Thermodynamic Formalism for Null Recurrent Potentials

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Abstract

We extend Ruelle's Perron-Frobenius theorem to the case of Hölder continuos functions on a topologically mixing topological Markov shift with a countable number of states. Let $P(\phi)$ denote the Gurevic pressure of ϕ and let L_ϕ be the corresponding Ruelle operator. We present a necessary and sufficient condition for the existence of a conservative measure ν and a continuous function h such that $L_\phi^* \nu = e^{P(\phi)}\nu$, $L_\phi h = e^{P(\phi)}h$ and characterize the case when $\int h d\nu < \infty$. In the case when $dm = hd\nu$ is infinite, we discuss the asymptotic behaviour of L_ϕ^k, and show how to interpret dm as an equilibrium measure. We show how the above properties reflect in the behaviour of a suitable dynmical zeta function. These results extend the results of [18] where the case $\int hd\nu < \infty$ was studied.

1 Introduction and statement of main results

Let S be a countable set of **states** and $\mathbf{A} = (t_{ij})_{S\times S}$ a matrix of zeroes and ones. We identify S with $\mathbf{N}$, and induce an order on S. Let $X = \left\{x \in S^{\mathbf{N}\cup\{0\}} \colon \forall i \;\; t_{x_i x_{i+1}} = 1\right\}$ and $T\colon X \to X$ be the left shift $(Tx)_i = x_{i+1}$. Fix $r \in (0,1)$ and set $t(x,y) = \inf\{i\colon x_i \neq y_i\}$. We endow X with the topology induced by the metric $d_r(x,y) = r^{t(x,y)}$. The **cylinder sets**

$$[\underline{a}] = [a_0, \ldots, a_{n-1}] = \{x \in X \colon \forall i \quad x_i = a_i\}$$

form a base for this topology and generate the corresponding Borel σ-algebra $\mathcal{B}$. Let α be the partition $\{[a] : a \in S\}$. The elements of α are called **partition sets,** and the members of α_0^{n-1} are called cylinders of **length** n. We denote the length of a cylinder $[\underline{a}]$ by $|\underline{a}|$.

X is called **topologically mixing** if (X,T) is topologically mixing. This means that $\forall a, b \in S\ \exists N_{ab}\ \forall n > N_{ab}\ \ [a] \cap T^{-n}[b] \neq \phi$. Throughout this paper, a function $\phi \colon X \to \mathbf{R}$ is called **locally Hölder continuous** (with parameter r), if it is uniformly Lipschitz continuous with respect to d_r on all cylinders of length 2. This is equivalent to the requirement that $\exists A > 0, r \in (0,1)$ such that $\forall n \geq 2\ \ V_n[\phi] < Ar^n$ where $V_n[\phi] = \sup\{|\phi(x) - \phi(y)| : x_0 = y_0, \ldots, x_{n-1} = y_{n-1}\}$. This notion of Hölder continuity extends the one considered in [18], where $V_n[\phi] < Ar^n$ was also assumed for $n = 1$. Indeed, every function of the form $\phi = \phi(x_0, x_1)$ is locally Hölder continuous, even when $V_1(\phi) = \infty$ (in which case it does not satisfy the condition used in [18]). A close reading of [18] shows that the seemingly greater generality does not affect the arguments in sections 1-4 there.

The **Ruelle Operator** [15] is given by $(L_\phi f)(x) = \sum_{Ty=x} e^{\phi(y)} f(y)$. If $\|L_\phi 1\|_\infty < \infty$ this is a bounded linear operator on the Banach space of bounded continuous functions on X. Note that for a countable Markov shift the sum which defines L_ϕ may be infinite, in which case ϕ must be unbounded in order for it to converge. This is not a problem since local Hölder continuity on a non compact space does not imply boundness.

In this paper the term 'measure' refers to any σ-finite Borel measure μ which is not trivial in the sense that there is some $A \in \mathcal{B}$ for which $\mu(A) > 0$. We use the notation $\mu(f)$ for the integral of the function f with respect to μ, when it exists. The measure $\mu \circ T$ is the measure given on cylinders by

$$(\mu \circ T)(A) = \sum_{a \in S} \mu(T(A \cap [a])) \tag{1}$$

Integrals with respect to $\mu \circ T$ are given by

$$\int f \, d\mu \circ T = \sum_{a \in S} \int_{T[a]} f(ax) \, d\mu(x)$$

If μ is non singular (i.e. $\mu \sim \mu \circ T^{-1}$) then $\mu \ll \mu \circ T$ and the function $g_\mu = d\mu / d\mu \circ T$ is well defined $\mu \circ T$ almost everywhere. It is characterized $mod \, \mu \circ T$ by the property that $L_{\log g_\mu}$ acts as the transfer operator of μ, i.e $\mu(\varphi_1 L_{\log g} \varphi_2) = \mu(\varphi_1 \circ T \cdot \varphi_2)$ for every $\varphi_1 \in L^\infty(\mu)$, $\varphi_2 \in L^1(\mu)$. We will also make use of the measures $\mu \circ T^n$ defined by induction by $\mu \circ T^n = (\mu \circ T^{n-1}) \circ T$.

For every $a \in S$, $n \in \mathbf{N}$ set $Z_n(\phi, a) = \sum_{T^n x = x} e^{\phi_n(x)} 1_{[a]}(x)$ where $\phi_n = \sum_{k=0}^{n-1} \phi \circ T^k$. It was shown in [18] that if X is topologically mixing and ϕ is locally Hölder continuous then the limit

$$P_G(\phi) = \lim_{n \to \infty} \frac{1}{n} \log Z_n(\phi, a)$$

exists, is independent of a and belongs to $(-\infty, \infty]$. If $\|L_\phi 1\|_\infty < \infty$, this limit is finite and satisfies

$$P_G(\phi) = \sup \left\{ h_\mu(T) + \int \phi \, d\mu \colon \mu \in \mathcal{P}_T(X) \; , \; \mu(-\phi) < \infty \right\} \quad (2)$$

where $\mathcal{P}_T(X)$ denotes the set of all invariant Borel probability measures. $P_G(\phi)$ is called the **Gurevic Pressure** of ϕ, and is a generalization of the Gurevic topological entropy (Gurevic [7]). (The above results were stated in [18] only for locally Hölder continuous functions for which $V_1(\phi) < \infty$ but are also true, with the same proofs, without that latter assumption. Indeed, the proofs only require that $\sum_{n \geq 2} V_n(\phi)$ be finite.)

In [18] a necessary and sufficient condition was given for Ruelle's Perron-Frobenius theorem to hold: there exist a positive number λ, a positive continuous function h and a $\sigma-$finite Borel measure ν such that $L_\phi h = \lambda h$, $L_\phi^* \nu = \lambda \nu$, $\int h \, d\nu = 1$ and such that for every cylinder $[\underline{a}]$, $\lambda^{-n} L_\phi^n 1_{[\underline{a}]} \underset{n \to \infty}{\longrightarrow} h\nu [\underline{a}]$ uniformly on compacts. If this is the case, $P_G(\phi) = \log \lambda$ and $dm = h \, d\nu$ is an invariant probability measure which can be interpreted as the 'equilibrium' measure of ϕ in a certain sense (see [18] for details).

In this paper we study the case when Ruelle's Perron-Frobenius theorem fails. The main theme of this work is that the phenomenology of this situation is analogous to that one encounters in the case of a null recurrent or a transient probabilistic Markov chain (see [6], [10], [20]). In this situation $\lambda^{-n} L_\phi^n 1_{[\underline{a}]} \underset{n\to\infty}{\longrightarrow} 0$, but there may exist constants $a_n \nearrow \infty$ for which for every cylinder $a_n^{-1} \sum_{k=1}^n \lambda^{-n} L_\phi^n 1_{[\underline{a}]} \underset{n\to\infty}{\longrightarrow} h\nu\,[\underline{a}]$ pointwise where $L_\phi h = \lambda h, L_\phi^* \nu = \lambda \nu$, $\int h\, d\nu = \infty$. In this case, the measure $dm = hd\nu$ is an infinite invariant measure which can be described as the appropriate 'equilibrium measure' of ϕ. Given ν, the construction of h is done using the techniques of [1] (see also [2], [13], [21], [22], [28], [29], [30], [31]). The main point of this paper is the construction of a conformal measure ν with respect to which these methods can be applied.

We proceed to make our statements more precise. Set

$$Z_n(\phi, a) = \sum_{\substack{T^n x = x \\ x_0 = a}} e^{\phi_n(x)} \quad ; \quad Z_n^*(\phi, a) = \sum_{\substack{T^n x = x \\ x_0 = a; x_1, \ldots, x_{n-1} \neq a}} e^{\phi_n(x)}$$

We introduce the following definition, in analogy with the theory of Markov chains:

Definition 1 *Let X be topologically mixing and ϕ be locally Hölder continuous with finite Gurevic pressure* $\log \lambda$. *ϕ is called:*

1. **recurrent** *if for some (hence all) $a \in S$* $\sum \lambda^{-n} Z_n(\phi, a) = \infty$, *and* **transient** *otherwise.*
2. **positive recurrent** *if it is recurrent and for some (hence all[1]) $a \in S$* $\sum n \lambda^{-n} Z_n^*(\phi, a) < \infty$
3. **null recurrent** *if it is recurrent and for some (hence all) $a \in S$* $\sum n \lambda^{-n} Z_n^*(\phi, a) = \infty$.

