring{P}$ be the intersection of $\widehat{\gamma}$ with $\text{Hor}_{-e^{i\theta}}(P)$. We have:

$$\begin{aligned}
\text{dist}(\breve{P}, o) &\leq \text{the length of the arc } \breve{P}o \text{ on } \text{Hor}_{-e^{i\theta}}(o) \\
&\leq e^{-\text{dist}(o,P)} \times \text{the length of the arc } \mathring{P}P \text{ on } \text{Hor}_{-e^{i\theta}}(o) \\
&\leq 2e^{-\text{dist}(o,P)}[1 - \tanh \text{dist}(\mathring{P}, P)]^{-1} \quad \text{(see the footnote on page 26);} \\
\text{dist}(\mathring{P}, P) &\leq \text{dist}(\mathring{P}, \widehat{P}) + \text{dist}(\widehat{P}, P) = |B_{-e^{i\theta}}(P, \widehat{P})| + \text{dist}(\widehat{P}, P) \leq 2\,\text{dist}(P, \widehat{P}) \\
&< 2C.
\end{aligned}$$

Thus $\text{dist}(\breve{P}, o) \leq 2e^{-\text{dist}(o,P)}[1 - \tanh 2C]^{-1}$.

Let P^* to be the intersection of $\widehat{\gamma}$ with $\text{Hor}_{e^{i\tilde{\theta}}}(o)$. Then

$$\begin{aligned}
\text{dist}(P^*, o) &\leq \text{dist}(P^*, \breve{P}) + \text{dist}(\breve{P}, o) = |B_{e^{i\tilde{\theta}}}(o, \breve{P})| + \text{dist}(\breve{P}, o) \leq 2\,\text{dist}(\breve{P}, o) \\
&\leq 4e^{-\text{dist}(o,P)}[1 - \tanh 2C]^{-1}.
\end{aligned}$$

By the footnote on page 26, the length of the arc P^*o on $\mathrm{Hor}_{e^{i\tilde{\theta}}}(o)$ is bounded by

$$\frac{2\tanh[\mathrm{dist}(o,P^*)/2]}{1-\tanh[\mathrm{dist}(o,P^*)/2]} \leq \frac{2\,\mathrm{dist}(o,P^*)}{1-\tanh[\mathrm{dist}(o,P^*)/2]} \leq K(C)e^{-\mathrm{dist}(o,P)},$$

where $K(C) := 10(1-\tanh 2C)^{-1}[1-\tanh(2/(1-\tanh 2C))]^{-1}$.

This means that if $\widehat{\omega}$ is the unit tangent vector to $\widehat{\gamma}$ at P^*, then

$$\mathrm{dist}(g^s\widehat{\omega}, g^s\omega(e^{i\widetilde{\theta}})) \leq K(C)e^{-(\mathrm{dist}(o,P)+s)}.$$

Recall that it takes at least $\eta(\ell_{\max})$ units of time to cross a pair of pants with norm no larger than $\ell_{\max}$, and that $\Gamma\gamma^+$ crosses $j-1$ such pants between o and P. Therefore $\mathrm{dist}(o,P) > (j-1)\eta(\ell_{\max})$.

Choose $j_0 = j_0(\ell_{\max}, C)$ such that if $j > j_0$, then

$$\begin{aligned} e^{-(j-1)\eta(\ell_{\max})}K(C) &< \frac{1}{10}\min\left\{\varepsilon_{sh}^{col}, \eta(\ell_{\max})\right\}, \\ (j-1)\eta(\ell_{\max}) &\geq 4[1-\tanh C]^{-1} + C. \end{aligned}$$

(ε_{sh}^{col} is given by proposition 3.4.2).

Assume from now on that $j > j_0$ and $N > 10$. By the first condition on j_0,

$$\mathrm{dist}(g^s\widehat{\omega}, g^s\omega(e^{i\widetilde{\theta}})) \leq \frac{1}{10}e^{-s}\min\{\varepsilon_{sh}^{col}, \eta(\ell_{\max})\} \text{ for all } s > 0. \tag{4.11}$$

The second condition on j_0 implies that $\widehat{\gamma}$ enters the collar of the short boundary component of Y_j at positive time, because the base point of $\widehat{\omega}$, P^*, is $\mathrm{dist}(o,P^*) < 4[1-\tanh C]^{-1}$ away from o, whereas the entrance point of $\widehat{\gamma}$ to O_j, $\widehat{P}$, is at least $d(o,P) - d(P,\widehat{P}) \geq (j-1)\eta(\ell_{\max}) - C > 4[1-\tanh C]^{-1}$ away from o.

Thus $\Gamma\widetilde{\gamma}^+$ ε_{sh}^{col}–shadows $\Gamma\widehat{\gamma}^+$ during its sojourn in the collar of the short boundary component of $Y_j(e^{i\theta})$.

By proposition 3.4.2 and claim 1, the length of the cutting sequence of $\Gamma\widetilde{\gamma}$ in that collar is at least $N + k_j(e^{i\theta}) - 25 \geq N - 25$.

We show that if N is large enough, then $Y_j(e^{i\theta})$ is the first pair of pants where $\Gamma\widetilde{\gamma}^+$ behaves this way.

Let $\breve{\omega}$ be the unit tangent vector to $\widehat{\gamma}$ at $\breve{P}$. We saw above that $\mathrm{dist}(o,\breve{P}) \leq 2e^{-\mathrm{dist}(o,P)}[1-\tanh 2C]^{-1}$. Therefore the length of the arc $\breve{P}o$ on $\mathrm{Hor}_{-e^{i\theta}}(o)$ is less than $K(C)e^{-\mathrm{dist}(o,P)}$. It follows that

$$\mathrm{dist}(g^s\omega(e^{i\theta}), g^s\breve{\omega}) \leq K(C)e^{s-\mathrm{dist}(o,P)} \text{ for all } s > 0. \tag{4.12}$$

We also clearly have

$$\begin{aligned} \mathrm{dist}(g^s\breve{\omega}, g^s\widehat{\omega}) &= \mathrm{dist}(\breve{\omega},\widehat{\omega}) \quad (\because \breve{\omega},\widehat{\omega} \text{ are on the same geodesic}) \\ &= |B_{-e^{i\theta}}(P^*,\breve{P})| = |B_{-e^{i\theta}}(P^*,o)| \leq \mathrm{dist}(P^*,o) \\ &< 2K(C)e^{-\mathrm{dist}(o,P)}. \end{aligned} \tag{4.13}$$

By (4.11), (4.12), and (4.13), for all $0 \le s \le \text{dist}(P^*, \widehat{P})$ =the entrance time of $g^s\widehat{\omega}$ to the collar of the short boundary component in $Y_j(e^{i\theta})$,

$$\begin{aligned}\text{dist}(g^s\omega(e^{i\widetilde{\theta}}), g^s\omega(e^{i\theta})) &\le \text{dist}(g^s\omega(e^{i\widetilde{\theta}}), g^s\widehat{\omega}) + \text{dist}(g^s\widehat{\omega}, g^s\breve{\omega}) \\ &\qquad + \text{dist}(g^s\breve{\omega}, g^s\omega(e^{i\theta})) \\ &< \tfrac{1}{10}\min\{\varepsilon_{sh}^{col}, \eta(\ell_{\max})\} + 2e^{-\text{dist}(o,P)}K(C) \\ &\qquad + e^{s-\text{dist}(o,P)}K(C) \\ &\le \tfrac{3}{10}\min\{\varepsilon_{sh}^{col}, \eta(\ell_{\max})\} + e^{s-\text{dist}(o,P)}K(C) \qquad (4.14) \\ &\le \eta(\ell_{\max}) + e^{C+K(C)}K(C) =: \delta(C),\end{aligned}$$

where the last inequality is because

$$\begin{aligned}s - \text{dist}(o,P) &\le \text{dist}(P^*, \widehat{P}) - \text{dist}(o,P) < \text{dist}(P^*, o) + \text{dist}(P, \widehat{P}) \\ &\le K(C)e^{-\text{dist}(o,P)} + C < K(C) + C.\end{aligned}$$

In summary, $\exists\delta = \delta(C)$ s.t. $\{g^s\omega(e^{i\widetilde{\theta}})\}_{s=0}^{\text{dist}(P^*,\widehat{P})}$ δ–shadows $\{g^s\omega(e^{i\theta})\}_{s=0}^{\text{dist}(P^*,\widehat{P})}$.

We now appeal to corollary 3.4.1: $\exists K_0 := K_0(\delta(C))$ s.t. if $\{\Gamma g^s\omega(e^{i\widetilde{\theta}})\}_{s=0}^{\text{dist}(P^*,\widehat{P})}$ enters the collar of the short boundary component of some $Y \in \mathcal{Y}$ and has cutting sequence of length k there, then the length of the cutting sequence of any other geodesic which enters that collar and $\delta(C)$–shadows it there is at least $k - K_0$. Thus by (4.14), the length of the cutting sequence of $\{\Gamma g^s\omega(e^{i\theta})\}_{s=0}^{\text{dist}(P^*,\widehat{P})}$ in the collar of the short boundary component is at least $k - K_0$.

By assumption $\omega(e^{i\theta}) \in \Omega(n_0, I)$, so $k - K_0 \le n_0$. Thus: *the cutting sequence of* $\{\Gamma g^s\omega(e^{i\widetilde{\theta}})\}_{s=0}^{\text{dist}(P^*,\widehat{P})}$ *in the short boundary component of any* $Y \in \mathcal{Y}$ *which it crosses through the long boundary components, has length no larger than* $n_0 + K_0$.

We claim that the previous statement remains true if we replace $\text{dist}(P^*, \widehat{P})$ (the entrance time of $\Gamma g^s(\widehat{\omega})$ to the collar of the short boundary component of Y_j) by the entrance time of $\Gamma g^s\omega(e^{i\widetilde{\theta}})$ to Y_j. Indeed, (4.11) says that $\Gamma g^s\widehat{\omega}$ is $\frac{1}{4}\eta(\ell_{\max})$–shadowing $\Gamma g^s\widehat{\omega}$, which at time $s = \text{dist}(P^*, \widehat{P})$ is at position $\Gamma\widehat{P} \in \overline{Y_j(e^{i\theta})}$ on its way *in* to that pair of pants. This means at time $s = \text{dist}(P^*, \widehat{P})$, $\Gamma g^s\omega(e^{i\widetilde{\theta}})$ is either already in $Y_j(e^{i\theta})$, in the inner half of the collar of one of its boundary components, or on its way to Y_j in the external half of the collar of one of the boundary components of Y_j. In either case there is no way for it to cross any $Y \in \mathcal{Y}$ other than $Y_j(e^{i\theta})$.

