Thermodynamic Formalism for Countable Markov Shifts

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Abstract

We establish a generalized thermodynamic formalism for topological Markov shifts with a countable number of states. We offer a definition of topological pressure and show that it satisfies a variational principle for the metric entropies. The pressure of $\phi = 0$ is the Gurevic entropy (see [7]). This pressure may be finite even if the topological entropy is infinite. Let L_ϕ denote the Ruelle operator for ϕ. We offer a definition of positive recurrence for ϕ and show that it is a necessary and sufficient condition for a Ruelle-Perron-Frobenius theorem to hold: there exist an σ-finite measure ν, a continuous function $h > 0$ and $\lambda > 0$ such that $L_\phi^* \nu = \lambda \nu$, $L_\phi h = \lambda h$ and $\lambda^{-n} L_\phi^n f \to h \int f d\nu$ for suitable functions f. We show that under certain conditions this convergence is uniform and exponential. We prove a decomposition theorem for positive recurrent functions and construct conformal measures and equilibrium measures. We give complete characterization of the situation when the equilibrium measure is a Gibbs measure. We end by giving examples where positive recurrence can be verified. These include functions of the form $\phi = \log f\left(\frac{1}{x_0 + \frac{1}{x_1 + \cdots}}\right)$ where f is a suitable function on a suitable shift X.

1 Introduction

The theory of thermodynamic formalism was developed by Ruelle [16] and Bowen [4] for topological Markov shifts with a finite number of states. The main result of this theory is the identification of the topological pressure of a Hölder continuous function ϕ as the supremum of $h_\mu(T) + \mu(\phi)$ where μ ranges over the set of invariant probabilities and the construction of a (unique) measure where this supremum is attained. This measure, being the measure that minimizes the 'free energy', represents the 'equilibrium state' of the system. It is also shown that just like in physics, it is given by a 'Gibbs distribution'. The main tool in the construction is the Ruelle operator (Ruelle [15])

$$L_\phi f = \sum_{Ty=x} e^{\phi(y)} f(y) . \tag{1}$$

In fact the equilibrium state is given by $h\, d\nu$ where $L_\phi h = \lambda h$, $L_\phi^* \nu = \lambda \nu$ and $\log \lambda$ is the pressure of ϕ. The fact that h and ν exist for topologically mixing Markov shifts with a finite number of states and Hölder continuous functions was proved by Ruelle [15] and is frequently referred to as the Ruelle - Perron - Frobenius (RPF) theorem. The assumption that the number of states is finite is crucial: if we allow a countable number of states, the RPF theorem may be false even for the function $\phi = 0$ (Gurevic [7]).

The case $\phi = 0$ was studied for countable Markov shifts by Gurevic [7], [8] using ideas from renewal theory (see [12], [18]). Gurevic showed that the Perron value of the transition matrix is equal to the supremum of the metric entropies, and may thus be called the topological entropy of the shift. Using a sophisticated version of the Perron-Frobenius theorem due to Vere-Jones [20], [21], he showed that a maximal measure exists iff the transition matrix is R-positive and R-recurrent (see Vere-Jones [20], [21] for an explanation of these notions). This paper adopts the approach of Gurevic and Vere-Jones and treats the case $\phi \not\equiv 0$.

In section 3 we generalize Gurevic's definition of topological entropy and define the topological pressure of a locally Hölder continuous function (see below) on a topologically mixing countable Markov shift. Various authors had considered in the past the problem of defining topological entropy (Bowen [5]) and topological pressure (Pesin and Pitskel [14]) for non compact metric spaces. These definitions adopt the point of view of *dimension theory* and define the topological entropy or pressure as some kind of dimension (weighted dimension in the case of pressure) of the metric space in question. Different metrics imposed on a non compact space may have different dimensions even if they induce the same Borel structure (see Salama [17] or Kitchens [12] for examples). Thus, these definitions do not necessarily satisfy the variational principle (the supremum of metric entropies or metric pressures depends only on the *Borel* structure and is not sensitive to the *metric* structure of the underlying space). Indeed, unless a suitable metric is chosen (e.g. a compact metric when such exists) one can only expect to get an inequality $P_{top}(\phi) \geq \sup\{h_\mu + \mu(\phi) : \mu$ invariant Borel probability$\}$ (see [5], [14]). Our definition does satisfy a variational principle (theorem 3), and is therefore more suitable to the context of the thermodynamic formalism.