[1]The independence of positive recurrence and null recurrence from the choice of a follows from theorem 1 below

The notion of positive recurrence was given a different, though equivalent, definition in [18]. The equivalence follows from theorem 1 below. It can be easily verified that if $\phi = \phi(x_0, x_1)$ then recurrence, positive recurrence and null recurrence are equivalent to the matrix $\left(e^{\phi(i,j)}\right)_{S\times S}$ being R-recurrent, R-positive and R-null in the terminology of Vere-Jones [24], [25]. The main results of this paper are contained in the following theorem:

Theorem 1 *Let X be topologically mixing and ϕ locally Hölder continuous with finite Gurevic pressure. ϕ is recurrent iff there exist $\lambda > 0$, a conservative measure ν, finite and positive on cylinders, and a positive continuous function h such that $L_\phi^*\nu = \lambda\nu$ and $L_\phi h = \lambda h$. In this case $\lambda = \exp P_G(\phi)$ and $\exists a_n \nearrow \infty$ such that for every cylinder $[\underline{a}]$ and $x \in X$*

$$\frac{1}{a_n}\sum_{k=1}^{n}\lambda^{-k}\left(L_\phi^k 1_{[\underline{a}]}\right)(x) \underset{n\to\infty}{\longrightarrow} h(x)\,\nu[\underline{a}] \tag{3}$$

where $\{a_n\}_n$ satisfies $a_n \sim \left(\int_{[a]} h\,d\nu\right)^{-1}\sum_{k=1}^{n}\lambda^{-k}Z_k(\phi, a)$ for every $a \in S$. Furthermore,

1. *if ϕ is positive recurrent then $\nu(h) < \infty$, $a_n \sim n\cdot const$, and for every $[\underline{a}]$ $\lambda^{-n}L_\phi^n 1_{[\underline{a}]} \underset{n\to\infty}{\longrightarrow} h\nu[\underline{a}]/\nu(h)$ uniformly on compacts.*
2. *if ϕ is null recurrent then $\nu(h) = \infty$, $a_n = o(n)$, and for every $[\underline{a}]$ $\lambda^{-n}L_\phi^n 1_{[\underline{a}]} \underset{n\to\infty}{\longrightarrow} 0$ uniformly on cylinders.*

Remark 1. In the case when ϕ depends on a finite number of coordinates, this theorem can be derived from the work of Vere-Jones on countable matrices ([24],[25]). The case when ϕ depends on an infinite number of coordinates, however, requires techniques which are essentially different. The main new ingredient in the proof is a tightness argument (see proposition 2).

Remark 2. It follows from the proof that $\log h, \log h\circ T$ are both locally Hölder continuous (in particualr h is uniformly bounded away from zero and infinity on partition sets). It follows from (3) that ν and h are uniquely determined up to a multiplicative factor.

Remark 3. The measure $dm = h\,d\nu$ is invariant and conservative, and its transfer operator is given by $\hat{T}f = \lambda^{-1}h^{-1}L_\phi(hf)$. It follows from local Hölder continuity and results in [1] that dm is exact, pointwise dual ergodic and that for dm, every cylinder $[\underline{a}]$ is a Darling-Kac set with an exponential continued fraction mixing return time process. See [1], [2] for definitions and a survey of limit theorems for such measures m.

We now show how to formulate the results of theorem 1 in terms of suitable **dynamical zeta functions**.

Assume that X is topologically mixing and that ϕ is locally Hölder continuous such that $\|L_\phi 1\|_\infty < \infty$. In this case, by the results of [18], $P_G(\phi)$ is finite and (2) holds. Recall that Ruelle's dynamical zeta function [15] is given by

$$\zeta(t) = \exp\left(\sum_{n=1}^{\infty} \frac{t^n}{n} Z_n(\phi)\right)$$

where $Z_n(\phi) = \sum_{a\in S} Z_n(\phi, a) = \sum_{T^n x = x} e^{\phi_n(x)}$. The radius of convergence of ζ is equal to $e^{-P(\phi)}$ where $P(\phi) = \overline{\lim_{n\to\infty}} \frac{1}{n}\log Z_n(\phi)$.

If S is finite, $P(\phi) = P_G(\phi)$ whence ζ is holomorphic in $[|z| < e^{-P}]$ where $P = \sup\{h_\mu + \mu(\phi)\}$ (in this case X is compact so ϕ is bounded and the condition $\mu(-\phi) < \infty$ in (2) is empty). It is also known that in this case ζ has a simple pole in e^{-P} [15].

If S is infinite $P(\phi)$ *may* be strictly larger than P (for examples in the case $\phi = 0$ see [7] and [16]). Therefore, the disc of convergence of ζ may be strictly smaller than $\left\{z\colon |z| < e^{-P}\right\}$. We are naturally led to the consideration of the following **local dynamical zeta functions** defined for each $a \in S$

$$\zeta_a(t) = \exp\left(\sum_{n=1}^{\infty} \frac{t^n}{n} Z_n(\phi, a)\right)$$

Note that at least formally, $\zeta = \prod_{a\in S}\zeta_a$. The radius of convergence of ζ_a is independent of a, and is equal to $e^{-P_G(\phi)}$ where $P_G(\phi)$ satisfies (2). Obviously, ζ_a has a singularity in $e^{-P_G(\phi)}$.

As the following corollary shows, the behavior of ζ_a near this singularity determines the recurrence properties of ϕ (this is similar to the role of generating functions in renewal theory [6]). The following corollary is obtained from theorem 1.

Corollary 1 *Let X be topologically mixing and ϕ locally Hölder continuous such that $\|L_\phi 1\|_\infty < \infty$. Fix $a \in S$ and let $R = e^{-P_G(\phi)}$ be the radius of convergence of ζ_a.*

1. *ϕ is recurrent iff $(\log \zeta_a)'(R) = \infty$. In this case, if $dm = hd\nu$ is the corresponding invariant measure and $\{a_n\}_n$ is a return sequence of m, then*
$$(\log \zeta_a)'(t) \sim \frac{m[a]}{R}\left(1 - \frac{t}{R}\right)\sum_{n=1}^{\infty} a_n R^{-n} t^n \quad \textit{as } \ t \nearrow R$$
2. *ϕ is positive recurrent iff there exists $C_a > 0$ such that $(\log \zeta_a)' \sim C_a (1 - t/R)^{-1}$ as $t \nearrow R$. In this case $C_a = e^{P_G(\phi)} m[a]$ where m is the equilibrium probability measure of ϕ.*
3. *ϕ is null recurrent iff $(\log \zeta_a)' = o(1/(1 - t/R))$ as $t \nearrow R$ and ϕ is recurrent.*

It follows from the corollary that in the positive recurrent case

$$\zeta_a(t) = \left(\frac{1}{1 - e^{P_G(\phi)} t}\right)^{m[a](1+o(1))} \quad \text{as } \ t \nearrow e^{-P_G(\phi)}$$

where m is the equilibrium *probability* measure of ϕ. If S is finite, we retrieve the well known property of $\zeta = \prod_{a \in S} \zeta_a$ that $\zeta_a(t) = \left(1 - e^{P_G(\phi)} t\right)^{-(1+o(1))}$ as $t \nearrow e^{-P_G(\phi)}$ (in fact $e^{-P_G(\phi)}$ is a simple pole [15]). In broad terms, the degree of singularity for the full zeta function is distributed among the various local zeta functions according to the equilibrium measure.

In section 2 we apply theorem 1 to the theory of equilibrium states by describing the measure $dm = h\,d\nu$ as an equilibrium measure in a certain weak sense, when it is infinite. Section 3 contains a proof of theorem 1.

Notational Convention: We use the following short hand notation for double inequalities: $\forall a, b > 0 \; B > 1 \quad a = B^{\pm 1} b \Leftrightarrow B^{-1} b \leq$

$a \le Bb$. We write $a = A^{\pm 1}B^{\pm 1}b$ for $a = (AB)^{\pm 1} b$, and $a = A^{\pm k}b$ for $a = \left(A^k\right)^{\pm 1} b$.

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2 Application to the theory of equilibrium states

Let X be topologically mixing and ϕ be a locally Hölder continuous function with finite Gurevic pressure. Assume ϕ is recurrent. Let λ, ν, h denote the eigenvalue, eigenmeasure and eigenfunction given by theorem 1. It is easy to verify that the measure $dm = h\,d\nu$ is an invariant conservative measure. This is a **Gibbs measure** for ϕ in the following sense: $\forall a, b \in S\ \ \exists M_{ab} > 1$ such that for $m-$almost all $x \in X$

$$m\left(x_0, \ldots, x_{n-1} \middle| x_n, x_{n+1}, \ldots\right) = \frac{h(x)\, e^{\phi_n(x)}}{\lambda^n h(T^n x)} = M_{x_0,x_n}^{\pm 1} e^{\phi_n(x) - nP_G(\phi)} \tag{4}$$

This is weaker than the Gibbs property used by Bowen in [5], because the bound M_{x_0,x_n} may depend on x. To prove (4), check that the transfer operator of m is given by $\hat{T}f = \lambda^{-1}h^{-1}L_\phi(hf)$ and that $\mathbf{E}_m\left(f \middle| T^{-n}\mathcal{B}\right) = \left(\hat{T}^n f\right) \circ T^n$. The rest follows by direct computation from the fact that h is bounded away from zero and infinity on partition sets. Note that if ϕ is null recurrent, m is infinite.

We want to describe the measure m as a solution of a suitable variational problem. This was done for the positive recurrent case in [18] so we focus on null recurrent potentials. For such potentials m is infinite and the notion of entropy requires explanation.

We recall the definition given in [11], following the approach of [3]. Let $(X, \mathcal{B}, \mu, T)$ be an ergodic probability preserving transformation. For every measurable set with positive measure A one can define the

induced transformation T_A: $A \to A$ by $T_A x = T^{\varphi_A(x)} x$ where $\varphi_A(x) = \inf\{n \geq 1 \colon T^n x \in A\}$ (the Poincarè Recurrence theorem guaranties that $\varphi_A < \infty$ almost everywhere on A). It is known that the probability measure $\mu_A(E) = \mu(E \cap A)/\mu(A)$ is T_A-invariant and ergodic, and that its entropy is given by the **Abramov Formula** [4]:

$$h_\mu(T) = \mu(A)\, h_{\mu_A}(T_A)$$

If μ is infinite, ergodic and conservative, the same method of inducing applies (in this case Poincarè's theorem is replaced by the conservativity assumption). Applying the Abramov formula to T_A, T_B as induced versions of $T_{A\cup B}$ one sees that

$$0 < \mu(A), \mu(B) < \infty \quad \Rightarrow \quad \mu(A)\, h_{\mu_A}(T_A) = \mu(B)\, h_{\mu_B}(T_B).$$

Thus, the number $\mu(A)\, h_{\mu_A}(T_A)$ is independent of the choice of $A \in \mathcal{B}$ (as long as $0 < \mu(A) < \infty$) and may therefore be used as the *definition* of the entropy of the infinite conservative ergodic measure μ.