Thus $\Gamma\widetilde{\gamma}^+$ does not cross a $Y \in \mathcal{Y}$ where it enters the collar of the short boundary component with cutting sequence of length more than $n_0 + K_0$, before arriving to $Y_j(e^{i\theta})$.

Thus if we choose N to be larger than $n_0 + K_0(\delta(C)) + 25$, and j to be larger than $j_0(\ell_{\max}, C)$, then $Y_j(e^{i\theta})$ will always be the first pair of pants crossed by $\Gamma\widetilde{\gamma}^+$ through its long boundary components, where the cutting sequence in the collar of the short boundary component is longer than $N - 25$. Step 1 (a) is done.

Claim 5. Proof of step 1, part (b).

Proof. By (4.14), there is a constant $T = T(C)$ s.t. $\text{dist}(g^s\omega(e^{i\theta}), g^s\omega(e^{i\widetilde{\theta}})) \le \frac{1}{3}\eta(\ell_{\max})$ for all $s < \text{dist}(P^*, \widehat{P}) - T$. Thus $\Gamma g^s\omega(e^{i\theta})$ and $\Gamma g^s\omega(e^{i\widetilde{\theta}})$ cross exactly the same elements of $\mathcal{Y}$ during the time window $[0, \text{dist}(P^*, \widehat{P}) - T]$.

These geodesics can cross at most $T/\eta(\ell_{\max})$ elements of $\mathcal{Y}$ during the time interval $[\mathrm{dist}(P^*,\widehat{P})-T,\mathrm{dist}(P^*,\widehat{P})]$.

At time $s=\mathrm{dist}(P^*,\widehat{P})$ (as explained above), they are both either in $Y_j(e^{i\theta})$ or on their way to cross $Y_j(e^{i\theta})$ without the possibility of crossing any other element of $\mathcal{Y}$ on their way.

Thus $Y_j(e^{i\theta})=Y_{j'}(e^{i\widetilde{\theta}})$ with $|j'-j|\leq T/\eta(\ell_{\max})$. Since $T=T(C)$, and $C=C(\ell_{\min},\ell_{\max},n_0)$, $N_0:=T/\eta(\ell_{\max})$ only depends on $\ell_{\min},\ell_{\max}$, and n_0.

Step 2. κ_j is injective, whence measure preserving.

Proof. Suppose we know $\kappa_i(e^{i\theta},s)=(e^{i\widetilde{\theta}},*)$. In order to find $e^{i\theta}$, we need to identify $\varphi_{e^{i\theta}}$.

To do this follow the projection of $g^s\omega(e^{i\widetilde{\theta}})$ on $\Gamma\backslash\mathbb{D}$ until the first time it crosses an element of $\mathcal{Y}$ through its long boundary components, and where its cutting sequence in the collar of the short boundary component has length more than $N-25$. By step 1 this pair of pants is $Y_j(e^{i\theta})$.

Label the seams and boundary components of $Y_j(e^{i\theta})$ so that a is the entrance component, b the exit component, and c the short component. Let $\underline{w}$ be the cutting sequence in $Y_j(e^{i\theta})$, write $\underline{w}=\underline{w}'\beta^k$ or $\underline{w}=\underline{w}'\overline{\beta}^k$ with k maximal, and let η be the last symbol of $\underline{w}'$ if this word is not empty and $\eta=a$ otherwise. Lift $Y_j(e^{i\theta})$ to the octagon in $\mathbb{D}$ which contains the segment of $g^s\omega(e^{i\widetilde{\theta}})$ from η to β^k or $\overline{\beta}^k$. This is $O_j(e^{i\theta})$.

The short boundary component of $O_j(e^{i\theta})$ has length $c_j(e^{i\theta})$, and lifts to a geodesic $\gamma_p\subset\mathbb{D}$. There are two ways to orient γ_p, choose the one which is compatible with the orientation of $g^s\omega(e^{i\widetilde{\theta}})$. Let φ be the hyperbolic transformation such that

- the axis of φ is γ_p;
- the translation length of φ is $Nc_j(e^{i\theta})$;
- φ translates points on its axis in the positive direction (w.r.t. the orientation of γ_p).

Then $\varphi=\varphi_{e^{i\theta}}$. Having identified $\varphi_{\varepsilon^{i\theta}}$ it is no problem to invert κ_j at $(e^{i\widetilde{\theta}},*)$.

Thus κ_j is injective. It is obviously Borel, and it clearly preserves Γ–orbits. Thus it is measure preserving.

Step 3. There exists j_0^* such that $\bigcup_{j>j_0^*}\kappa_j[\Omega(n,I)]$ is precompact.

Proof. As in case (a), everything boils down to finding j_0^* so that if $j>j_0^*$, and $e^{i\widetilde{\theta}}$ is defined by $(e^{i\widetilde{\theta}},*)=\kappa_j(e^{i\theta},*)$, then $|R(e^{i\theta},e^{i\widetilde{\theta}})|$ is bounded by some constant which does not depend on $e^{i\theta}$ or j.

Starting from (4.14), we see that $\mathrm{dist}(g^s\omega(e^{i\theta}),g^s\omega(e^{i\widetilde{\theta}}))<\delta(C)$ for

$$\begin{aligned} s=\mathrm{dist}(P^*,\widehat{P}) &> \mathrm{dist}(o,P)-\mathrm{dist}(o,P^*)-\mathrm{dist}(\widehat{P},P)\\ &> \mathrm{dist}(o,P)-[C+K(C)]>(j-1)\eta(\ell_{\max})-[C+K(C)]. \end{aligned}$$

Thus there exists j_0^* which only depends on $\ell_{\min},\ell_{\max}$, and n_0 such that $j>j_0^*$ implies $|e^{i\theta}-e^{i\widetilde{\theta}}|<1$. For such j, $|R(e^{i\theta},e^{i\widetilde{\theta}})|<|B(\Gamma\gamma,\Gamma\widehat{\gamma})|+4$.

We show that $|B(\Gamma\gamma, \Gamma\widehat{\gamma})|$ is bounded by some constant which only depends on $\ell_{\min}, \ell_{\max}, n_0$, and N. Recall the definitions of $P, Q, \widehat{P}, \widehat{Q}$ from claim 3 above.

$$\begin{aligned} |B(\Gamma\gamma, \Gamma\widehat{\gamma})| &\leq |B_{-e^{i\theta}}(P, \widehat{P})| + |d_{\Gamma\gamma}(\Gamma P, \Gamma Q) - d_{\Gamma\widehat{\gamma}}(\Gamma\widehat{P}, \Gamma\widehat{Q})| + |B_{e^{i\widetilde{\theta}}}(Q, \widehat{Q})| \\ &\leq \operatorname{dist}(P, \widehat{P}) + [d_\gamma(P, \varphi_{e^{i\theta}}^{-1} Q) + d_{\widehat{\gamma}}(\widehat{P}, \widehat{Q})] + \operatorname{dist}(Q, \widehat{Q}) \end{aligned}$$

which is bounded by a constant only depending on $\ell_{\min}, \ell_{\max}, n_0$ by claim 3.

Step 4. Completion of the proof.

One shows, just like in case (a), that the family $\{\kappa_j[\Omega(n_0, I)]\}_{i \geq j_0}$ contains an infinite collection of pairwise disjoint elements. Since κ_j is measure preserving (a Γ–holonomy), $\bigcup_{i>j_0} \kappa_i[\Omega(n_0, I)]$ has infinite measure. But step 3 says that this set is pre-compact, so we get a contradiction to the Radon property of m.

This contradiction shows that $m[\Omega(n_0, I)] = 0$ for all n_0 and I, therefore for almost every $(e^{i\theta}, s)$ and every n_0, there exists j such that $k_j(e^{i\theta}) > n_0$. □

4.4.3. *Proof of the holonomy lemma in case (b).* Fix $\varepsilon, c_0 > 0$, and let $\ell_{\max}, \ell_{\min}, c_{\max}$ be positive constants as in the previous sections. Assume w.l.o.g. that $\varepsilon < 1$. Choose $n_0^* \in \mathbb{N}$ so large, and $c^*_{\max} > 0$ so small, that

(a) $n_0^* > 10/\varepsilon$,
(b) $0 < c^*_{\max} < \min\{1, \ell_{\min}, c_{\max}, c_0^*\}$,
(c) $100(\frac{1}{n_0^*} + 2c^*_{\max}) + \sup\{f'_{c,\varepsilon}(t) : |\varepsilon| \leq 2, 0 < c < c^*_{\max}, t \geq n_0^*\} < \varepsilon/2$

(see lemma 4.4.1).

Let $\mathcal{Y}^*$ denote the collection of pants with two boundary components of length in $(\ell_{\min}, \ell_{\max}]$ and one boundary component with length in $[0, c^*_{\max}]$ (henceforth referred to as the 'long' and 'short' components).

Let $Y_j(e^{i\theta})$ be the ordered list of pants in $\mathcal{Y}^*$ which $\Gamma g^s\omega(e^{i\theta})$ crosses through the long components. In case (b), this is an infinite sequence for a.e. $e^{i\theta}$.

Let $c_j(e^{i\theta})$ denote the length of the short boundary component of $Y_j(e^{i\theta})$. Suppose $\Gamma g^s\omega(e^{i\theta})$ enters the collar of the short boundary component. It may do so several times. Let $k_j(e^{i\theta})$ denote the length of the longest cutting sequence in the collar of the short boundary component done by $\Gamma g^s\omega(e^{i\theta})$ during its sojourn in $Y_j(e^{i\theta})$. Let s_0 be the time when $\Gamma g^s\omega(e^{i\theta})$ starts this 'maximal' sojourn (if there is more than one maximal sojourn, take the smallest possible s_0).

If $\Gamma g^s\omega(e^{i\theta})$ does not enter the collar of the short boundary component, let s_0 be the time $\Gamma g^s\omega(e^{i\theta})$ enters $Y_j(e^{i\theta})$.

Lift $Y_j(e^{i\theta})$ to the unique octagon $O_j(e^{i\theta}) \subset \mathbb{D}$ containing $g^{s_0}\omega(e^{i\theta})$. Label the sides of $O_j(e^{i\theta})$ in accordance to figure 1 in such a way that $\Gamma g^s\omega(e^{i\theta})$ enters $Y_j(e^{i\theta})$ through side a and leaves it through b_2.