In section 4 we generalize the notions of R-positivity and R-recurrence for matrices due to Vere-Jones (see [20], [21], [18], [12]) and define a condition which we call *positive recurrence* (definition 2). We then show that this condition is necessary and sufficient for a RPF theorem for countable Markov shifts to hold (theorem 4). The main difficulties in the proof are (a) the shift space is not compact so it is no longer true that every sequence of probability measures has a w^* convergent subsequence (which is crucial for the works of Ruelle [15] and Bowen [4]); (b) it is also not possible, in general, to embed the shift space inside a larger compact metric space without losing the continuity properties of ϕ (which is important in Gurevic [7] and Walters [24]); (c) we do not assume that ϕ depends only on a finite number of coordinates, and thus cannot use direct renewal theoretic arguments (like Gurevic [7],[8] and Vere-Jones [20],[21]).The main tool in the proof is a tightness argument (lemma 8).

The RPF theorem of section 4 can be used to construct an invariant probability measure closely connected to ϕ. In sections 5 to 8 we study the properties of this measure: in section 5 we discuss the rate of decay of correlations, in sections 6 and 7 we describe this measure as an equilibrium measure in the sense of Walters [24], and in section 8 we describe it as a Gibbs measure in the sense of Bowen [4]. Section 9 contains examples.

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2 Preliminaries

Let S be a countable set and $\mathbf{A} = (t_{ij})_{S\times S}$ a matrix of zeroes and ones. S is called the set of *states*. $\mathbf{A}$ is called a *topological transition matrix* if $\forall a \in S\ \exists i,j\ \ (t_{ai} = t_{ja} = 1)$. If this is the case then one defines the (one sided) *countable Markov shift* generated by $\mathbf{A}$ to be

$$X = \Sigma_{\mathbf{A}}^{+} = \left\{ x \in S^{\mathbf{N}\cup\{0\}} : \forall i\ t_{x_i x_{i+1}} = 1 \right\}.$$

If S is finite we call X a *finite Markov shift*. The *cylinder sets* are the subsets of X given by $[a_0, \ldots, a_{n-1}] = \{x \in X : x_i = a_i\}$. We denote the natural partition $\{[a] : a \in S\}$ by α. The members of α are called *partition sets*. A *word* $\underline{a}$ is a finite concatenation of symbols from S. A word is called $\mathbf{A}$-*admissible* or simply *admissible* if the cylinder $[\underline{a}]$ is non empty.

Let $T : X \to X$ be given by $T(x_0, x_1, x_2, \ldots) = (x_1, x_2, x_3, \ldots)$. X is called *topologically mixing* if

$$\forall a, b \in S\ \exists N_{ab} \forall n > N_{ab}\ \left([a] \cap T^{-n}[b] \neq \emptyset \right).$$

Set $t(x,y) = \inf(\{k : x_k \neq y_k\} \cup \{\infty\})$. Fix $r \in (0,1)$ and set $d(x,y) = d_r(x,y) = r^{t(x,y)}$. This is a metric on X and the topology it generates has the cylinder sets as a base. T is continuous with respect to this topology. If S is finite, this is a compact topology. If S is infinite, it may not even be locally compact. Let $\mathcal{B}$ be the corresponding Borel $\sigma-$algebra. $\mathcal{B}$ is generated by the cylinders. Henceforth, unless mentioned otherwise, we always refer to this topology and Borel structure.

For every function $\phi : X \to \mathbf{R}$ set

$$V_n(\phi) = \sup\{|\phi(x) - \phi(y)| : x_i = y_i \;\; i = 0, \ldots, n-1\}.$$

Throughout this paper we work with a fixed $r \in (0,1)$. A function ϕ is called *locally Hölder continuous with parameter* r, or simply *locally Hölder continuous*, if it is Lipschitz continuous on partition sets with respect to d_r, i.e.

$$\exists A \;\; \forall n \geq 1 \; V_n(\phi) \leq Ar^n.$$

Note that nothing is required from $V_0(f)$: if S is infinite then ϕ may be both locally Hölder continuous and unbounded.

Let L_ϕ be given by (1). For every word $\underline{a}$ and $x \in X$ define the concatenation $\underline{a}x$ in the obvious way. Let $P^n(x)$ be the set of words of length n such that this concatenation is admissible -

$$P^n(x) = \{\underline{a} = (a_0, \ldots, a_{n-1}) : \underline{a}x \in X\}.$$

Set $\phi_n = \sum_{k=0}^{n-1} \phi \circ T^k$. The iterations of L_ϕ are given by $\left(L_\phi^n f\right)(x) = \sum_{T^n y = x} e^{\phi_n(y)} f(y) = \sum_{\underline{a} \in P^n(x)} e^{\phi_n(\underline{a}x)} f(\underline{a}x)$.