Example 1. (Krengel [11]) Let (p_{ij}) be a null recurrent irreducible stochastic matrix and (p_i) its stationary vector. Let μ be the corresponding invariant infinite Markovian measure. Then $h_\mu = -\sum_{s,t} p_s p_{st} \log p_{st}$.

For examples arising from interval maps, see [21].

Theorem 2 *Let X be topologically mixing and ϕ a recurrent locally Hölder continuous function with finite Gurevic pressure.*

1. *For every conservative ergodic invariant measure μ which is finite on partition sets, if $\mu(P_G(\phi) - \phi) < \infty$ then $h_\mu(T) \leq \mu(P_G(\phi) - \phi)$.*
2. *Let h and ν be as in theorem 1 and set $dm = h\, d\nu$. If $m(P_G(\phi) - \phi) < \infty$ then $h_m(T) = m(P_G(\phi) - \phi)$.*

Proof. Without loss of generality assume that $P_G(\phi) = 0$ (we can always pass to the potential $\phi - P_G(\phi)$). Fix some invariant measure μ which satisfies the assumptions of the theorem and choose some partition set A of (finite) positive measure.

Let μ_A be the probability measure $\mu_A(E) = \mu(A \cap E)/\mu(A)$. Let T_A: $A \to A$ be the induced map $T_A x = T^{\varphi_A(x)}x$ where $\varphi_A(x) = 1_A \inf\{n > 0\colon T^n x \in A\}$. Then μ_A is T_A invariant. Let

$$\overline{S}\colon = \{[\underline{a}] \subseteq A\colon \ A \text{ appears only once in } \underline{a} \text{ and } [\underline{a}, A] \neq \emptyset\}.$$

This is a generating Markov partition for T_A ($\mu_A\left(\cup\overline{S}\right) = 1$ by conservativity). Set $\overline{X} = \left(\overline{S}\right)^{\mathbf{N}\cup\{0\}}$ and let π: $\overline{X} \to A \subseteq X$ be the natural injection $\pi\left([\underline{a}]_1 [\underline{a}]_2 \ldots\right) = (\underline{a}_1; \underline{a}_2; \ldots)$. For every μ as in the above set $\overline{\mu} = \mu_A \circ \pi$. It is easy to check that the map π: $\overline{X} \to X$ is a measure theoretic isomorphism between the systems $(A, \mathcal{B} \cap A, \mu_A, T_A)$ and $\left(\overline{X}, \mathcal{B}(\overline{X}), \overline{\mu}, \overline{T}\right)$ where $\overline{T}$: $\overline{X} \to \overline{X}$ is the left shift. Let $\overline{\phi}$: $\overline{X} \to \mathbf{R}$ be the induced version of the potential ϕ given by

$$\overline{\phi} = \left(\sum_{i=0}^{\varphi_A - 1} \phi \circ T^i\right) \circ \pi$$

This is a locally Hölder continuous function (in fact, it even satisfies that $V_1\left(\overline{\phi}\right) < \infty$, since if $x_0 = [\underline{a}] \in \overline{S}$ then $\pi(x) \in [\underline{a}, A]$). The proof of local Hölder continuity is standard, and is therfore omitted.

Let $L_{\overline{\phi}}$ denote the Ruelle operator of $\overline{\phi}$, $L_{\overline{\phi}} f = \sum_{\overline{T}y = x} e^{\overline{\phi}(y)} f(y)$. Set $\overline{\nu} = \nu \circ \pi$ and $\overline{h} = h \circ \pi$. We claim that $L^*_{\overline{\phi}}\overline{\nu} = \overline{\nu}$, $L_{\overline{\phi}}\overline{h} = \overline{h}$. To see this note that

$$\log \frac{dm}{dm \circ T} = \phi + \log h - \log h \circ T$$

(because $f \mapsto h^{-1} L_\phi(hf)$ acts as the transfer operator of m). Let m_A denote the normalized restriction of m to A and $\overline{m} = m_A \circ \pi$. Then since $T_A = T^{\varphi_A}$

$$\log \frac{dm_A}{dm_A \circ T_A} = \sum_{i=0}^{\varphi_A - 1} \phi \circ T^i + \log h - \log h \circ T_A$$

whence

$$\log \frac{d\overline{m}}{d\overline{m} \circ \overline{T}} = \overline{\phi} + \log \overline{h} - \log \overline{h} \circ \overline{T}. \tag{5}$$

Since m is T invariant, m_A is T_A invariant. It follows that $\overline{m}$ is $\overline{T}$ invariant whence $L_{\log \overline{g}} 1 = 1$ where $\overline{g} = \log d\overline{m}/d\overline{m} \circ \overline{T}$. It follows from (5) that

$$\sum_{\overline{T}y=x} e^{\left(\overline{\phi} + \log \overline{h} - \log \overline{h} \circ \overline{T}\right)(y)} = 1$$

whence $L_{\overline{\phi}} \overline{h} = \overline{h}$. We show that $L_{\overline{\phi}}^* \overline{\nu} = \overline{\nu}$. Without loss of generality, $d\overline{\nu} = \overline{h}^{-1} d\overline{m}$ (the only difference is a normalizing constant). Using (5) and the fact that $L_{\log \overline{g}}$ acts as the transfer operator of $\overline{m}$, we have that for every $f \in L^1(\overline{\nu})$

$$\int L_{\overline{\phi}} f \, d\overline{\nu} = \int \overline{h}^{-1} L_{\overline{\phi}} f \, d\overline{m} = \int L_{\log \overline{g}} \left(\overline{h}^{-1} f \right) \, d\overline{m} = \int f \, d\overline{\nu}$$

as required.

It follows from theorem 1 and the relations $L_{\overline{\phi}} \overline{h} = \overline{h}$, $L_{\overline{\phi}}^* \overline{\nu} = \overline{\nu}$ and $\overline{\nu}\left(\overline{h}\right) = \nu(1_A h) < \infty$ that $\overline{\phi}$ is positive recurrent and that $P_G\left(\overline{\phi}\right) = 0$. Since $\overline{h} = h \circ \pi$ and $\pi(X) \subseteq A$, $\overline{h}$ is uniformly bounded away from zero and infinity. It follows that $\left\| L_{\overline{\phi}} 1 \right\|_\infty < \infty$. By (2),

$$\sup \left\{ h_\mu\left(\overline{T}\right) + \int \overline{\phi} \, d\mu \colon \mu \text{ is } \overline{T} \text{ invariant}, \ \mu\left(\overline{X}\right) = 1, \ \mu\left(-\overline{\phi}\right) < \infty \right\}$$
$$= P_G\left(\overline{\phi}\right) = 0$$

Since for every conservative invariant (possibly infinite) ergodic measure μ such that $\mu(A) < \infty$ and $\mu(-\phi) < \infty$ the measure $\overline{\mu} = \mu_A \circ \pi$ is a $\overline{T}$ invariant ergodic probability measure such that

$$\mu(A) \overline{\mu}\left(-\overline{\phi}\right) = -\int_A \sum_{k=0}^{\varphi_A - 1} \phi \circ T^k \, d\mu = \mu(-\phi) < \infty$$

we have that $h_\mu(T) + \mu(\phi) = \mu(A) \left[h_{\overline{\mu}}\left(\overline{T}\right) + \overline{\mu}\left(\overline{\phi}\right) \right] \leq 0$.

We now assume that $\mu = m$ and that $m(-\phi) < \infty$, and show that $h_m(T) + m(\phi) = 0$. $\overline{X}$ clearly satisfies the big images property: $\exists b_1, \ldots, b_N \in \overline{S}$ such that for every $a \in \overline{S}$ there is some b_i such that $[a, b_i]$ is not empty (in fact for every $a, b \in \overline{S}$ $[a, b]$ is non empty). Since $\overline{h}$ is uniformly bounded away from zero and infinity, $\overline{m}$ is a Gibbs measure for $\overline{\phi}$ in the sense of Bowen [5]: there is some *global* constant $M > 1$ such that for every $\underline{a}_0, \ldots, \underline{a}_{n-1} \in \overline{S}$ and $x \in [\underline{a}_0, \ldots, \underline{a}_{n-1}] \subseteq \overline{X}$

$$\overline{m}\left[\underline{a}_0, \ldots, \underline{a}_{n-1}\right] = M^{\pm 1} \exp \sum_{k=0}^{n-1} \overline{\phi}\left(\overline{T}^k x\right) \tag{6}$$

(see [18], theorem 8). Let $\overline{\alpha} = \left\{[\underline{a}] : \underline{a} \in \overline{S}\right\}$ denote the natural partition of $\overline{X}$. By the continuity properties of $\overline{\phi}$ and by (6)

$$\begin{aligned} H_{\overline{m}}(\overline{\alpha}) &= -\sum\nolimits_{[\underline{a}]\in\overline{\alpha}} \overline{m}\left[\underline{a}\right] \log \overline{m}\left[\underline{a}\right] \\ &\leq -\sum\nolimits_{[\underline{a}]\in\overline{\alpha}} \overline{m}\left[\underline{a}\right] \frac{1}{\overline{m}\left[\underline{a}\right]} \int_{[\underline{a}]} \overline{\phi}\, d\overline{m} + \log M \\ &= -\int_{\overline{X}} \overline{\phi}\, d\overline{m} + \log M \\ &= -\frac{1}{m(A)} \int_A \sum_{k=0}^{\varphi_A - 1} \phi \circ T^k \, dm + \log M \\ &= -\frac{1}{m(A)} \int \phi \, dm + \log M \end{aligned}$$

whence $H_{\overline{m}}(\overline{\alpha}) < \infty$. Since $\overline{\alpha}$ is a generator with finite entropy, we have by the Rokhlin formula [14] that

$$h_{\overline{m}}\left(\overline{T}\right) = -\int \log \frac{d\overline{m}}{d\overline{m} \circ \overline{T}}\, d\overline{m} = -\int \overline{\phi}\, d\overline{m} = -\frac{1}{m(A)} \int \phi\, dm$$

Multiplying both sides by $m(A)$ we have that $h_m(T) = -m(\phi)$ as required. ∎

Remark 4. It follows from the proof that m is the unique up to a constant conservative ergodic invariant measure such that $H_{\overline{m}}(\overline{\alpha}) < \infty$ and $h_m(T) = m(P_G(\phi) - \phi)$, since by a trivial generalization of an argument of Bowen's if there exists a probability measure which is Gibbs in the sense of Bowen, with a generator which has finite entropy, then this measure is the unique solution of the variational problem (see [5]).