Define two numbers $\widehat{k}_j(e^{i\theta}), \widehat{\varepsilon}_j(e^{i\theta})$ as follows. If $k_j(e^{i\theta}) = 0$, set $\widehat{k}_j = \widehat{\varepsilon}_j = 0$. Otherwise $\Gamma g^s\omega(e^{i\theta})$ enters the collar of the short boundary component during its sojourn in $O_j(e^{i\theta})$. Map $O_j(e^{i\theta})$ to the upper half plane in such a way that $\varphi_\beta^{O_j(e^{i\theta})}$ is conjugated to $z \mapsto e^{-c_j(e^{i\theta})}z$ when $c \neq 0$, or $z \mapsto z + 1$ if $c = 0$.[12] The image of

[12]This determines the conjugacy up to application of a map of the form $z \mapsto kz$ ($c \neq 0$) or $z \mapsto z + 1$ ($c = 0$). The definitions which follow are invariant under such transformations.

$\{g^s\omega(e^{i\theta})\}_{s>0}$ is some $\mathbb{H}$–geodesic from x to y, and $x, y \neq 0, \infty$. Define:

$$\widehat{k}_j(e^{i\theta}) \quad := \quad \begin{cases} \left\lfloor \left| \frac{1}{c_j(e^{i\theta})} \ln|x/y| \right| \right\rfloor & (c_j(e^{i\theta}) \neq 0) \\ \lfloor |x-y| \rfloor & (c_j(e^{i\theta}) = 0) \end{cases} \tag{4.15}$$

$$\widehat{\varepsilon}_j(e^{i\theta}) \quad := \quad \begin{cases} \left\{ \frac{1}{c_j(e^{i\theta})} \ln|x/y| \right\} & (c_j(e^{i\theta}) \neq 0) \\ \{y-x\} & (c_j(e^{i\theta}) = 0) \end{cases} \tag{4.16}$$

$$\mathrm{sgn}_j(e^{i\theta}) \quad := \quad \begin{cases} \mathrm{sgn}\ln(|x/y|) & (c_j(e^{i\theta}) \neq 0) \\ \mathrm{sgn}(y-x) & (c_j(e^{i\theta}) = 0) \end{cases} \text{ (where } \mathrm{sgn}\, 0 := 0). \tag{4.17}$$

By (3.3), $|\widehat{k}_j - k_j| \leq 4$.

Recall the definition of $f_{c,\varepsilon}(\cdot)$ from proposition 4.4.1, and define for all $N \in \mathbb{R}^+$,

$$g_{c,\varepsilon,N}(n) := \min\{n' > n : f_{c,\varepsilon}(n') - f_{c,\varepsilon}(n) \geq N\}.$$

The minimum exists because $f_{c,\varepsilon}(t) \xrightarrow[t\to\infty]{} \infty$ by lemma 4.4.1.

By our choice of n_0^*, $f'_{c,\varepsilon}(t) < 1$ for all $t \geq n_0^*$. Consequently, $g_{c,\varepsilon,N}(n) > n + N$ for all $n \geq n_0$. Moreover, $g_{c,\varepsilon,N}(\cdot)$ is injective, even strictly increasing: Set $f = f_{c,\varepsilon}$, $g = g_{c,\varepsilon,N}$, then

$$\begin{aligned} f(g(n)-1) - f(n-1) &= [f(g(n)) - f(n)] + [f(n) - f(n-1)] - [f(g(n)) - f(g(n)-1)] \\ &\geq N + \inf_{t<n} f'(t) - \sup_{t>g(n)-1} f'(t) \\ &\geq N + \inf_{t<n} f'(t) - \sup_{t>n} f'(t) \quad (\because g(n) > n) \\ &\geq N \quad (\because f' \text{ is decreasing (lemma 4.4.1)}). \end{aligned}$$

But $g(n-1)$ is the smallest n' s.t. $f(n') - f(n-1) \geq N$, so $g(n-1) \leq g(n) - 1 < g(n)$, proving that $g_{c,\varepsilon,N}$ is strictly increasing, whence injective.

Fix two numbers j_0^*, N_0^* (to be determined later), and define for $\mathbb{R} \ni N > N_0^*$:

$$\begin{aligned} j(e^{i\theta}) &:= \min\{j \geq j_0^* : \widehat{k}_j(e^{i\theta}) > n_0^*\}, \\ N(e^{i\theta}) &:= g_{c,\widehat{\varepsilon},N}(\widehat{k}_j(e^{i\theta})) - \widehat{k}_j(e^{i\theta}), \text{ where } \begin{cases} j = j(e^{i\theta}) \\ c = c_j(e^{i\theta}) \\ \widehat{\varepsilon} = \widehat{\varepsilon}_j(e^{i\theta}) \end{cases}, \\ \varphi_{e^{i\theta}}(e^{i\theta}) &:= \begin{cases} (\varphi_\beta^{O_j(e^{i\theta})})^{N(e^{i\theta})} & \mathrm{sgn}_j(e^{i\theta}) \geq 0 \\ (\varphi_\beta^{O_j(e^{i\theta})})^{-N(e^{i\theta})} & \mathrm{sgn}_j(e^{i\theta}) < 0 \end{cases}, \\ \kappa(e^{i\theta}) &:= \varphi_{e^{i\theta}}(e^{i\theta}). \end{aligned}$$

These definition make sense for almost every $e^{i\theta}$, because of proposition 4.4.2 and the inequality $|\widehat{k}_j - k_j| \leq 4$.

We claim that if j_0^*, N_0^* are sufficiently large, then κ is injective.

The argument is similar to the one we used for the proof of proposition 4.4.2. Suppose $e^{i\widetilde{\theta}} = \kappa(e^{i\theta})$, and let γ be the geodesic from $-e^{i\theta}$ to $e^{i\theta}$; $\widetilde{\gamma}$ be the geodesic from $-e^{i\widetilde{\theta}}$ to $e^{i\widetilde{\theta}}$, and $\widehat{\gamma}$ the geodesic from $-e^{i\theta}$ to $e^{i\widetilde{\theta}}$. Set $j := j(e^{i\theta})$, $c = c(e^{i\theta})$, $\widehat{k} := \widehat{k}_j(e^{i\theta})$, $\widehat{\varepsilon}_j = \widehat{\varepsilon}_j(e^{i\theta})$.

Claim 1. $\Gamma\widehat{\gamma}$ enters the collar of the short boundary component of $Y_j(e^{i\theta})$ and its cutting sequence there has length at least $n_0^* + N_0^* - 4$.

Proof. By (3.3), this length is at least $\widehat{k}_j(e^{i\theta}) + N(e^{i\theta}) - 4 = g_{c,\widehat{\varepsilon},N}(\widehat{k}_j) - 4 > \widehat{k}_j + N - 4$, because $g_{c,\widehat{\varepsilon},N}(n) > n + N$ for all $n \geq n_0^*$. Since $\widehat{k}_j \geq n_0^*$, $\widehat{k}_j(e^{i\theta}) + N(e^{i\theta}) - 4 \geq n_0^* + N_0^* - 4$.

Claim 2. If P, $\widehat{P}$ are the points of entry of $g^s\omega(e^{i\theta})$ and $\widehat{\gamma}$ to the collar of the short boundary component in $O_j(e^{i\theta})$, then $\mathrm{dist}(P, \widehat{P}) \leq C := 25\left(\frac{1}{n_0^*} + c^*_{\max}\right)$.

Proof. See step 3 in the proof of proposition 4.4.1.

Claim 3. There are constants $j_0 = j_0(\ell_{\max}, C)$ and $N_0(C)$ s.t. if $j_0^* > j_0$, then

(a) $g^s\omega(e^{i\widetilde{\theta}})$ enters the collar of the short boundary component of $O_j(e^{i\theta})$, and the length of its cutting sequence in the projection of that collar is at least $n_0 + N_0^* - 25$;

(b) the length of the cutting sequence of $\Gamma\widetilde{\gamma}^+$ in the collars of the short boundary components of the pants in $\mathcal{Y}^*$ it crosses before $g^s\omega(e^{i\widetilde{\theta}})$ gets to $O_j(e^{i\theta})$ does not exceed $n_0 + N_0(C)$.

Consequently, if $N_0^* > N_0(C) + 100$, then $\kappa(e^{i\theta})$ determines $O_j(e^{i\theta})$.

Proof. See the proof of step 1 in the proof of proposition 4.4.2.

Now suppose that j_0^*, N_0^* are large enough to make the previous claim valid. Then we can 'read' $O_j(e^{i\theta})$ from $\kappa(e^{i\theta})$. This is enough to determine $c_j(e^{i\theta})$ and

- $\widehat{k}_j(e^{i\theta}) + N(e^{i\theta})$;
- $\widehat{\varepsilon}_j(e^{i\theta})$;
- $\mathrm{sgn}_j(e^{i\theta})$.

Indeed, these are the parameters of $\widetilde{\gamma}$ when we lift it to the upper half plane in a way which conjugates $\varphi_\beta^{O_j(e^{i\theta})}$ to $z \mapsto e^{-c_j(e^{i\theta})}z$.

Since we know $\widehat{k}_j(e^{i\theta})+N(e^{i\theta})$, we know $g_{c,\widehat{\varepsilon},N}(\widehat{k}_j(e^{i\theta}))$. We have seen above that $g_{c,\widehat{\varepsilon},N}(\cdot)$ is injective, therefore $g_{c,\widehat{\varepsilon},N}(\widehat{k}_j(e^{i\theta}))$ determines $\widehat{k}_j(e^{i\theta})$. But this means that we can also determine $N(e^{i\theta})$.

This is enough information to identify $\varphi_{e^{i\theta}} \equiv (\varphi_\beta^{O_j(e^{i\theta})})^{N(e^{i\theta})\,\mathrm{sgn}_j(e^{i\theta})}$, and to invert κ at $\kappa(e^{i\theta})$.

It follows that if j_0^*, N_0^* are large enough, and $N > N_0^*$, then κ is injective, whence a Γ–holonomy.

We now choose j_0^* and N. As in the proof of claim 4 in proposition 4.4.2, there are constants $\delta(C)$, $d(C)$ such that $\mathrm{dist}(g^s\omega(e^{i\theta}), g^s\omega(e^{i\widetilde{\theta}})) < \delta(C)$ for all $0 < s < d(o,P) - d(C)$, and $\mathrm{dist}(o,P) > (j_0^* - 1)\eta(\ell_{\max})$. It follows that if j_0^* is sufficiently large, then for all $e^{i\theta}$, $|e^{i\theta} - \kappa(e^{i\theta})| \leq \sqrt{\varepsilon}/4$.