Notational Convention: We use the following notation to handle double inequalities. If $C \geq 0$ then

$$a = b \pm C \Leftrightarrow b - C \leq a \leq b + C$$

and if $C \geq 1$, $a, b > 0$ then

$$a = C^{\pm 1} b \Leftrightarrow C^{-1} b \leq a \leq Cb.$$

We also write $C^{\pm n}$ for $(C^n)^{\pm 1}$ and $C_1^{\pm 1} C_2^{\pm 1}$ for $(C_1 C_2)^{\pm 1}$.

For every pair of cylinders $[a_0, \ldots, a_n], [b_0, \ldots, b_m]$ where $b_0 = a_n$ set

$$[a_0, \ldots, a_n] \cdot [b_0, \ldots, b_m] = [a_0, \ldots, a_{n-1}; b_0, \ldots, b_m].$$

Note that if $[\underline{a}]$,$[\underline{b}]$ are non empty, then so is $[\underline{a}] \cdot [\underline{b}]$. For every set E, let $E \cap \alpha_0^{k-1}$ denote the set of all cylinders of length k that are included in E. If $A \subseteq T^{-n}[a] \cap \alpha_0^n$ and $B \subseteq [a] \cap \alpha_0^m$, set

$$A \cdot B = \bigcup_{[\underline{a}] \subseteq A \,;\, [\underline{b}] \subseteq B} [\underline{a}] \cdot [\underline{b}]$$

The following lemma collects some useful consequences of local Hölder continuity and will be used frequently throughout this paper.

Lemma 1 *Let ϕ be locally Hölder continuous. Then for $B_n = B_n(\phi) = \exp \sum_{k>n} V_k(\phi)$.*

1. $\forall n \leq m \quad V_m(\phi_n) \leq \log B_{m-n}$.
2. *set* $\phi_n[\underline{a}] = \sup\{\phi_n(x) : x \in [\underline{a}]\}$. *Then*
$$\forall n \leq m \quad \phi_n(x) = \phi_n[x_0, \ldots, x_{m-1}] \pm \log B_{m-n}$$
3. *let* $A \subseteq T^{-n}[a] \cap \alpha_0^n$ *and* $B \subseteq [a] \cap \alpha_0^m$. *Then*
$$\left(\sum_{[\underline{a}] \in A \cap \alpha_0^n} e^{\phi_n[\underline{a}]}\right)\left(\sum_{[\underline{b}] \in B \cap \alpha_0^m} e^{\phi_m[\underline{b}]}\right) = B_1^{\pm 2} \sum_{[\underline{c}] \in (A \cdot B) \cap \alpha_0^{n+m}} e^{\phi_{n+m}[\underline{c}]}.$$

3 The Gurevic Pressure

Throughout this section X is a fixed topologically mixing countable Markov shift and $\phi : X \to \mathbf{R}$ a locally Hölder continuous function. Consider the following *partition functions* defined for a fixed $a \in S$

$$Z_n(\phi, a) = \sum_{\substack{T^n x = x \\ x_0 = a}} e^{\phi_n(x)}.$$

Lemma 2 *Let X be topologically mixing and ϕ locally Hölder continuous.*

1. $\exists \Lambda_1 > 0$ *and* $N \in \mathbf{N}$ *such that* $\forall n > N \;\; Z_n(\phi, a) \geq \Lambda_1^n$.
2. *If* $\|L_\phi 1\|_\infty < \infty$ *then* $\exists \Lambda_2$ *such that* $\forall n \;\; Z_n(\phi, a) \leq \Lambda_2^n$.

Proof. Let $Y \subseteq X$ be a topologically mixing finite topological Markov shift such that $Y \cap [a] \neq \emptyset$. Set
$$Z_n(Y, \phi, a) = \sum_{\substack{T^n x = x, x \in [a] \\ x \in Y}} e^{\phi_n(x)}. \tag{2}$$
By lemma 1,
$$\forall x \in Y \cap [a] \quad Z_n(Y, \phi, a) = B_1^{\pm 1}\left(L_{\phi|_Y}^n 1_{[a]}\right)(x). \tag{3}$$
It follows from the Ruelle - Perron - Frobenius theorem for finite topological Markov shifts (see Bowen [4]) that $Z_n(Y, \phi, a)$ is exponentially bounded from below, whence so is $Z_n(\phi, a)$. The second statement follows from the estimation