The problem with the last theorem is that frequently both $h_m(T)$ and $m(P_G(\phi) - \phi)$ are infinite. In this situation, the sum $h_m(T) +$

$m\left(\phi - P_G\left(\phi\right)\right)$ is meaningless. The following theorem completes our discussion by treating this case as well.

Set

$$I_\mu = -\sum_{a\in S} 1_{[a]} \log \mu\left([a] \middle| T^{-1}\mathcal{B}\right)$$

This is well defined for every μ which is finite on partition sets. The following theorem generalizes theorem 7 in [18] (see also [12],[26],[28]).

Theorem 3 *Let X be topologically mixing and ϕ locally Hölder continuous with finite Gurevic pressure. Assume that ϕ is recurrent, let h and ν be as in theorem 1 and set $\phi' = \phi + \log h - \log h \circ T$. Then for every conservative invariant measure μ which is finite on partition sets, $I_\mu + \phi' - P_G\left(\phi'\right)$ is one sided integrable with respect to μ and*

$$-\infty \le \int \left(I_\mu + \phi' - P_G\left(\phi'\right)\right) d\mu \le 0. \tag{7}$$

if $\mu \sim \mu\circ T$, the integral in (7) is equal to zero iff μ is proportional to $h\,d\nu$.

Proof. Fix a conservative invariant measure μ, finite on partition sets, and set $g_\mu = d\mu/d\mu\circ T$ where $\mu \circ T$ is given by (1). Recall that the transfer operator of μ is given by $L_{\log g_\mu}$ and that $\mathbf{E}_\mu\left(f \middle| T^{-1}\mathcal{B}\right) = \left(L_{\log g_\mu} f\right)\circ T$. It follows that

$$I_\mu = -\log g_\mu$$

Set $g = \lambda^{-1} e^\phi h/h \circ T$ where $\lambda = \exp P_G\left(\phi\right)$. One checks that $\sum_{Ty=x} g\left(y\right) = 1$ and that $\sum_{Ty=x} g_\mu\left(y\right) = 1$ for μ almost all $x \in X$ (the first equality follows from the equation $L_\phi h = \lambda h$; the second follows from the identity $\mu\left(f\sum_{Ty=x} g_\mu\left(y\right)\right) = \mu\left(L_{\log g_\mu}\left(f\circ T\right)\right) = \mu\left(f\right)$ which is satisfied for every $f \in L^1\left(\mu\right)$).

We show that $I_\mu + \phi' - P_G\left(\phi'\right)$ is one sided integrable. We use the notation $\psi^+ = \psi 1_{[\psi>0]}$ and show that $\left(I_\mu + \phi' - P_G\left(\phi'\right)\right)^+$ is integrable. Fix a sequence of measurable sets $A_n \nearrow X$ such that $0 < \mu\left(A_n\right) < \infty$. Fix an arbitrary integrable function $f \ge 0$. Set $A_{s,t,n} = A_n \cap [s < g/g_\mu < t]$. Using the inequality $\log x \le x - 1$ we see that for every s, t, n,

$$\int_{A_{s,t,n}} (I_\mu + \phi' - P_G(\phi'))^+ \ f \circ T d\mu =$$

$$\begin{aligned}
&= \int (-\log g_\mu + \log g)^+ 1_{A_{s,t,n}} f \circ T \, d\mu \\
&= \int [\log (g/g_\mu)]^+ 1_{A_{s,t,n}} f \circ T \, d\mu \\
&\leq \int \left(\frac{g}{g_\mu} - 1\right)^+ 1_{A_{s,t,n}} f \circ T d\mu \\
&= \int f \circ T \cdot \mathbf{E}_\mu \left(\left(\frac{g}{g_\mu} - 1\right)^+ 1_{A_{s,t,n}} \middle| T^{-1}\mathcal{B} \right) \, d\mu \\
&= \int f \circ T \sum_{Ty=Tx} g_\mu(y) 1_{A_{s,t,n}}(y) \left(\frac{g(y)}{g_\mu(y)} - 1\right)^+ \, d\mu \\
&= \int f \circ T \sum_{Ty=Tx} 1_{A_{s,t,n}}(y) [g(y) - g_\mu(y)]^+ \ d\mu
\end{aligned}$$

The last integrand is bounded by $f \circ T$. Since this is true forall s, t, n the integral $\mu\left[(I_\mu + \phi' - P_G(\phi'))^+\right]$ is finite. This implies that $I_\mu + \phi' - P_G(\phi')$ is one sided integrable. Applying the same calculation to $I_\mu + \phi' - P_G(\phi')$ rather than $(I_\mu + \phi' - P_G(\phi'))^+$ yields the inequality

$$\int_{A_{s,t,n}} f \circ T (I_\mu + \phi' - P_G(\phi')) \ d\mu$$

$$\leq \int f \circ T \sum_{Ty=Tx} 1_{A_{s,t,n}}(y) [g(y) - g_\mu(y)] \ d\mu$$

The integrand on the left is bounded in absolute value by the integrable function $f \circ T$. Its pointwise limit when $s \to 0, t, n \to \infty$ is zero, because $\sum_{Ty=Tx} [g(y) - g_\mu(y)] = 0$. We may therefore apply the dominated convergence theorem and deduce

$$\int f \circ T [I_\mu + \phi' - P_G(\phi')] \ d\mu \leq 0.$$

Since f was arbitrary, (7) follows.

Assume that $\mu \sim \mu \circ T$. We show that the integral in (7) is equal to zero if and only if $d\mu$ is proportional to $h\,d\nu$. If $d\mu$ is proportional to $h\,d\nu$ the integrand in (7) is identically zero because then $I_\mu = -\log g$ where $g = \lambda^{-1} e^{\phi} h / h \circ T$ (this follows from the fact that the transfer operator of any measure proportional to $h\,d\nu$ is given by $f \mapsto \lambda^{-1} h^{-1} L_\phi(hf)$). We show the reverse implication. Assume that μ is such that $\mu \sim \mu \circ T$ and that there is an equality in (7). A close inspection of the proof shows that this is possible only if $\log(g/g_\mu) = (g/g_\mu) - 1$ μ almost everywhere. This is possible only if $g_\mu = g$ *mod* μ. Since $\mu \sim \mu \circ T$, this implies that $g_\mu = g$ *mod* $\mu \circ T$. It follows that $L_{\log g}$ is the transfer operator of μ. Consider the function $\psi = \log g = \phi + \log h - \log h \circ T - \log \lambda$. This is a locally Hölder continuous function (because by remark 2 after theorem 1 $\log h$, $\log h \circ T$ are both locally Hölder continuous). It is also clear that $L_\psi 1 = 1, L_\psi^* \mu = \mu$ whence ψ is recurrent. Since it is also true that $L_\psi^*(hd\nu) = L_{\log g}^*(hd\nu) = hd\nu$ we have by the convergence part of theorem 1 that μ and $hd\nu$ are proportional. ∎

3 Proof of Theorem 1

This section is devoted to the proof of theorem 1. Throughout the proof we assume that X is a topologically mixing countable Markov shift and that ϕ: $X \to \mathbf{R}$ is locally Hölder continuous with finite Gurevic pressure. Set

$$B_k = \exp \sum_{n=k+1}^{\infty} V_n(\phi) \quad (k = 1, 2, \ldots)$$

Local Hölder continuity implies that $\forall n \geq 1$ $B_n < \infty$ and $B_n \searrow 1$. The following inequality will be used constantly:

$$x_0 = y_0, \ldots, x_{n-1} = y_{n-1} \Rightarrow \forall m \leq n-1 \left(e^{\phi_m(x)} = B_{n-m}^{\pm 1} e^{\phi_m(y)} \right) \tag{8}$$

A frequently used corollary is that $\forall x_a \in [a]$,

$$Z_n(\phi, a) = B_1^{\pm 1} \left(L_\phi^n 1_{[a]} \right)(x_a)$$

The reader should note that assumption that the Gurevic pressure is finite implies that *all* of the $Z_n(\phi, a)$ are finite (because by local Hölder continuity $\exists C > 1$ such that $\forall m, n$ $C^{-m} Z_n(\phi, a)^m < Z_{mn}(\phi, a)$). This assumption also implies that the L_ϕ^n are all defined on bounded functions supported inside a finite union of partition sets.