Increase j_0^*, if necessary, to satisfy this condition, and fix $N \geq N_0^*$. We have

$$\begin{aligned}
R(e^{i\theta}, \kappa(e^{i\theta})) &= B(\Gamma\gamma, \Gamma\widehat{\gamma}) \pm 4|e^{i\theta} - \kappa(e^{i\theta})|^2 && \text{(proposition 4.1.1)} \\
&= [f_{c_j,\widehat{\varepsilon}_j}(\widehat{k}_j + N(e^{i\theta})) - f_{c_j,\widehat{\varepsilon}_j}(\widehat{k}_j)] \\
&\qquad \pm 100[c^*_{\max} + 1/n_0^*] \pm \frac{\varepsilon}{4} && \text{(proposition 4.4.1)} \\
&= [f_{c_j,\widehat{\varepsilon}_j}(g_{c_j,\widehat{\varepsilon}_j,N}(\widehat{k}_j)) - f_{c_j,\widehat{\varepsilon}_j}(\widehat{k}_j)] \pm 100\left(c^*_{\max} + \frac{1}{n_0^*}\right) \pm \frac{\varepsilon}{4}.
\end{aligned}$$

The first bracketed term is the first number in the sequence $\{f_{c,\widehat{\varepsilon}}(n)-f_{c,\widehat{\varepsilon}}(\widehat{k}_j)\}_{n\geq\widehat{k}_j}$ which is more than N. The increments of this sequence are less than

$$\sup\{f'_{c,\varepsilon}(t):|\varepsilon|\leq 2, 0<c<c^*_{\max}, t>n^*_0\},$$

because $\widehat{k}_j>n^*_0$ by the definition of $j=j(e^{i\theta})$. This means that the first bracketed term is equal to $N\pm\sup\{f'_{c,\varepsilon}(t):|\varepsilon|\leq 2, 0<c<c^*_{\max}, t>n^*_0\}$, whence

$$\begin{aligned}|R(e^{i\theta},\kappa(e^{i\theta}))-N| &\leq \sup\{f'_{c,\varepsilon}(t):|\varepsilon|\leq 2, 0<c<c^*_{\max}, t>n^*_0\}\\ &\qquad +100[c^*_{\max}+1/n^*_0]+\frac{\varepsilon}{4}\\ &< \varepsilon, \text{ by our choice of } c^*_0 \text{ and } n^*_0.\end{aligned}$$

Taking $N:=nN^*_0$ and $\varepsilon<1$ gives the first part of the holonomy lemma with $\alpha_0:=N^*_0$ and $M_0:=1$.

Taking $N\in c_0(\frac{1}{2}+\mathbb{Z})\cap[N^*_0,\infty)$ and $\varepsilon<c_0/4$ gives the lemma with $M(c_0):=N+1$ and $\varepsilon(c_0)=\frac{1}{4}c_0$. □

4.5. **Proof of the holonomy lemma in case (c).** In case (c) almost every $\{\Gamma g^s\omega(e^{i\theta})\}_{s>0}$ crosses infinitely many pairs of pants where it enters or leaves through a boundary component whose length tends to zero.

We label this boundary component by c, and the rest of the sides of the pants in accordance to figure 1 (the reader will hopefully excuse this deviation from the notational conventions of lemma 4.2.1). We abuse notation and write $c=\ell(c)$.

4.5.1. *Controlling the values of the Busemann cocycle in case (c).* Suppose γ is a geodesic which crosses the collar of c in a pair of pants Y in such a way that its cutting sequence inside the collar is β^k, with k maximal. Let O_Y be an octagon representing Y. Define a geodesic $\widetilde{\gamma}$ as follows: Fix $N\in\mathbb{N}$, lift γ to $\mathbb{D}$ in such a way that it crosses side c of O_Y. Define $\widetilde{\gamma}$ to be the projection of the geodesic with endpoints $\widetilde{\gamma}(-\infty)=\gamma(-\infty)$, $\widetilde{\gamma}(\infty)=\varphi^N_\beta[\gamma(\infty)]$.

Proposition 4.5.1. *$\forall\varepsilon\exists\ell_{\min}(\varepsilon)$, $K_{col}(\varepsilon)$, K_0,ε_0 s.t. if $0<c<\varepsilon_0$, then $|B(\gamma,\widetilde{\gamma})-Nc|<K_0$, and if $0<c<\ell_{\min}(\varepsilon)$ and $kc>K_{col}(\varepsilon)$, then $|B(\gamma,\widetilde{\gamma})-Nc|<\varepsilon$.*

Proof. We assume throughout that $0<c<1$.

Step 1. Calculation and comparison of the distance which $\gamma,\widetilde{\gamma}$ travel inside the collar of c.

Fix some $A>0$ (to be determined later). Draw the collar of c in the upper half plane in such a way that the fixed points of φ_β are ∞ (attracting) and 0 (repelling), and side c of O_Y is the segment $[iA,e^cAi]$. The geodesic γ_p is then the positive y–axis, φ_β becomes $z\mapsto e^cz$, and the boundary of the $\eta(c)$–collar of c is $y=s|x|$, where as before $s=\tan\theta$ and $\text{dist}(i,e^{i\theta})=\eta(c)$.

Draw γ in $\mathbb{H}$ so that its intersection point with γ_p is in O_Y. The endpoints of γ are on different sides of the y–axis, and there are unique $m,n\in\mathbb{Z}$, $x\in(-e^c,-1]$, $y\in[1,e^c)$ such that $\gamma(-\infty)=e^{-mc}x$ and $\gamma(+\infty)=e^{nc}y$. Choose A in such a way that $|m-n|\leq 1$ (apply the isometry $z\mapsto e^{\frac{m-n}{2}c}z$ when $2|(m-n)$ and $z\mapsto e^{\frac{m-n+1}{2}c}z$ when $2\not|(m-n)$, and choose A to ensure that $\{z:A\leq|z|\leq e^cA\}$ contains O_Y).

Note that $k:=m+n$, and $\frac{k}{2}-1\leq m,n\leq\frac{k}{2}+1$.

The entry and exit points of γ to the collar are $z_j = \xi_j + \eta_j i$ $(j = 1, 2)$, where

$$\xi_j = \frac{e^{nc}y + e^{-mc}x}{2(1+s^2)}\left[1 + (-1)^j\sqrt{1 + \frac{4|xy|e^{(n-m)c}(1+s^2)}{(e^{nc}y + e^{-mc}x)^2}}\right], \quad \eta_i = (-1)^j s\xi_j.$$

Using the identity $\sinh\left(\frac{1}{2}\operatorname{dist}(z_1, z_2)\right) \equiv |z_1 - z_2|/[2\sqrt{\mathrm{Im}z_1\mathrm{Im}z_2}]$, we see that

$$\begin{aligned}\sinh\left(\frac{1}{2}\operatorname{dist}(z_1, z_2)\right) &= \frac{|e^{nc}y + e^{-mc}x|}{2se^{\frac{n-m}{2}c}\sqrt{|xy|}}\sqrt{1 + \frac{4|xy|e^{(n-m)c}(1+s^2)}{(e^{nc}y + e^{-mc}x)^2}} \qquad (4.18)\\ &= \frac{e^{kc/2}}{2s}\frac{1 - e^{-kc}e^{\pm 2c}}{e^{\pm 2c}}\sqrt{1 + \frac{4e^{\pm 2c}(1+s^2)}{(1 - e^{-kc}e^{\pm 2c})^2}e^{-kc}}.\end{aligned}$$

Suppose $kc > 3$. We saw in the proof of proposition 3.4.2 that $s \sim \frac{c}{2}$ as $c \to 0^+$. Thus there exists two global constants A_0, B_0 such that

$$\sinh\left(\frac{1}{2}\operatorname{dist}(z_1, z_2)\right) = \frac{e^{kc/2}}{c} \times \exp\left[\pm\left(A_0 c + B_0 e^{-kc}\right)\right].$$

Since $c < 1$ and $e^x > 4\sinh^2\frac{x}{2}$ $(x > 0)$, $e^{\operatorname{dist}(z_1,z_2)} > C_0 e^{kc}$, for $C_0 := 4e^{-2(A_0+B_0)}$. Substituting $x = \operatorname{dist}(z_1, z_2)$ in $x \equiv 2\ln\sinh\frac{x}{2} + \ln 4 - 2\ln(1 - e^{-x})$ thus gives,

$$\begin{aligned}\operatorname{dist}(z_1, z_2) &= kc + 2\ln\frac{2}{c} \pm \left(A_0 c + B_0 e^{-kc}\right) - 2\ln(1 - e^{-\operatorname{dist}(z_1,z_2)})\\ &= kc + 2\ln\frac{2}{c} \pm \left(A_0 c + B_0 e^{-kc} - 2\ln(1 - C_0^{-1}e^{-kc})\right).\end{aligned}$$

We see that if kc is sufficiently large, and c sufficiently small, then $\operatorname{dist}(z_1, z_2)$ is arbitrarily close to to $kc + 2\ln\frac{2}{c}$.

Now suppose $kc \le 3$. Write (4.18) in the following form:

$$\sinh\left(\frac{1}{2}\operatorname{dist}(z_1, z_2)\right) = \frac{e^{\frac{m-n}{2}c}}{2s\sqrt{|xy|}}\sqrt{(e^{nc}y + e^{-mc}x)^2 + 4|xy|e^{(n-m)c}(1+s^2)}.$$

Since $kc \le 3$ and $c < 1$, the term in the square root is globally bounded away from zero and infinity, so $\sinh\frac{1}{2}\operatorname{dist}(z_1, z_2) \asymp \frac{1}{c}$. As before, this implies that $\operatorname{dist}(z_1, z_2) = kc + 2\ln\frac{2}{c} \pm \text{const}$, with a universal constant.

In summary, for all k and $0 < c < 1$, the time γ spends in the collar is

$$kc + 2\ln\frac{2}{c} \pm [\varepsilon^{(1)}_{col}(c) + \varepsilon^{(2)}_{col}(kc)] \qquad (4.19)$$

where $\varepsilon^{(i)}_{col} : [0, \infty) \to \mathbb{R}$ are bounded universal functions such that $\varepsilon^{(1)}_{col}(t) \xrightarrow[t\to 0^+]{} 0$ and $\varepsilon^{(2)}_{col}(t) \xrightarrow[t\to\infty]{} 0$. It is easy to arrange for $\varepsilon^{(1)}_{col}$ to be increasing, and for $\varepsilon^{(2)}_{col}$ to be decreasing.

To compare the sojourns of γ and $\widetilde{\gamma}$, we note that $|k[\widetilde{\gamma}] - k[\gamma]| = N$, so

$$[\text{length of } \widetilde{\gamma} \text{ in collar}] - [\text{length of } \gamma \text{ in collar}] = Nc \pm 2[\varepsilon^{(1)}_{col}(c) + \varepsilon^{(2)}_{col}(kc)]. \qquad (4.20)$$

Step 2. There is a global constant K such that distance between the entrance points of $\gamma, \widetilde{\gamma}$ to the collar of c is less than or equal to Kc, and the distance between the exit points of $\widetilde{\gamma}, \varphi^N_\beta[\gamma]$ from the collar is less than or equal to Kc.