3.1 Existence of ν

Proposition 1 *If there exists $\lambda > 0$ and a conservative $\sigma-$ finite measure ν which is finite on some cylinder such that $L_\phi^* \nu = \lambda \nu$ then ϕ is recurrent and $\lambda = e^{P_G(\phi)}$.*

Proof. Choose a cylinder $[\underline{b}]$ with finite positive measure. It easy to verify that $\lambda^{-1} L_\phi$ acts as the transfer operator of ν whence by conservativity $\sum_{n\geq 1} \lambda^{-n} L_\phi^n 1_{[\underline{b}]} = \infty$ ν -a.e. on $[\underline{b}]$ (see [2]). Thus, for $\nu-$almost all $x \in [\underline{b}]$

$$\sum_{n=1}^{\infty} \lambda^{-n} Z_n(\phi, b_0) \geq B_1^{-1} \sum_{n=1}^{\infty} \lambda^{-n} \left(L_\phi^n 1_{[\underline{b}]} \right)(x) = \infty$$

We show that $\lambda = e^{P_G(\phi)}$. It follows from what we just proved that $\lambda \leq e^{P_G(\phi)}$ because the radius of convergence of the series $\sum_{k\geq 1} Z_k(\phi, b_0) x^k$ is $e^{-P_G(\phi)}$. Consider $Z_n(\phi, \underline{b}) = \sum_{T^n x = x} e^{\phi_n(x)} 1_{[\underline{b}]}(x)$. By local Hölder continuity,

$$\lambda^{-n} Z_n(\phi, \underline{b}) \leq B_1 \left[\frac{1}{\nu[\underline{b}]} \int_{[\underline{b}]} \left(\lambda^{-n} L_\phi^n 1_{[\underline{b}]} \right) d\nu \right] \leq B_1$$

By topological mixing and local Hölder continuity $n^{-1} \log Z_n(\phi, \underline{b}) \to P_G(\phi)$, whence $\lambda \geq e^{P_G(\phi)}$. ∎

Proposition 2 *If ϕ is recurrent there exist $\lambda > 0$ and a conservative measure ν, finite and positive on cylinders, such that $L_\phi^* \nu = \lambda \nu$.*

Proof. Fix $a \in S$, set $\lambda = e^{P_G(\phi)}$ and let $a_n = \sum_{k=1}^n \lambda^{-k} Z_k(\phi, a)$. For every $y \in X$ let δ_y denote the probability measure concentrated on $\{y\}$. Fix a periodic point $x_a \in [a]$ and set for every $b \in S$

$$\nu_n^b = \frac{1}{a_n} \sum_{k=1}^n \lambda^{-k} \sum_{T^k y = x_a} e^{\phi_k(y)} 1_{[b]}(y) \delta_y$$

Clearly $\nu_n^b(X) = \nu_n^b([b]) = \frac{1}{a_n}\sum_{k=1}^n \lambda^{-k}\left(L_\phi^k 1_{[b]}\right)(x_a)$. It follows from local Hölder continuity, topological mixing and the definition of the Gurevic pressure that for every $b \in S$

$$0 < \varliminf_{n\to\infty} \nu_n^b(X) \leq \varlimsup_{n\to\infty} \nu_n^b(X) < \infty$$

(It is enough to show that $a_n^{-1}\sum_{k=1}^n \lambda^{-k} Z_n(\phi, b)$ is bounded away from zero and infinity for every b. To see this note that $\exists C, c$ such that $Z_n(\phi, b) < C Z_{n+c}(\phi, a)$ and that $\forall k$ $\lambda^{-k} Z_k(\phi, a) < 2B_1$. The last inequality follows from $\lambda^{-km} Z_{km}(\phi, a) > B_1^{-m}\left(\lambda^{-k} Z_k(\phi, a)\right)^m$.)

We show how to choose a subsequence $\{m_k\}_{k\geq 1}$ such that for every $b \in S$, $\left\{\nu_{m_k}^b\right\}$ is w^* convergent, and show that the non trivial measure ν given by $\nu_{m_k}^b \xrightarrow{w^*} \nu|_{[b]}$ satisfies $L_\phi^*\nu = \lambda\nu$. Since X is not compact, to do this we have to prove that $\left\{\nu_{m_k}^b\right\}_{k\geq 1}$ are all **tight** , i.e.,

$$\forall b\, \forall \varepsilon > 0\, \exists F = F_{b,\varepsilon} \text{ compact such that } \forall n \quad \nu_n^b(F^c) < \varepsilon$$

It follows from the topological mixing of X that if $\left\{\nu_n^b\right\}_{n\geq 1}$ is tight for some b, then it is tight for *every* b. Therefore, we may restrict ourselves to the case $b = a$ and set $\nu_n^a = \nu_n$.

Step 1. We show that $\sum_{k\geq 1} \lambda^{-k} Z_k^*(\phi, a) < \infty$. To see this, set $T(x) = 1 + \sum_{k\geq 1} Z_k(\phi, a) x^k$ and $R(x) = \sum_{k\geq 1} Z_k^*(\phi, a) x^k$. It is not difficult to verify that $\forall x \in (0, \lambda^{-1})$, $T(x) - 1 = B_1^{\pm 2} R(x) T(x)$. Therefore $\forall x \in (0, \lambda^{-1})$ $R(x) \leq B_1^2$ whence $R(\lambda^{-1}) < \infty$.

Step 2. Set

$$\tau_1(x) = \begin{cases} \inf\{n \geq 1 \colon T^n x \in [a]\} & x \in [a] \\ 0 & x \notin [a] \end{cases} \tag{9}$$

where $\inf \varnothing = \infty$. Define by induction $\tau_n(x) = \tau_1\left(T^{\tau_1(x)+\ldots+\tau_{n-1}(x)} x\right)$ if $\tau_{n-1}(x) < \infty$ and $\tau_n(x) = \infty$ else. Note that $\tau_n > 0$ only if $x_0 = a$. For every sequence of natural numbers $\{n_i\}_{i\geq 1}$ set

$$R(\{n_i\}) = \{x \in [a] : \forall i \;\; \tau_i(x) \leq n_i\}.$$

We show that $\forall \varepsilon > 0\ \exists \{n_i\}$ such that $\forall n\ \ \nu_n\left(R\{n_i\}^c\right) < \varepsilon$ To see this set

$$Z^*_{k_1,\ldots,k_m} = \sum \left\{ e^{\phi_{k_1+\ldots+k_m}(x)} : x_0 = a\ ;\ T^{k_1+\ldots+k_m}x = x \right.$$
$$\left. ;\ \forall j \le m \quad \tau_j(x) = k_j \right\}$$

For $\{n_i\}_{i\ge 1}$ such that n_i is larger than the period of x_a,

$$\nu_n\left(R\{n_i\}^c\right) \le \sum_{i=1}^{\infty} \nu_n\left[\tau_i > n_i\right]$$

$$\begin{aligned}
&= \sum_{i=1}^{\infty} \frac{1}{a_n} \sum_{k=1}^{n} \lambda^{-k} \sum_{\substack{T^k y = x_a \\ y_0 = a}} e^{\phi_k(y)} 1_{[\tau_i > n_i]}(y) \\
&\le \sum_{i=1}^{\infty} \frac{1}{a_n} \sum_{k=n_i+1}^{n} \lambda^{-k} \sum_{\substack{T^k y = x_a \\ y_0 = a}} \sum_{\substack{k_1+\ldots+k_N = k \\ k_i > n_i\ ,\ N \le k}} e^{\phi_k(y)} 1_{[\forall j \le N\ \ \tau_j(y) = k_j]}(y) \\
&\le B_1^3 \sum_{i=1}^{\infty} \frac{1}{a_n} \sum_{k=n_i+1}^{n} \lambda^{-k} \sum_{\substack{k_1+\ldots+k_N = k \\ k_i > n_i\ ,\ N \le k}} Z^*_{k_1,\ldots,k_{i-1}} Z^*_{k_i} Z^*_{k_{i+1},\ldots,k_N} \\
&\le \frac{B_1^3}{a_n} \sum_{i=1}^{\infty} \sum_{k_i=n_i+1}^{\infty} \lambda^{-k_i} Z^*_{k_i} \sum_{k=k_i}^{n} \lambda^{-(k-k_i)} \sum_{\substack{k_1+\ldots+k_N = k \\ N \le k}} Z^*_{k_1,\ldots,k_{i-1}} Z^*_{k_{i+1},\ldots,k_N} \\
&\le B_1^5 \sum_{i=1}^{\infty} \sum_{k_i=n_i+1}^{\infty} \lambda^{-k_i} Z^*_{k_i} \left(\frac{1}{a_n} \sum_{k=k_i}^{n} \lambda^{-(k-k_i)} Z_{k-k_i}(\phi, a) \right) \\
&\le B_1^5 \sum_{i=1}^{\infty} \sum_{k_i=n_i+1}^{\infty} \lambda^{-k_i} Z^*_{k_i}
\end{aligned}$$

It remains to apply the previous step and choose n_i such that

$$\sum_{k_i=n_i+1}^{\infty} \lambda^{-k_i} Z^*_{k_i} < \frac{\varepsilon}{2^i B_1^5}.$$

Step 3. Fix $\{n_i\}_i$ such that $\forall n\ \ \nu_n\left(R\{n_i\}^c\right) < \varepsilon$. For every sequence of natural numbers $\{k_i\}$ set

$$S(\{k_i\}) = \{x \in [a] : \forall i \quad \tau_i(x) = k_i\}$$

We show that for every $\varepsilon > 0$ there exists a compact set $F \subseteq [a]$ such that

$$\forall i \;\; k_i \leq n_i \;\Rightarrow\; \forall n \;\; \nu_n \left(F^c \cap S\left\{k_i\right\}\right) \leq \varepsilon\, \nu_n \left(S\left\{k_i\right\}\right). \tag{10}$$

This is enough to prove tightness, because (10) implies that for every n $\nu_n\left(F^c\right) \leq \varepsilon(1+\nu_n(R))$ and we already know that the total mass of ν_n is uniformly bounded from above. The F we will construct will be of the form $F = \{x \in [a] : \forall i \;\; x_i \leq N_i\}$ where $N_i \in S$ (we are using an order on S induced by the identification $S \approx \mathbf{N}$). Clearly, this is a compact set. We show how to choose $\{N_i\}$. Set

$$Z_k^*\left(N\right) = \sum \left\{ e^{\phi_k(x)} \colon x \in [a] \;\; ; \; T^k x = x \; ; \; \tau_1\left(x\right) = k \; ; \; \exists i \; x_i > N \right\}$$

Obviously, $Z_k^*\left(N\right) \searrow 0$ as $N \nearrow \infty$. For every i, we choose N_i in a way such that for every $k \leq n_i$

$$Z_k^*\left(N_i\right) \leq \frac{\varepsilon}{2^i B_1^7} Z_k^*.$$

We make sure that $\{N_i\}$ are chosen in an increasing way and that $N_1 > \sup_{i\geq 0}\left\{x_a\left(i\right)\right\}$ (recall that x_a was chosen to be periodic, so its coordinates are bounded).