Draw everything in the upper half plane as before. Since $\gamma, \widetilde{\gamma}$ intersect the positive y–axis, they are above the geodesic γ_1 connecting $x' := \gamma(-\infty)$ to 0, and to the right of the geodesic γ_2 connecting x' to ∞. Thus the entrance points of $\gamma, \widetilde{\gamma}$ to the collar are between the entrance points of γ_1 and γ_2. Call the first point z, and the second w.

It is clear that $w = x' - isx'$, and $z = \frac{x'}{1+s^2} - \frac{isx'}{1+s^2}$, so the hyperbolic length of the euclidean segment $[z, w]$ can be easily found to be asymptotic to $s \sim c/2$ as $c \to 0$, see (4.10). It follows that $\exists K$ such that for all $0 < c < 1$, this length is less than Kc. Since the entry points of $\gamma, \widetilde{\gamma}$ are inside this segment, the distance between them is at most Kc.

The statement on the exit points follows by symmetry.

Step 3. Calculation of $B(\gamma, \widetilde{\gamma})$ and proof of the proposition.

This is the same as in case (b), so we omit the details. □

4.5.2. *The support of m in case (c).* Let $\ell_{\min}(\varepsilon)$ be as in proposition 4.5.1. W.l.o.g. $\ell_{\min}(\varepsilon) < \varepsilon$.

Proposition 4.5.2. *In case (c), for every $\varepsilon > 0$ and almost every $e^{i\theta}$, $\{g^s\omega(e^{i\theta})\}_{s>0}$ crosses infinitely many collars of closed geodesics of length $0 < c < \ell_{\min}(\varepsilon)$, so that if k is the length of the collar cutting sequence, then $kc > K_{col}(\varepsilon)$.*

Proof. Let $\mathcal{C}$ denote the collection of collars of closed geodesics of length less than $\min\{\varepsilon_0, \ell_{\min}(\varepsilon), c_0^*, \frac{1}{100}\}$, where $c_0^*, \varepsilon_0, \ell_{\min}(\varepsilon)$ are given by propositions 3.4.2 and 4.5.1. In case (c), for a.e. $e^{i\theta}$, $\Gamma g^s\omega(e^{i\theta})$ crosses an infinite sequence of collars $C_j(e^{i\theta}) \in \mathcal{C}$. Let $k_j(e^{i\theta})$ denote the length of the cutting sequence of $\Gamma g^s\omega(e^{i\theta})$ in $C_j(e^{i\theta})$, and let $c_j(e^{i\theta})$ denote the length of the closed geodesic at the center of $C_j(e^{i\theta})$.

We prove the proposition by showing that the following set has measure zero for all $n_0 \in \mathbb{N}$ and $I \subset \mathbb{R}$ compact:

$$\Omega(n_0, I) := \{(e^{i\theta}, s) \in \partial\mathbb{D} \times I : \nexists j \geq 0 \text{ s.t. } k_j(e^{i\theta})c_j(e^{i\theta}) > n_0\}.$$

Lift $C_j(e^{i\theta}) \subset \Gamma\backslash\mathbb{D}$ to the connected subsets $R_j(e^{i\theta}) \subset \mathbb{D}$ which $g^s\omega(e^{i\theta})$ crosses. The closed geodesic at the center of $C_j(e^{i\theta})$ lifts to a geodesic segment γ_p which crosses $R_j(e^{i\theta})$. There are two ways to orient γ_p. Choose the one that is counterclockwise, when viewed from the entrance boundary component to the collar. The endpoints of this segment differ by a hyperbolic element $\varphi^{R_j(e^{i\theta})} \in \Gamma$, and the translation length of this element is $c_j(e^{i\theta})$.

Define $\widehat{k}_j(e^{i\theta})$, $\widehat{\varepsilon}_j(e^{i\theta})$, and $\mathrm{sgn}_j(e^{i\theta})$ as in (4.15–4.17). Fix some N (to be determined later), and define for $j \in \mathbb{N}$

$$\begin{aligned} N_j(e^{i\theta}) &:= \lceil N/c_j(e^{i\theta})\rceil, \\ \varphi_{e^{i\theta}} &:= \begin{cases} (\varphi^{R_j(e^{i\theta})})^{N_j(e^{i\theta})} & \mathrm{sgn}_j(e^{i\theta}) \geq 0 \\ (\varphi^{R_j(e^{i\theta})})^{-N_j(e^{i\theta})} & \mathrm{sgn}_j(e^{i\theta}) < 0 \end{cases} \\ \kappa_j(e^{i\theta}, s) &:= \big(\varphi_{e^{i\theta}}(e^{i\theta}), s - \log|(\varphi_{e^{i\theta}})'(e^{i\theta})|\big). \end{aligned}$$

As before the idea is to choose N so that κ_j are injective (whence measure preserving), and such that $\{\kappa_j[\Omega(n_0, I)] : j \geq 1\}$ contains an infinite pairwise disjoint collection with pre-compact union. Once this is done, $m[\Omega(n_0, I)] = 0$ by the Radon property of m. The construction is essentially the same as in case (b), so we restrict ourselves to a brief sketch, pausing only at points where the details are different from what we did above.

From now on assume that $(e^{i\theta}, s) \in \Omega(n_0, I)$, and that $\{C_j(e^{i\theta})\}$ is infinite.

Step 1. Define $e^{i\widetilde{\theta}}$ by $(e^{i\widetilde{\theta}}, *) := \kappa_j(e^{i\theta}, *)$, and let $\widetilde{\gamma}^+ = \{g^s\omega(e^{i\widetilde{\theta}})\}_{s\geq 0}$. If j and N are large enough (how large depends on ε, but not on $e^{i\theta}$), then

(a) $C_j(e^{i\theta})$ is the first collar in $\mathcal{C}$ crossed by $\Gamma\widetilde{\gamma}^+$, where the length of the cutting sequence times the length of the closed geodesic in the collar is more than N;

(b) $\exists N(\varepsilon)$ independent of $e^{i\theta}$ s.t. $C_j(e^{i\theta}) = C_{j'}(e^{i\widetilde{\theta}})$ with $|j' - j| \leq N(\varepsilon)$.

Proof. Define γ to be the geodesic from $-e^{i\theta}$ to $e^{i\theta}$; $\widehat{\gamma}$ the geodesic from $-e^{i\theta}$ to $e^{i\widetilde{\theta}}$; and $\widetilde{\gamma}$ the geodesic from $-e^{i\widetilde{\theta}}$ to $e^{i\widetilde{\theta}}$.

$\Gamma\gamma$ crosses $C_j(e^{i\theta})$ by assumption. $\Gamma\widehat{\gamma}$ crosses $C_j(e^{i\theta})$, because if we lift $\Gamma\gamma$ and $\Gamma\widehat{\gamma}$ to the upper half plane in such a way that the closed geodesic at the center of $C_j(e^{i\theta})$ is mapped to the y–axis, and $\varphi^{R_j(e^{i\theta})}$ is conjugated to $z \mapsto e^{-c_j(e^{i\theta})}z$, then the lift of $\widehat{\gamma}$ satisfies $\widehat{\gamma}(-\infty) = \gamma(-\infty)$, and $\widehat{\gamma}(\infty) = e^{\pm N_j(e^{i\theta})c_j(e^{i\theta})}\gamma(\infty)$, and these numbers have opposite signs, because γ crosses the collar.

Let P and $\widehat{P}$ be the points where γ and $\widehat{\gamma}$ enter $C_j(e^{i\theta})$. Let Q and $\widehat{Q}$ be the points where $\varphi_{e^{i\theta}}[\gamma]$ and $\widehat{\gamma}$ leave the lift of the collar. One shows, as in step 2 in the proof of proposition 4.5.1, that $\text{dist}(P, \widehat{P})$ and $\text{dist}(Q, \widehat{Q})$ are bounded by some constant C, which is independent of $e^{i\theta}$ and j.

Let $\widehat{\omega}$ denote the unit tangent vector to $\widehat{\gamma}$ at its point of intersection with $\text{Hor}_{e^{i\widetilde{\theta}}}(o)$. Repeating the argument of claim 4 in the proof of proposition 4.4.2, we find $j_0 = j_0(C), \delta = \delta(C), d^* = d^*(C)$ such that if $j \geq j_0$, then

$$\text{dist}(g^s\widehat{\omega}, g^s\omega(e^{i\widetilde{\theta}})) \leq \frac{1}{10}e^{-s}\min\{\varepsilon_{sh}^{col}, \eta(\ell_{\min}(\varepsilon))\} \text{ for all } s > 0 \quad (4.21)$$

$$\text{dist}(g^s\omega(e^{i\theta}), g^s\omega(e^{i\widetilde{\theta}})) \leq \delta(C) \text{ for all } 0 < s < d(o, P) - d^*(C). \quad (4.22)$$

The first inequality can be used to show that $\Gamma\widetilde{\gamma}^+$ crosses $C_j(e^{i\theta})$, and that its cutting sequence there has length $\widetilde{k} \geq N_j(e^{i\theta}) - 10$ (proposition 3.4.2), whence $\widetilde{k}c_j(e^{i\theta}) > N - 1$. The second inequality and the assumption that $\exists s$ s.t. $(e^{i\theta}, s) \in \Omega(n_0, I)$ can be used as in claim 4 of proposition 4.4.2 to show that there exists some global constant $N_0(\delta(C))$ such that $\Gamma\widehat{\gamma}^+$ does not cross a collar $C \in \mathcal{C}$ where its cutting sequence times the length of the closed geodesic in C is more than $K_{col}(\varepsilon) + n_0 + N_0(\delta(C))$, before crossing $C_j(e^{i\theta})$.

Part (a) follows, with any choice of N so that $N > K_{col}(\varepsilon) + n_0 + N_0(\delta(C)) + 1$. Part (b) is proved as in claim 5 in the proof of proposition 4.4.2.

Step 2. κ_j are injective, whence measure preserving for all j large enough.

Proof. The previous step shows that $C_j(e^{i\theta})$ can be read from $e^{i\widetilde{\theta}}$, and this is enough information to invert κ_j.

Step 3. There exists j_0 such that $\bigcup_{j>j_0} \kappa_j[\Omega(n_0, I)]$ is precompact.