Fix $\{k_i\} \leq \{n_i\}$ such that $\nu_n\left(S\left\{k_i\right\}\right) > 0$. Fix $N = N\left(n, \{k_i\}\right)$ such that $k_1 + \ldots + k_N \geq n$. Since $N_i > \sup\left\{x_a\left(i\right)\right\} \geq a$,

$$\begin{aligned}
&\nu_n\left(F^c \cap S\left\{k_i\right\}\right) \\
&\leq \sum_{i=1}^{N} \nu_n \left\{ x \in S\left\{k_i\right\} : \exists j \in \left(\sum_{m=1}^{i-1} k_m, \sum_{m=1}^{i} k_m \right) \;\; x_j > N_j \right\} \\
&\leq \sum_{i=1}^{N} \nu_n \left\{ x \in S\left\{k_i\right\} : \exists j \in \left(\sum_{m=1}^{i-1} k_m, \sum_{m=1}^{i} k_m \right) \;\; x_j > N_i \right\} \\
&\leq \frac{B_1^3}{a_n} \sum_{i=1}^{N} \sum_{\ell=i}^{N} \lambda^{-(k_1+\ldots+k_\ell)} Z^*_{k_1,\ldots,k_{i-1}} Z^*_{k_i}\left(N_i\right) Z^*_{k_{i+1},\ldots,k_\ell} 1_{S(\{k_j\}_{i>\ell})}(x_a) \\
&\leq B_1^6 \sum_{i=1}^{N} \frac{\varepsilon}{2^i B_1^7} \left(\frac{1}{a_n} \sum_{\ell=i}^{N} \lambda^{-(k_1+\ldots+k_\ell)} Z^*_{k_1,\ldots,k_\ell} 1_{S(\{k_j\}_{i>\ell})}(x_a) \right) \\
&\leq \varepsilon\, \nu_n\left(S\left\{k_i\right\}\right)
\end{aligned}$$

Tightness is proved.

By tightness, there exists a subsequence m_k such that $\forall b \in S$, $\left\{\nu^b_{m_k}\right\}_{k\geq 1}$ is w^*-convergent. We denote its limit by ν^b and set $\nu = \sum_{b\in S} \nu^b$. It is not difficult to check that

$$\forall [\underline{b}] \quad 0 < \nu [\underline{b}] < \infty \tag{11}$$

We show that $L^*_\phi \nu = \lambda\nu$. By recurrence, $a_n \nearrow \infty$. A standard calculation shows that for every $[\underline{b}]$ and N, $\nu \left(1_{[x_0<N]} L_\phi 1_{[\underline{b}]}\right) = \lambda\nu \left(1_{[x_1<N]} 1_{[\underline{b}]}\right)$. It follows from the Lebesgue monotone convergence theorem that $\nu \left(L_\phi 1_{[\underline{b}]}\right) = \lambda\nu [\underline{b}]$. Since $[\underline{b}]$ was arbitrary, we have that $L^*_\phi \nu = \lambda\nu$.

We show that ν is conservative. One checks that the transfer opertor of ν is $\hat{T} = \lambda^{-1} L_\phi$. To prove conservativity it is enough to show that for some positive integrable function f $\sum_{k\geq 1} \hat{T}^k f = \infty$ almost everywhere. Set $f = \sum_{a\in S} f_a 1_{[a]}$ where $f_a > 0$ are chosen so that $\nu(f) < \infty$. For every $a \in S$ and $x \in [a]$

$$\sum_{k=1}^{\infty} \lambda^{-k} \left(L^k_\phi f\right)(x) \geq B_1^{-1} f_a \sum_{k=1}^{\infty} \lambda^{-k} Z_k (\phi, a) = \infty.$$

Conservativity follows. ∎

3.2 The Schweiger property

Let X be a topological Markov shift and μ be a measure supported on X such that $\mu \sim \mu \circ T^{-1}$ and $\mu \sim \mu \circ T$. μ is said to have the **Schweiger property** (see [1]) if there exists a collection of cylinders $\mathcal{R}$ such that:

1. the members of $\mathcal{R}$ have finite positive measures and $\cup\mathcal{R} = X \bmod \nu$;
2. for every $[\underline{b}] \in \mathcal{R}$ and arbitrary cylinder $[\underline{a}]$ if $[\underline{a}, \underline{b}] \neq \emptyset$ then $[\underline{a}, \underline{b}] \in \mathcal{R}$;

3. there exists a constant $C > 1$ such that for every $[\underline{b}] \in \mathcal{R}$ of length n and $\mu \times \mu$ almost all $x, y \in [\underline{b}] \times [\underline{b}]$

$$\left.\frac{d\mu}{d\mu \circ T^n}\right|_{[\underline{b}]}(x) = C^{\pm 1} \left.\frac{d\mu}{d\mu \circ T^n}\right|_{[\underline{b}]}(y) \qquad (12)$$

Aaronson, Denker and Urbanski proved in [1] that if μ has the Schweiger property, is supported on a topologically mixing topological Markov shift, and is conservative then:

1. μ is exact (hence ergodic);

2. there exists a σ-finite invariat measure $m \sim \mu$ such that $\log\left(\frac{dm}{d\nu}\right)$ is bounded on every $B \in \mathcal{R}$.

3. every $[\underline{b}] \in \mathcal{R}$ is a Darling-Kac set for m with a continued fraction mixing return time process (see [1] for definitions and implications).

4. m is **pointwise dual ergodic**: there exist $a_n > 0$ such that for every $f \in L^1(m)$

$$\frac{1}{a_n} \sum_{k=1}^{n} \hat{T}^k f \underset{n\to\infty}{\longrightarrow} m(f) \quad a.e.$$

   where $\hat{T}$ is the transfer operator of m.

Rényi's property states that (12) holds for *all* cylinders (see [2]). It follows from local Hölder continuity that ν satisfies Rényi's property with respect to the partition generated by cylinders of length two. It is not true in general, however, that ν satisfies this property with respect to all cylinders, including those of length one (see example 2 below). In order to obtain information on cylinders of length one as well, we need the following lemma, which was inspired by [1]. For every $c \in S$ set $\mathcal{R}_c = \{[b_0, \ldots, b_{n-1}] : n \in \mathbf{N}\ ,\ b_{n-1} = c\ \}$. Note that $[c] \in \mathcal{R}_c$.

Lemma 1 *Let X be topologically mixing and ϕ locally Hölder continuous. Suppose that ν is a conservative measure, finite and positive on cylinders such that $L_\phi^* \nu = \lambda \nu$. Then $\forall c \in S$ there exists a density function $q = q^{(c)}\colon X \to (0, \infty)$ such that $d\nu_c = q^{(c)} \, d\nu$ has the Schweiger property with respect to $\mathcal{R}_c$. q can be chosen to be constant on partition sets.*

Proof. For every $1 \le m \le n-1$ and $[\underline{b}]$ of length n set $\phi_m(\underline{b}) = \inf\{\phi_m(x) : x \in [\underline{b}]\}$. By (8) $\forall x \in [\underline{b}] \quad \phi_m(x) = \phi_m(x_0, \ldots, x_{n-1}) \pm \log B_{n-m}$. Set $q(x) = q^{(c)}(x) = q_{x_0}$ where

$$q_b = \begin{cases} e^{\phi(c,b)} & [b] \subseteq T[c] \\ 1 & \text{else} \end{cases}$$

and set $d\nu_c = q d\nu$. A calculation shows that $d\nu_c \circ T^n = q_c \circ T^n d\nu \circ T^n$ whence

$$\frac{d\nu_c}{d\nu_c \circ T^n} = \frac{q_c}{q_c \circ T^n} \lambda^{-n} e^{\phi_n}$$

It follows that for every $x \in [b_0, \ldots, b_{n-1}]$ such that $b_{n-1} = c$

$$\frac{d\nu_c}{d\nu_c \circ T^n}(x) = \frac{q_{b_0}}{q_{x_n}} e^{\phi_n(x)} = B_1^{\pm 1} \lambda^{-n} q_{b_0} e^{\phi_{n-1}(x)}$$

Thus (12) is proved.