Proof. Write $(e^{i\widetilde{\theta}}, *) := \kappa_j(e^{i\theta}, *)$, and recall that $d(o, P)$ is the time it takes $\Gamma g^s\omega(e^{i\theta})$ to reach $C_j(e^{i\theta})$. Since it crosses $j - 1$ collars in $\mathcal{C}$ before doing so, $d(o, P) \geq (j-1)\eta(\ell_{\min}(\varepsilon))$. If $j > j_0$, then (4.22) says that $d(g^s\omega(e^{i\theta}), g^s\omega(e^{i\widetilde{\theta}})) \leq \delta(C)$ for $0 < s < (j_0 - 1)\eta(\ell_{\min}(\varepsilon)) - d^*(C)$. Choosing $j_0 = j_0(C)$ sufficiently large (how large only depends on C), we see that this forces $|e^{i\theta} - e^{i\widetilde{\theta}}| < 1$, whence

$$\begin{aligned} |R(e^{i\theta}, e^{i\widetilde{\theta}})| &< |B(\Gamma\gamma, \Gamma\widehat{\gamma})| + 4 && \text{(proposition 4.1.1)} \\ &< N_j(e^{i\theta})c_j(e^{i\theta}) + K_0 + 4 && \text{(proposition 4.5.1)} \\ &< k_j(e^{i\theta})c_j(e^{i\theta}) + N + K_0 + 5, \end{aligned}$$

which is globally bounded since $k_j(e^{i\theta})c_j(e^{i\theta}) < n_0$ by the assumption on $e^{i\theta}$.

We obtained a uniform upper bound on $|B(\Gamma\gamma, \Gamma\widetilde{\gamma})|$, whence on $|R(e^{i\theta}, e^{i\widetilde{\theta}})|$. This shows the pre-compactness of $\bigcup_{j>j_0} \kappa_j[\Omega(n_0, I)]$. □

4.5.3. *Proof of the holonomy lemma in case (c).* Fix $\varepsilon < 1$ and c_0 as in the statement of the holonomy lemma. Let $\ell_{\min}(\varepsilon), K_{col}(\varepsilon)$ be as in propositions 4.5.1 and 4.5.2. W.l.o.g. $\ell_{\min}(\varepsilon) < \varepsilon$.

Fix N_0 and j_0 (to be determined later), and recall the definition of $\mathcal{C}$, $C_j(e^{i\theta})$, $c_j(e^{i\theta}), k_j(e^{i\theta}), \varphi^{R_j(e^{i\theta})}$ from the previous section. Let $j = j(e^{i\theta}, j_0, K_0)$ be the first $j > j_0$ s.t. $k_j(e^{i\theta})c_j(e^{i\theta}) > K_{col}(\varepsilon)$, and define

$$\begin{aligned} N_j(e^{i\theta}) &:= \lfloor (N_0 + \tfrac{1}{2})c_0/c_j(e^{i\theta}) \rfloor, \\ \varphi_{e^{i\theta}} &:= \begin{cases} (\varphi^{R_j(e^{i\theta})})^{N_j(e^{i\theta})} & \mathrm{sgn}_j(e^{i\theta}) \geq 0 \\ (\varphi^{R_j(e^{i\theta})})^{-N_j(e^{i\theta})} & \mathrm{sgn}_j(e^{i\theta}) < 0 \end{cases}, \\ \kappa(e^{i\theta}) &:= \varphi_{e^{i\theta}}(e^{i\theta}). \end{aligned}$$

One shows, as in the previous section, that if j_0 and N_0 are large enough, then κ is one-to-one, and $|\kappa(e^{i\theta}) - e^{i\theta}| < \varepsilon$, and

$$\begin{aligned} R(e^{i\theta}, \kappa(e^{i\theta})) &= B(\Gamma\gamma, \Gamma\widehat{\gamma}) \pm 4\varepsilon^2 && \text{(proposition 4.1.1)} \\ &= N_j(e^{i\theta})c_j(e^{i\theta}) \pm (4\varepsilon^2 + \varepsilon) && \text{(proposition 4.5.1)} \\ &= (N_0 + \tfrac{1}{2})c_0 \pm (4\varepsilon^2 + \varepsilon + c_j(e^{i\theta})) = (N_0 + \tfrac{1}{2})c_0 \pm (4\varepsilon^2 + 2\varepsilon), \end{aligned}$$

because $c_j(e^{i\theta}) < \ell_{\min}(\varepsilon) < \varepsilon$. Making ε smaller, if needed, it is no problem to ensure that $4\varepsilon^2 + \varepsilon + \ell_{\min}(\varepsilon) < \frac{1}{10}c_0$, whence $|R(e^{i\theta}, \kappa(e^{i\theta})) - (N_0 + \frac{1}{2})c_0| < \frac{1}{10}c_0$ for all N_0 sufficiently large. The holonomy lemma (both parts) easily follows. □

4.6. **Proof of the holonomy lemma in case (d).** There is a direct proof along the lines of the argument we used in case (a). Here is a shorter proof.

We show that case (d) M must have finite area. This means that Γ is a lattice. The holonomy lemma for lattices follows from lemma 1 in [LS]. (Of course, in this case we have no need of the holonomy lemma, as the classification of horocycle invariant Radon measures is known thanks to work of Dani and Smillie [DS].)

In case (d), there exists a finite connected union of pairs of pants S such that for almost every $e^{i\theta}$, $\{\Gamma g^s \omega(e^{i\theta})\}_{s>0}$ eventually enters S and stays there. If S is a complete hyperbolic surface, then $S = M$, which means that M has finite area. Suppose by way of contradiction that S is not complete.

This means that one of the pants pieces which makes S has a boundary component which borders a pants component Y_0 outside S. Let M_0 denote the complete hyperbolic surface obtained from $S \cup Y_0$ by extending it to funnels across each of its boundary components. The result is a complete hyperbolic surface which is isometric to $\Gamma_0 \setminus \mathbb{D}$, where Γ_0 is geometrically finite. The group Γ_0 can be taken to be a subgroup of Γ.[13]

In case (d), almost every $e^{i\theta}$ has the property that $\{\Gamma g^s \omega(e^{i\theta})\}_{s>0}$ eventually enters $S \subset S \cup Y_0$ and stays there. This means that almost every $e^{i\theta}$ belongs to

$$\bigcup_{g \in \Gamma} g[\Lambda(\Gamma_0)], \text{ where } \Lambda(\Gamma_0) := \text{limit set of } \Gamma_0.$$

It follows that $m[\Lambda(\Gamma_0) \times \mathbb{R}] \neq 0$.

Since $\Lambda(\Gamma_0)$ is Γ_0–invariant, $m_0 := m|_{\Lambda(\Gamma_0)\times\mathbb{R}}$ is an invariant Radon measure for the Radon Nikodym action of Γ_0. In particular, it corresponds to an invariant Radon measure of the horocycle flow on the geometrically finite hyperbolic surface M_0 (although it need not be ergodic).

The *ergodic* invariant Radon measures of the horocycle flow on M_0 are known [Ro], and fall into three classes:

(a) Ergodic components carried by single horocycles made of unit tangent vectors which escape to a funnel under the action of the geodesic flow,
(b) Ergodic components carried by single horocycles made of unit tangent vectors which tend to a cusp under the action of the geodesic flow,
(c) One g^s–quasi-invariant ergodic component, which enjoys the property that almost every unit tangent vector determines a geodesic which is forward dense in the forward non-wandering set Ω_0 of the geodesic flow on $T^1(M_0)$.

Our measure cannot have ergodic components of type (a) because, almost surely, $\Gamma g^s \omega(e^{i\theta})$ does not leave S. It cannot have ergodic components of type (b), because these ergodic components give $\mathrm{Par}(\Gamma_0) \times \mathbb{R} \subset \mathrm{Par}(\Gamma) \times \mathbb{R}$ positive measure, whereas we are assuming that $m[\mathrm{Par}(\Gamma) \times \mathbb{R}] = 0$. Thus our measure is equal to the ergodic component of type (c). But this ergodic component gives full measure to unit tangent vectors whose geodesics are dense in $\Omega_0 \cap \mathrm{int}[T^1(Y_0)]$, whereas our measure forbids the ω–limit set to intersect $\mathrm{int}[T^1(Y_0)]$. We get a contradiction. □

Appendix: Cutting sequences of geodesics crossing a pair of pants

The purpose of this appendix is to prove proposition 3.3.1. The proof is standard, but we decided to include it, because there are subtleties which do not appear in the classical case of non-wandering geodesics (as treated for example in [Se1]).

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[13] Proof: Write $M = \Gamma \setminus \mathbb{D}$, $M_0 = \Gamma_0 \setminus \mathbb{D}$, and let $p : \mathbb{D} \to M$, $p(z) = \Gamma z$; $p_0 : \mathbb{D} \to M_0$, $p_0(z) = \Gamma_0 z$. Fix two unit tangent vectors $\widetilde{\omega} \in T^1_o(\mathbb{D})$, $\omega \in T^1(S)$. Abusing notation we think of $T^1(S)$ as a subset of $T^1(M)$ and $T^1(M_0)$. Modify Γ, Γ_0 by a conjugacy to guarantee that p, p_0 both map $\widetilde{\omega}$ to ω. We show that $\Gamma_0 \subseteq \Gamma$.

Suppose $g_0 \in \Gamma_0$ is hyperbolic with axis $\widetilde{\gamma}_p$, then $\gamma_p := p_0[\widetilde{\gamma}_p]$ is a closed geodesic on M_0, and w.l.o.g. $\ell(\gamma_p)$ =translation length of g_0. Let $\widetilde{\gamma}_1$ be the shortest geodesic from o to $\widetilde{\gamma}_p$. It meets $\widetilde{\gamma}_p$ perpendicularly at some point $\widetilde{z}$. Closed geodesics do not enter funnels, so $\gamma_p \subset \overline{S \cup Y_0}$. Geodesics which enter a funnel never leave the funnel, so $\gamma_1 := p_0[\widetilde{\gamma}_1] \subset \overline{S \cup Y_0}$. Since $\gamma_1 \subset \overline{S \cup Y_0}$, we can lift it to $\mathbb{D}$ at o using p. The lift is completely determined by the length of γ_1, and the angle between $\widetilde{\omega}$ and the direction of γ_1 at its beginning point, therefore the p–lift is also equal to $\widetilde{\gamma}_1$. This means that $p(\widetilde{z}) = p_0(\widetilde{z}) \in \gamma_p$, and that p–lift of γ_p at $\widetilde{z}$ is equal to $\widetilde{\gamma}_p$ (the unique geodesic perpendicular to $\widetilde{\gamma}_1$ at $\widetilde{z}$). We conclude that $p[\widetilde{\gamma}_p] = \gamma_p$ is a closed geodesic. Thus $\widetilde{\gamma}_p$ is the axis of some hyperbolic element $h \in \Gamma$ with translation length $\ell(\gamma_p)$. But g_0 has these properties, so $g_0 = h^{\pm 1} \in \Gamma$. We just proved that all the hyperbolic elements of Γ_0 belong to Γ. The case of parabolic elements is handled in a similar way, using closed horocycles instead of closed geodesics.