Obviously for every $[\underline{b}]$ in $\mathcal{R}_c$ and for every $[\underline{a}]$, $[\underline{a}, \underline{b}]$ is either empty or in $\mathcal{R}_c$. We show that $X = \bigcup \mathcal{R}_c (\text{mod } \nu_c)$. Assume this were not the case. Then $\exists a \in S \ \exists A \subseteq [a]$ measurable of positive measure such that $\nu_c(A \cap \bigcup \mathcal{R}_c) = 0$. By topological mixing there exists a $[\underline{c}] \subseteq [c]$ such that $[\underline{c}, a] \ne \emptyset$. Choose such a $\underline{c}$ of minimal length. Set $[\underline{c}, A] = [\underline{c}] \cap T^{-|\underline{c}|} A$ where $|\underline{c}|$ denotes the length of $[\underline{c}]$. Then $[\underline{c}, A] \ne \emptyset$ and

$$\int_{[\underline{c}, A]} \frac{d\nu_c \circ T^{|\underline{c}|}}{d\nu_c} d\nu_c = \nu_c(A) > 0$$

whence $\nu_c[\underline{c}, A] > 0$. Since $|\underline{c}|$ is minimal, $[\underline{c}, A] \subseteq [c] \setminus T^{-1}(\cup \mathcal{R}_c) = [c] \setminus \bigcup_{n \ge 1} T^{-n}[c]$ so by conservativity $\nu_c[\underline{c}, A] = 0$. ∎

Example 2. Set $S = \{a, b, 1, 2, 3, \ldots\}$ and $\mathbf{A} = (t_{ij})_{S \times S}$ where $t_{ij} = 1$ if and only if $i \in \{a, b\}, j \in \mathbf{N}$ or $i \in \{a, b\}, j = i$ or $i =$

$1, j \in \{a,b\}$ or $i \neq a,b,1$ and $j = i-1$. Set $\phi(x) = \log p_{x_0,x_1}$ where $p_{aa} = p_{bb} = f_0$ and for all $i \in \mathbf{N}$ and $j \in S$, $p_{ai} = f_i$, $p_{bi} = f_i'$, $p_{ij} = 1$ where f_i, f_i' will be determined later. Then $Z^*_{n+1}(\phi,1) = \sum_{k=0}^{n-1}(f_{n-k} + f'_{n-k})f_0^k$ and

$$Z_n(\phi,1) = Z_n^*(\phi,1) + \sum_{k=1}^{n-1} Z^*_{n-k}(\phi,1)Z_k(\phi,k)$$

Now choose $f_0 = 1/4$, $f_i = C/2^i$ and $f_i' = C/4^i$ where $C > 0$ is a constant such that $\sum_{n\geq 1} Z_k^*(\phi,1) = 1$. It follows from the renewal theorem that $Z_n(\phi,1)$ tends to $1/\sum_{n\geq 1} nZ_n^*(\phi,1) > 0$ as n tends to infinity. Thus $P_G(\phi) = 0$ and ϕ is positive recurrent. Let ν be the corresponding eigenmeasure (the existence of which is guarantied by proposition 2). Then there is no density vector $\{p_k\}$ such that the resulting measure satisfies Rényi's condition because such a vector must satisfy $p_k \asymp p_{ak}, p_{bk}$ whereas $p_{ak} \not\asymp p_{bk}$.

3.3 Existence of h and $\{a_n\}_n$

Proposition 3 *If ϕ is recurrent then $\exists h > 0$ and $\exists \{a_n\}_{n=1}^{\infty}$ such that $L_\phi h = \lambda h$ and such that for every cylinder $[\underline{b}]$ and $x \in X$*

$$\frac{1}{a_n}\sum_{k=1}^{n} \lambda^{-k}\left(L_\phi^k 1_{[\underline{b}]}\right)(x) \underset{n\to\infty}{\longrightarrow} h(x)\,\nu\,[\underline{b}]\,.$$

Furthermore, h is bounded away from zero and infinity on partition sets, $\log h, \log h\circ T$ are locally Hölder continuous, and every cylinder is a Darling-Kac set for $dm = h\,d\nu$ with a continued fraction mixing return time process.

Proof. Since ϕ is recurrent, there exists a conservative measure ν, finite and positive on cylinders, such that $L_\phi^*\nu = \lambda\nu$. Fix an arbitrary $c \in S$ and set $\mathcal{R}_c = \{[b_0,\ldots,b_{n-1}] : n \in \mathbf{N}, b_{n-1} = c\}$. By lemma 1 $\exists \nu_c \sim \nu$ with the Schweiger property with respect to $\mathcal{R}_c$ such that $d\nu_c/d\nu$ is constant on partition sets. By the results cited in the last section, there exists an exact invariant measure m which is equivalent to ν_c, hence also to ν. Its derivative $dm/d\nu$ is bounded away from zero and infinity on members of $\mathcal{R}_c$ (because $d\nu_c/d\nu$ is constant on partition sets). This measure is pointwise dual ergodic: there exist $a_n > 0$ such that for every $f \in L^1(m)$

$$\frac{1}{a_n}\sum_{k=1}^{n}\hat{T}^k f \underset{n\to\infty}{\longrightarrow} \int f\,dm \quad a.e \tag{13}$$

Set $h = dm/d\nu$. Since ν is equivalent to m and m is exact, ν is conservative ergodic and can only have one invariant density (up to a constant). Thus h and m are independent of c. It also follows from (13) that $\{a_n\}$ is independent of c (up to a constant and asymptotic equivalence). The results of the previous section imply that every member of $\mathcal{R}_c$ is a Darling-Kac set for m with a continued fraction mixing return time process. Since m is independent of c and c is arbitrary, this is true for every member of $\bigcup_{c\in S}\mathcal{R}_c$, i.e. for all cylinders. The same reasoning shows that h is bounded away from zero and infinity on every cylinder. Thus, since ν is positive and finite on cylinders, so is m.

We show that h and $\{a_n\}$ are the required eigenfunction and sequence. The transfer operator of dm is given by $\hat{T}f = \lambda^{-1}h^{-1}L_\phi(hf)$, (because $dm = h\,d\nu$ and the transfer operator of ν is given by $\lambda^{-1}L_\phi$). Thus, for every cylinder $[\underline{b}]$

$$\frac{1}{a_n}\sum_{k=1}^{n}\lambda^{-k}L_\phi^k 1_{[\underline{b}]} = \frac{1}{a_n}h\sum_{k=1}^{n}\hat{T}^k\left(h^{-1}1_{[\underline{b}]}\right). \tag{14}$$

For every cylinder $[\underline{b}]$ the function $h^{-1}1_{[\underline{b}]}$ is m-integrable (because h is bounded away from zero on cylinders). Thus (14) implies that for $m-$almost every $x \in X$ for every cylinder $[\underline{b}]$

$$\frac{1}{a_n}\sum_{k=1}^{n}\lambda^{-k}\left(L_\phi^k 1_{[\underline{b}]}\right)(x) \underset{n\to\infty}{\longrightarrow} h(x)\,\nu[\underline{b}]. \tag{15}$$

Since ν is positive on cylinders, and $m \sim \nu$, there is a dense set of points $x \in X$ for which (15) is valid for every cylinder $[\underline{b}]$. By (8) $\forall m \geq 1\ \forall k\ \ V_m\left[\log\left(L_\phi^k 1_{[\underline{b}]}\right)\right] < \log B_m$ and we have that the logarithm of each of the summands in the left hand of (15) is uniformly continuous in x. It follows that h has a version for which (15) holds *everywhere* for every cylinder $[\underline{b}]$. This version must satisfy

$$\forall m \geq 1 \; V_m \left[\log h\right] < \log B_m \tag{16}$$

whence $\log h, \log h \circ T$ are locally Hölder continuous. We see, again, that h is uniformly bounded away from zero and infinity on partition sets, because the last estimation is also valid for the case $m = 1$.

It is now possible to show that h is an eigenfunction. Applying L_ϕ on both hands of (15) (and noting that by conservativity $a_n \to \infty$) it is easy to see that $L_\phi h \leq \lambda h$. Set $f = h - \lambda^{-1} L_\phi h$. This is a non negative function which satisfies $\sum_{k>0} \lambda^{-k} L_\phi^k f < \infty$. Since ν is ergodic conservative with transfer operator $\lambda^{-1} L_\phi$, this is impossible unless $f = 0 \; \nu - a.e.$ Since f is continuous and ν supported everywhere, $f = 0$ whence $L_\phi h = \lambda h$. ∎

3.4 Identification of $\{a_n\}_n$

Proposition 4 *Let m and $\{a_n\}_n$ be as in proposition 3. Then for every $a \in S$*

$$a_n \sim \frac{1}{m\left[a\right]} \sum_{k=1}^{n} \lambda^{-k} Z_n\left(\phi, a\right)$$

Proof. Let $\hat{T}$ denote the transfer operator of m. For every cylinder $[\underline{a}]$ of length N set $Z_n\left(\phi, \underline{a}\right) = \sum_{T^n x = x} e^{\phi_n(x)} 1_{[\underline{a}]}\left(x\right)$ and choose some $x_{\underline{a}} \in [\underline{a}]$. By (16) for every $N \geq 1$ and almost all $x_{\underline{a}} \in [\underline{a}]$

$$\lambda^{-n} Z_n\left(\phi, \underline{a}\right) = B_N^{\pm 1} \left(\lambda^{-n} L_\phi^n 1_{[\underline{a}]}\right)\left(x_{\underline{a}}\right) = B_N^{\pm 2} \left(\hat{T}^n 1_{[\underline{a}]}\right)\left(x_{\underline{a}}\right) \tag{17}$$

By (13)

$$\underline{\lim}_{n\to\infty}, \overline{\lim}_{n\to\infty} \left[\frac{1}{a_n} \sum_{k=1}^{n} \lambda^{-k} Z_k\left(\phi, \underline{a}\right)\right] = B_N^{\pm 2} m\left[\underline{a}\right] \tag{18}$$

The idea is to sum over $[\underline{a}] \subseteq [a]$ and deduce that

$$\underline{\lim}_{n\to\infty}, \overline{\lim}_{n\to\infty} \left[\frac{1}{a_n} \sum_{k=1}^{n} \lambda^{-k} Z_k\left(\phi, a\right)\right] = B_N^{\pm 2} m\left[a\right]$$

which implies, since N is arbitrary, that both limits coincide and are equal to $m[a]$. We need a regularity argument to deal with the possibility that there may be an infinite number of $[\underline{a}] \subseteq [a]$ such that $|\underline{a}| = N$.

Let $\varepsilon > 0$ and $F = F_\varepsilon$ be a compact such that $m([a] \setminus F) < \varepsilon$. We denote by $[a] \cap \alpha_0^{N-1}$ the set of all cylinders of length N that are included in $[a]$. Then,

$$\begin{aligned} \frac{1}{a_n}\sum_{k=1}^{n}\lambda^{-k}Z_k(\phi,a) &= \frac{1}{a_n}\sum_{k=1}^{n}\lambda^{-k}\sum_{[\underline{a}]\subseteq[a]\cap\alpha_0^{N-1}} Z_k(\phi,\underline{a}) \\ &= \frac{1}{a_n}\sum_{k=1}^{n}\lambda^{-k}\sum_{\substack{[\underline{a}]\subseteq[a]\cap\alpha_0^{N-1}\\ [\underline{a}]\cap F\neq\phi}} Z_k(\phi,\underline{a}) \\ &\qquad + \frac{1}{a_n}\sum_{k=1}^{n}\lambda^{-k}\sum_{\substack{[\underline{a}]\subseteq[a]\cap\alpha_0^{N-1}\\ [\underline{a}]\subseteq[a]\setminus F}} Z_k(\phi,\underline{a}) \end{aligned}$$

Using (16), (17) and the pointwise dual ergodicity of m we have that for almost every $z_a \in [a]$

$$\begin{aligned} &\frac{1}{a_n}\sum_{k=1}^{n}\lambda^{-k}\sum_{\substack{[\underline{a}]\subseteq[a]\cap\alpha_0^{N-1}\\ [\underline{a}]\subseteq[a]\setminus F}} Z_k(\phi,\underline{a}) \\ &\quad\le B_N^2\frac{1}{a_n}\sum_{k=1}^{n}\lambda^{-k}\sum_{\substack{[\underline{a}]\subseteq[a]\cap\alpha_0^{N-1}\\ [\underline{a}]\subseteq[a]\setminus F}}\left[h^{-1}L_\phi^k\left(h1_{[\underline{a}]}\right)\right]\left(x_{\underline{a}}\right) \\ &\quad\le B_N^2 B_1\frac{1}{a_n}\sum_{k=1}^{n}\left[\lambda^{-k}h^{-1}L_\phi^k\left(h1_{[a]\setminus F}\right)\right](z_a) \\ &\quad\le B_N^2 B_1\frac{1}{a_n}\sum_{k=1}^{n}\left(\widehat{T}^k 1_{[a]\setminus F}\right)(z_a) \underset{n\to\infty}{\longrightarrow} B_N^2 B_1 m([a]\setminus F) \end{aligned}$$

Thus,

$$\frac{1}{a_n}\sum_{k=1}^{n}\lambda^{-k}Z_k(\phi,a) = \sum_{\substack{[\underline{a}]\subseteq[a]\cap\alpha_0^{N-1}\\ [\underline{a}]\cap F\neq\phi}}\left[\frac{1}{a_n}\sum_{k=1}^{n}\lambda^{-k}Z_k(\phi,\underline{a})\right] + O(\varepsilon)$$

The sum on the right is finite, because F is compact. It follows from this and (18) that

$$\underline{\lim}_{n\to\infty}, \overline{\lim}_{n\to\infty} \left[\frac{1}{a_n}\sum_{k=1}^{n}\lambda^{-k}Z_k(\phi,a)\right] = B_N^{\pm 2} m \left(\bigcup_{\substack{[\underline{a}]\subseteq[a]\cap\alpha_0^{N-1}\\ [\underline{a}]\cap F_\varepsilon\neq\phi}}[\underline{a}]\right) + O(\varepsilon)$$

Letting ε tend to zero, and then N tend to infinity we have that the upper and lower limits coincide and are equal to $m[a]$. ∎

3.5 Positive recurrence and Null recurrence

Throughout this subsection we assume that X is topologically mixing, ϕ is locally Hölder continuous and recurrent and that λ, ν and h are its corresponding eigenvalue, eigenmeasure and eigenfunction respectfully. As usual, $dm = h\,d\nu$ and $\hat{T}f = \lambda^{-1}h^{-1}L_\phi(hf)$ is its transfer operator.

Proposition 5 *Under the above assumptions, $\nu(h) < \infty$ iff ϕ is positive recurrent, and $\nu(h) = \infty$ iff ϕ is null recurrent.*

Proof. Fix $a \in S$ and let $\tau_1(x)$ be given by (9). By conservativity, τ_1 is well defined and finite $\nu-$almost everywhere in $[a]$. Set $\psi_N = 1_{[\tau_1=N]}$. By (16) $\forall N\,\forall k > N$

$$\left(\hat{T}^k\psi_N\right)1_{[a]} = B_1^{\pm 2}\lambda^{-N}Z_N^*(\phi,a)\left(\hat{T}^{k-N}1_{[a]}\right)1_{[a]}$$

Taking limits in both sides, using pointwise dual ergodicity, we see that

$$\lambda^{-N}Z_N^*(\phi,a) = B_1^{\pm 2}m[\tau_1 = N]/m[a].$$

It follows that

$$\sum_{n=1}^{\infty} n\lambda^{-n}Z_n^*(\phi,a) = B_1^{\pm 2}\frac{1}{m[a]}\int_{[a]}\tau_1\,dm.$$

The result follows from the ergodicity and conservativity of m and the Kac formula $\int_{[a]}\tau_1\,dm = m(X)$. ∎

Proposition 6 *Under the above assumptions, for every cylinder* $[\underline{a}]$

1. *if* ϕ *is null recurrent then* $\lambda^{-n}L_\phi^n 1_{[\underline{a}]} \underset{n\to\infty}{\longrightarrow} 0$ *uniformly on cylinders whence* $a_n = o(n)$*;*

2. *if* ϕ *is positive recurrent then* $\lambda^{-n}\left(L_\phi^n 1_{[\underline{a}]}\right)(x) \underset{n\to\infty}{\longrightarrow} \frac{h(x)}{\nu(h)}\nu[\underline{a}]$ *uniformly on compacts whence* $a_n \sim n\cdot const.$

Proof. Assume that ϕ is null recurrent and fix some $a\in S$. Since L_ϕ is positive and h is uniformly bounded away from zero and infinity on $[a]$, it is enough to show that $\lambda^{-n}h^{-1}L_\phi^n\left(h1_{[a]}\right) \underset{n\to\infty}{\longrightarrow} 0$ uniformly on cylinders. Choose unions of partition sets F_n such that $F_n \nearrow X$ and $0 < m(F_n) < \infty$. ϕ is null recurrent so $m(F_N) \nearrow \infty$. Set $f_N = 1_{[a]} - 1_{F_N}\cdot m[a]/m(F_N)$. For every $b\in S$ the usual estimations yield (for $\|\cdot\|_1 = \|\cdot\|_{L^1(m)}$)

$$\begin{aligned}
\left\|1_{[b]}\hat{T}^n 1_{[a]}\right\|_\infty &\le B_1^3\frac{1}{m[b]}\left\|1_{[b]}\hat{T}^n 1_{[a]}\right\|_1 \\
&\le \frac{B_1^3}{m[b]}\left(\left\|1_{[b]}\hat{T}^n f_N\right\|_1 + \frac{m[a]}{m(F_N)}\left\|1_{[b]}\hat{T}^n 1_{F_N}\right\|_1\right) \\
&\le \frac{B_1^3}{m[b]}\left(\left\|\hat{T}^n f_N\right\|_1 + \frac{m[a]\,m[b]}{m(F_N)}\right)
\end{aligned}$$

$\hat{T}$ is the transfer operator of m. Since $m(f_N) = 0$ and m is exact (it is equivalent to ν, and ν has the Schweiger property), it follows from a theorem of M. Lin (see theorem 1.3.3 in [2]) that $\left\|\hat{T}^n f_N\right\|_{L^1(m)} \to 0$. It follows from this and from the fact that $m(F_N)\uparrow\infty$ that $\left\|1_{[b]}\hat{T}^n 1_{[a]}\right\|_\infty \underset{n\to\infty}{\longrightarrow} 0$ as required.

Assume now that ϕ is positive recurrent. Without loss of generality, assume that $\nu(h) = 1$. For every cylinder $[\underline{a}]$ the family $\left\{\lambda^{-n}L_\phi^n 1_{[\underline{a}]}\right\}_n$ is equicontinuous and uniformly bounded on partition sets $[b]$ (by $C\left\|h1_{[b]}\right\|_\infty$ where $C = 1/\inf\{h(x) : x\in[\underline{a}]\}$). It follows that every subsequence has a subsequence of its own which converges uniformly on compacts. It is enough to show that the only possible limit point is $h\nu[\underline{a}]$, because it will then immediately follow from the equicontinuity of $\left\{\lambda^{-n} L_\phi^n 1_{[\underline{a}]}\right\}_n$ that this sequence tends uniformly on compacts to $h\nu\,[\underline{a}]$.

Assume that $\lambda^{-n_k} L_\phi^{n_k} 1_{[\underline{a}]}$ tends to φ pointwise. Since for every k $\lambda^{-n_k} L_\phi^{n_k} 1_{[\underline{a}]} \leq Ch$ and Ch is integrable, we have by the dominated convergence theorem that

$$\begin{aligned} \int \left|\varphi - h\nu\,[\underline{a}]\right| d\nu &= \lim_{k\to\infty} \int \left|\lambda^{-n_k} L_\phi^{n_k} 1_{[\underline{a}]} - h\nu\,[\underline{a}]\right| d\nu \\ &= \lim_{k\to\infty} \int \left|\hat{T}^{n_k}\left(h^{-1} 1_{[\underline{a}]} - \nu\,[\underline{a}]\right)\right| dm \end{aligned}$$

Since m is exact, the last limit is equal to zero and we have that $\varphi = h\nu\,[\underline{a}]$ almost everywhere. Since φ must be continuous, it must be equal to $h\nu\,[\underline{a}]$ everywhere. (Note that this argument does not work if ϕ is null recurrent, because in this case $h^{-1}1_{[\underline{a}]} - \nu\,[\underline{a}]$ is not integrable.) ∎

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