Throughout this section O_Y is a fixed right–angled octagon, representing a pair of pants Y. The sides of O_Y are labeled as in figure 1, $H(x)$ denotes the half space under side x, and $Q(x) :=$ relative closure in $H(x)$ of $H(x) \setminus \bigcup_{x \neq z \in \mathcal{S} \cup \{a, b_1, b_2, c\}} H(z)$, where $\mathcal{S} := \{\alpha, \beta, \overline{\alpha}, \overline{\beta}\}$.

Lemma 4.6.1. *If x, y are different nonadjacent sides of O_Y, then $H(x) \cap H(y) = \varnothing$.*

Proof. Suppose $x = \alpha$. In hyperbolic geometry the angles of every triangle add up to less than $180°$, so two geodesics with a common perpendicular do not intersect. Thus α does not intersect the common perpendiculars (seams) connecting a to c and b_1 to b_2. Considering the connected components of O_Y minus each of these seams, we see that α cannot meet the sides of O_Y which are not adjacent to it. □

Lemma 4.6.2. *$\varphi_x[Q(y)] \subset Q(\overline{x})$ whenever $x, y \in \mathcal{S}$ are different, and*

$$\begin{array}{llll}
\varphi_\alpha[H(a)] = H(\alpha) & \varphi_\beta[H(a)] \subset Q(\overline{\beta}) & \varphi_{\overline{\alpha}}[H(a)] = H(a) & \varphi_{\overline{\beta}}[H(a)] \subset Q(\beta) \\
\varphi_\alpha[H(b_1)] = H(b_2) & \varphi_\beta[H(b_1)] = H(b_2) & \varphi_{\overline{\alpha}}[H(b_1)] \subset Q(\alpha) & \varphi_{\overline{\beta}}[H(b_1)] \subset Q(\beta) \\
\varphi_\alpha[H(b_2)] \subset Q(\overline{\alpha}) & \varphi_\beta[H(b_2)] \subset Q(\overline{\beta}) & \varphi_{\overline{\alpha}}[H(b_2)] = H(b_1) & \varphi_{\overline{\beta}}[H(b_2)] = H(b_1) \\
\varphi_\alpha[H(c)] \subset Q(\overline{\alpha}) & \varphi_\beta[H(c)] = H(c) & \varphi_{\overline{\alpha}}[H(c)] \subset Q(\alpha) & \varphi_{\overline{\beta}}[H(c)] = H(c).
\end{array}$$

Proof. The key to the proof are the following observations

- Lemma 4.6.1;
- Möbius transformations map geodesics to geodesics;
- Möbius transformations preserve angles and orientation.

Using these observations it is easy to see that the combinatorics of the action of $\{\varphi_x : x \in \mathcal{S}\}$ on the sides of O_Y is as depicted in figure 1, in the sense that the ordering of the endpoints of the images of the geodesic extensions of the sides of O_Y is as depicted there. The lemma follows by inspection. □

Lemma 4.6.3. *$Q(x) \cup O_Y \cup Q(y)$ is geodesically convex whenever $x, y \in \mathcal{S}$.*

Proof. $Q(x) \cup O_Y \cup Q(y) = \mathbb{D} \setminus \bigcup_{z \in (\mathcal{S} \cup \{a, b_1, b_2, c\}) \setminus \{x, y\}} H(z)$, an intersection of hyperbolic half planes. □

Lemma 4.6.4. *$\bigcup_{g \in \Gamma(O_Y)} g(O_Y)$ is geodesically convex.*

Proof. (Compare with [Se1]). It is routine to check that

$$H(a) = \bigcup_{n \in \mathbb{Z}} \varphi_\alpha^n[Q(a)], \quad H(b_1) = \bigcup_{n \in \mathbb{Z}} (\varphi_\beta^{-1} \circ \varphi_\alpha)^n [Q(b_1) \cup \varphi_\beta^{-1} Q(b_2)],$$
$$H(c) = \bigcup_{n \in \mathbb{Z}} \varphi_\beta^n[Q(c)], \quad H(b_2) = \bigcup_{n \in \mathbb{Z}} (\varphi_\beta \circ \varphi_\alpha^{-1})^n [Q(b_2) \cup \varphi_\beta Q(b_1)].$$

Recall that $\Gamma(O_Y) = \langle \varphi_x : x \in \mathcal{S} \rangle$ has fundamental domain $\widetilde{O}_Y := O_Y \cup Q(a) \cup Q(b_1) \cup Q(b_2) \cup Q(c)$ (section 3.1).

Suppose $g \in \Gamma(O_Y)$, then we claim that $g(O_Y) \cap H(a) = \varnothing$: Otherwise, $g(\widetilde{O}_Y) \cap \bigcup_{n \in \mathbb{Z}} \varphi_\alpha^n(\widetilde{O}_Y) \supset g(O_Y) \cap H(a) \neq \varnothing$, so $g = \varphi_\alpha^n$ for some n ($\because \widetilde{O}_Y$ is a fundamental domain). But if $g = \varphi_\alpha^n$, then $g[O_Y] \cap H(a) = \varnothing$, because $\varphi_\alpha[H(a)] = H(a)$, and $O_Y \cap H(a) = \varnothing$.

In the same way, one shows that there is no $g \in \Gamma(O_Y)$ such that $g(O_Y)$ intersects $H(b_1)$, $H(b_2)$, or $H(c)$. We conclude that

$$\bigcup_{g \in \Gamma(O_Y)} g(O_Y) \cap \bigcup_{g \in \Gamma(O_Y)} g[H(a) \cup H(b_1) \cup H(b_2) \cup H(c)] = \varnothing.$$

This means that

$$\begin{aligned}\bigcup_{g\in\Gamma(O_Y)} g(O_Y) &= \bigcup_{g\in\Gamma(O_Y)} g(O_Y)\setminus \bigcup_{g\in\Gamma(O_Y)} g[H(a)\cup H(b_1)\cup H(b_2)\cup H(c)]\\ &= \bigcup_{g\in\Gamma(O_Y)} g(\widetilde{O}_Y)\setminus \bigcup_{g\in\Gamma(O_Y)} g[H(a)\cup H(b_1)\cup H(b_2)\cup H(c)]\\ &= \mathbb{D}\setminus \bigcup_{g\in\Gamma(O_Y)} g[H(a)\cup H(b_1)\cup H(b_2)\cup H(c)],\end{aligned}$$

an intersection of hyperbolic half planes. □

Proof of Proposition 3.3.1, part (1). Let $\underline{w}=(w_1,\ldots,w_k)$ be a reduced word on the alphabet $\mathcal{S}$, and let γ be a geodesic in $\mathbb{D}$ such that $\gamma(-\infty)\in\varphi_\alpha^k[Q(a)]$, $\gamma(\infty)\in\varphi_\alpha^\ell\varphi_{w_1}\cdots\varphi_{w_k}\varphi_\beta^m[Q(c)]$, $w_1\notin\{\alpha,\overline{\alpha}\}$, $w_k\notin\{\beta,\overline{\beta}\}$, and $|\ell|,|m|$ are maximal. We are asked to prove that:

1. $\gamma^*:=\varphi_\alpha^{-\ell}[\gamma]$ enters O_Y through one of the sides $\alpha,a,\overline{\alpha}$;
2. $\gamma^\#:=\varphi_{\overline{w}_k}\cdots\varphi_{\overline{w}_1}\varphi_\alpha^{-\ell}[\gamma]$ leaves O_Y through one of the sides $\beta,c,\overline{\beta}$;
3. the cutting sequence of γ in Y contains the word $\underline{w}$.

Step 1. For every $i=1,\ldots,k$, $\varphi_{w_i}\cdots\varphi_{w_k}\varphi_\beta^{\pm m}[Q(c)]\subseteq Q(\overline{w_i})$, and for every j, $\varphi_\alpha^j\varphi_{w_1}\cdots\varphi_{w_k}\varphi_\beta^{\pm m}[Q(c)]\subseteq Q(\overline{\alpha})$.

Proof. By lemma 4.6.2, and since $\underline{w}$ is reduced,

$$\begin{aligned}\varphi_\beta^{\pm m}[Q(c)]&\subset\varphi_\beta^{\pm m}[H(c)]=H(c)\\ \varphi_{w_k}\varphi_\beta^{\pm m}[Q(c)]&\subseteq\varphi_{w_k}[H(c)]\subseteq Q(\overline{w}_k) && (\because w_k\neq\beta,\overline{\beta})\\ \varphi_{w_{k-1}}\varphi_{w_k}\varphi_\beta^{\pm m}[Q(c)]&\subset\varphi_{w_{k-1}}[Q(\overline{w}_k)]\subseteq Q(\overline{w}_{k-1}) && (\because w_{k-1}\neq\overline{w}_k)\end{aligned}$$

The step follows by induction.

Step 2. $\gamma^*:=\varphi_\alpha^{-\ell}[\gamma]$ enters O_Y through the union of its $\alpha,a,\overline{\alpha}$–sides, and leaves it entering $\varphi_{w_1}[O_Y]$ through side w_1 of that octagon.

Proof. Let $A:=H(\alpha)\cup H(a)\cup H(\overline{\alpha})$. The endpoints of γ^* are

$$\begin{aligned}\gamma^*(-\infty)=&\varphi_\alpha^{-\ell}[\gamma(-\infty)]\in\varphi_\alpha^{-\ell}[\varphi_\alpha^kQ(a)]\subset H(a)\subset A && (\because\ \varphi_\alpha[H(a)]=H(a)),\\ \gamma^*(\infty)\in&\varphi_{w_1}\cdots\varphi_{w_k}\varphi_\beta^{\pm m}[Q(c)]\subseteq Q(\overline{w}_1)\subset\mathbb{D}\setminus A && (\because\text{step 1}),\end{aligned}$$

so γ^* starts in A and ends outside A. It leaves A through sides α,a, or $\overline{\alpha}$ of O_Y, because the only way to leave A without crossing these sides is to enter $H(b_1)\cup H(b_2)$, which cannot happen since the endpoints of γ^*, whence γ^* itself, are in $\mathbb{D}\setminus[H(b_1)\cup H(b_2)]$.

Since γ^* is not equal to the geodesic extensions of α,a, or $\overline{\alpha}$ it must enter O_Y when it crosses α,a, or $\overline{\alpha}$.

Let P be the entry point of γ^* to O_Y. Then $P\in\overline{Q(\eta)}$ where $\eta=\alpha,a$, or $\overline{\alpha}$. Since $\gamma^*(\infty)\in Q(\overline{w}_1)$, the points P and $\gamma^*(\infty)$ belong to the euclidean closure of $Q(\eta)\cup O_Y\cup Q(\overline{w}_1)$. This union is geodesically convex (lemma 4.6.3), so the geodesic ray from P to $\gamma^*(\infty)$ is in $\overline{Q(\eta)\cup O_Y\cup Q(\overline{w}_1)}$. It follows that γ^* leaves O_Y through side $\overline{w}_1$. Since side $\overline{w}_1$ of O_Y is side w_1 of $\varphi_{w_1}[O_Y]$, step 2 is proved.

Step 3. The cutting sequence of γ in Y includes the word $\underline{w}$.

Proof. We study the behavior of γ^* in $\varphi_{w_1}[O_Y]$ by studying the behavior of $\gamma_1^* := \varphi_{w_1}^{-1}[\gamma^*]$ in O_Y. The geodesic γ_1^* connects the points

$$P_1 := \varphi_{w_1}^{-1}(P) \in \varphi_{\overline{w}_1}[\overline{Q(\eta)}] \subseteq \overline{Q(w_1)}, \qquad \text{(Lemma 4.6.2)}$$
$$\gamma_1^*(\infty) \in \varphi_{w_2} \cdots \varphi_{w_k} \varphi_\beta^{\pm m}[Q(c)] \subseteq Q(\overline{w}_2). \qquad \text{(step 1)}$$

Therefore it is contained in the geodesically convex set $Q(w_1) \cup O_Y \cup Q(\overline{w}_2)$. As before, this means that γ_1^* leaves O_Y by entering $\varphi_{w_2}[O_Y]$ through side w_2 of that octagon. Returning to γ^* we see that γ^* enters O_Y, then enters to $\varphi_{w_1}[O_Y]$ through side w_1, then enters $\varphi_{w_1}\varphi_{w_2}[O_Y]$ through side w_2.

Continuing in the manner we see that γ^* moves from O_Y to $\varphi_{w_1}[O_Y]$ cutting side w_1, then moves to $\varphi_{w_1}\varphi_{w_2}[O_Y]$ cutting side w_2, then moves to $\varphi_{w_1}\varphi_{w_2}\varphi_{w_3}[O_Y]$ cutting side w_3 and so on until it enters $\varphi_{w_1} \cdots \varphi_{w_k}[O_Y]$ through side w_k of that octagon. This means that the cutting sequence of γ in Y contains the word $\underline{w}$.

We have proved (1) and (3). Part (2) follows by time reversal. □

Proof of proposition 3.3.1, part (2) Let $\underline{w} = (w_1, \ldots, w_k)$ be a reduced word of length $k \geq 2$ on the alphabet $\mathcal{S}$, and let γ be a geodesic in $\mathbb{D}$ s.t. $\gamma(-\infty) \in \varphi_\alpha^k[Q(a)]$, $\gamma(\infty) \in \varphi_\alpha^\ell \varphi_{w_1} \cdots \varphi_{w_k}(\varphi_\beta\varphi_{\overline{\alpha}})^m[Q(b_2) \cup \varphi_\beta Q(b_1)]$,where $|\ell|, |m|$ are maximal. We are asked to prove that

(1) $\gamma^* := \varphi_\alpha^{-\ell}[\gamma]$ enters O_Y through one of the sides $\alpha, a, \overline{\alpha}$;
(2) $\gamma^\# := (\varphi_\alpha^\ell \varphi_{w_1} \cdots \varphi_{w_k})^{-1}[\gamma]$ leaves O_Y through one of the sides $\overline{\alpha}, b_2, \overline{\beta}$;
(3) the cutting sequence of γ in Y contains the word $\underline{w}$.

Step 1. For every $i = 1, \ldots, k-1$, $\varphi_{w_i} \cdots \varphi_{w_k}(\varphi_\beta\varphi_{\overline{\alpha}})^m[Q(b_2) \cup \varphi_\beta Q(b_1)] \subseteq Q(\overline{w_i})$, and for every j, $\varphi_\alpha^j \varphi_{w_1} \cdots \varphi_{w_k}\varphi_\beta^{\pm m}[Q(b_2) \cup \varphi_\beta Q(b_1)] \subseteq Q(\overline{\alpha})$.

Proof. By lemma 4.6.2, $Q(b_2) \cup \varphi_\beta Q(b_1) \subset H(b_2)$, and $\varphi_\beta\varphi_{\overline{\alpha}}[H(b_2)] = H(b_2)$, therefore $(\varphi_\beta\varphi_{\overline{\alpha}})^m[Q(b_2) \cup \varphi_\beta Q(b_1)] \subset H(b_2)$. The same lemma shows that

(a) If $w_k = \overline{\alpha}$, then $\varphi_{w_k}(\varphi_\beta\varphi_{\overline{\alpha}})^m[Q(b_2) \cup \varphi_\beta Q(b_1)] \subset \varphi_{\overline{\alpha}}[H(b_2)] = H(b_1)$. Since $w_k = \overline{\alpha}$ and $|m|$ is maximal, $w_{k-1} \neq \beta$. Since (w_{k-1}, w_k) is reduced, $w_{k-1} \neq \alpha$. Thus $w_{k-1} \in \{\overline{\alpha}, \overline{\beta}\}$. In both cases, lemma 4.6.2 says that
$$\varphi_{w_{k-1}}\varphi_{w_k}(\varphi_\beta\varphi_{\overline{\alpha}})^m[Q(b_2) \cup \varphi_\beta Q(b_1)] \subset \varphi_{w_{k-1}}[H(b_1)] \subset Q(\overline{w}_{k-1}).$$

(b) If $w_k = \overline{\beta}$, then $\varphi_{w_k}(\varphi_\beta\varphi_{\overline{\alpha}})^m[Q(b_2) \cup \varphi_\beta Q(b_1)] \subset \varphi_{\overline{\beta}}[H(b_2)] = H(b_1)$. Note that m must be non-positive (otherwise $\overline{\beta}(\beta\overline{\alpha})^m$ is not reduced). Since $|m|$ is maximal, $w_{k-1} \neq \alpha$. Since (w_{k-1}, w_k) is reduced, $w_{k-1} \neq \beta$. Thus $w_{k-1} \in \{\overline{\alpha}, \overline{\beta}\}$. In both cases, lemma 4.6.2 says that
$$\varphi_{w_{k-1}}\varphi_{w_k}(\varphi_\beta\varphi_{\overline{\alpha}})^m[Q(b_2) \cup \varphi_\beta Q(b_1)] \subset \varphi_{w_{k-1}}[H(b_1)] \subset Q(\overline{w}_{k-1}).$$

(c) If $w_k \neq \overline{\alpha}, \overline{\beta}$, then $\varphi_{w_k}(\varphi_\beta\varphi_{\overline{\alpha}})^m[Q(b_2) \cup \varphi_\beta Q(b_1)] \subset \varphi_{w_k}[H(b_2)] \subset Q(\overline{w}_k)$. Since $w_{k-1} \neq \overline{w}_k$, $\varphi_{w_{k-1}}\varphi_{w_k}(\varphi_\beta\varphi_{\overline{\alpha}})^m[Q(b_2) \cup \varphi_\beta Q(b_1)] \subset \varphi_{w_{k-1}}[Q(\overline{w}_k)] \subset Q(\overline{w}_{k-1})$.

In all cases, step 1 holds for $i = k-1$. The reminder of the step follows as before.

Step 2. $\gamma^* := \varphi_\alpha^{-\ell}[\gamma]$ enters O_Y through one of the sides $\alpha, a, \overline{\alpha}$, and leaves it entering $\varphi_{w_1}[O_Y]$ through side w_1 of that octagon.

Proof. $\gamma^*(-\infty) = \varphi_\alpha^{-\ell}[\gamma(-\infty)] \in \varphi_\alpha^{k-\ell}[Q(\alpha)] \subset H(\alpha)$, and (step 1)
$$\gamma^*(\infty) \in \varphi_{w_1} \cdots \varphi_{w_k}(\varphi_{\overline{\beta}}\varphi_\alpha)^m[Q(b_2) \cup \varphi_\beta Q(b_1)] \subseteq Q(\overline{w}_1) \subset Q(\beta) \cup Q(\overline{\beta}).$$

The geodesic γ^* thus starts at $H(\alpha)$, ends at $Q(\overline{w}_1)$ where $w_1 \in \{\beta, \overline{\beta}\}$, and stays outside $H(b_1) \cup H(c) \cup H(b_2)$ (an easy convexity argument).

Such a geodesic must cross O_Y, entering through one of the sides α, a or $\overline{\alpha}$, and leaving through side $\overline{w}_1$ on its way to $\varphi_{w_1}[O_Y]$. The side of $\varphi_{w_1}[O_Y]$ through which γ^* enters is labeled by the inverse of the label of the side of O_Y through which γ^* exits. This label is $\overline{(\overline{w}_1)} = w_1$.

Step 3. The cutting sequence of the projection of γ inside Y contains the word $\underline{w}$.

Proof. As the proof of part (1) of the proposition, one shows that after γ^* enters $\varphi_{w_1}[O_Y]$ cutting side w_1, it moves to $\varphi_{w_1}\varphi_{w_2}[O_Y]$ cutting side w_2, and then to $\varphi_{w_1}\varphi_{w_2}\varphi_{w_3}[O_Y]$ cutting side w_3 and so on until it enters $\varphi_{w_1} \cdots \varphi_{w_k}[O_Y]$, cutting side w_k. Thus the cutting sequence of the projection of γ contains the word $\underline{w}$.

Step 4. $\gamma^{\#} := (\varphi_\alpha^\ell \varphi_{w_1} \cdots \varphi_{w_k})^{-1}[\gamma]$ leaves O_Y through one of the sides $\overline{\alpha}, b_2, \overline{\beta}$.

Proof. The previous step implies that $\gamma^{\#}$ enters O_Y through side w_k. By construction $\gamma^{\#}(\infty) \in (\varphi_\beta \varphi_{\overline{\alpha}})^m [Q(b_2) \cup \varphi_\beta Q(b_1)] \subset H(b_2)$.

Thus $\gamma^{\#}$ crosses O_Y, starts outside $H(b_2)$, and ends in $H(b_2)$. Such geodesics must leave O_Y through one of the sides $\overline{\alpha}, b_2$, or $\overline{\beta}$. □

Proof of proposition 3.3.1, part (3). The same as the proof of part (2). □

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