$$Z_n(\phi, a) \leq B_1\left(L_\phi^n 1\right)(x) \leq B_1 \|L_\phi 1\|_\infty^n.$$

which holds for every $x \in [a]$. $\square$

Lemma 3 *Assume that X is a topologically mixing countable Markov shift, and that ϕ is locally Hölder continuous. Then*

$$\forall a, b\, \exists C, c \;\; Z_n(\phi, a) < C\, Z_{n+c}(\phi, b)$$

$$\forall a\, \forall m, n \;\; Z_n(\phi, a)\, Z_m(\phi, a) < B_1^3\, Z_{n+m}(\phi, a)$$

Proof. Both statements follows from the irreducibility of X and lemma 1. □

Theorem 1 *Assume that X is topologically mixing and that ϕ is locally Hölder continuous. Then the limit $\lim\limits_{n\to\infty} \frac{1}{n} \log Z_n(\phi, a)$ exists and is independent of a. This limit is never $-\infty$. If $\|L_\phi 1\|_\infty < \infty$, it is not $+\infty$.*

Proof. Fix a and set $a_n = \log Z_n(\phi, a)$. By lemma 3 part 2, $\{a_n\}_{n\geq 1}$ is almost subadditive, in the sense that for some constant c (in our case $c = 3\log B_1$)

$$a_n + a_m < a_{n+m} + c.$$

It follows that the series $\left\{\frac{a_n}{n}\right\}_n$ converges to a limit. By lemma 3 part 1 this limit is independent of a. Lemma 2 implies that it is always larger than $-\infty$ and that if $\|L_\phi 1\|_\infty < \infty$ then it is finite. □

The following definition, therefore, makes sense:

Definition 1 *Let X is topologically mixing and ϕ be locally Hölder continuous. The Gurevic Pressure of ϕ is the number*

$$P_G(\phi) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(\phi, a).$$

The Gurevic pressure is a generalization the Gurevic entropy $h_G(T)$ (Gurevic [7], [8]):

Proposition 1 *Let $X = \sum_{\mathbf{A}}^{+}$ be a topologically mixing countable Markov shift. Then $P_G(0) = h_G(T)$.*

Proof. The Gurevic entropy is equal to $\log\lambda$ where λ is the Perron value of the transition matrix $\mathbf{A}$ (see Kitchens [12]). Denote the powers of $\mathbf{A}$ by $\mathbf{A}^n = \left((\mathbf{A}^n)_{ij}\right)$. By the definition of the Perron value, $\log\lambda = \lim\limits_{n\to\infty} \frac{1}{n} \log(\mathbf{A}^n)_{aa} = \lim\limits_{n\to\infty} \frac{1}{n} \log Z_n(0, a) = P_G(0)$. □

The following theorem extends a property of the Gurevic entropy (Gurevic [7]).

Theorem 2 *Let X be a topologically mixing, and ϕ locally Hölder continuous. Then*

$$P_G(\phi) = \sup\left\{P_{top}\left(\phi|_Y\right) : Y \subseteq X \;\; \textit{top. mixing finite top. Markov shift}\right\}$$

where $P_{top}\left(\phi|_Y\right)$ is the topological pressure of the restriction of ϕ to the compact metric space Y.

Proof. Let $Y \subseteq X$ be any topologically mixing finite topological Markov shift and $Z_n(Y,\phi,a)$ be given by (2). By (3),

$$P_{top}\left(\phi|_Y\right) = \lim_{n\to\infty}\frac{1}{n}\log Z_n\left(Y,\phi,a\right) \le P_G\left(\phi\right).$$

whence $P_G(\phi) \ge \sup\left\{P_{top}\left(\phi|_Y\right) : Y \subseteq X \text{ top. mixing finite top. Markov shift}\right\}$.

We show the reverse inequality. We prove this under the assumption that $P_G(\phi) < \infty$ (the case $P_G(\phi) = \infty$ is similar and is left to the reader). Fix $\varepsilon > 0$ and let $m > \frac{3\log B_1}{\varepsilon}$ be large enough so that

$$P_G\left(\phi\right) < \frac{1}{m}\log Z_m\left(\phi,a\right) + \varepsilon. \tag{4}$$

Choose an M big enough such that $\frac{1}{m}\log Z_m(\phi,a) \le \frac{1}{m}\log Z_m\left(\{1,\ldots,M\}^{\mathbf{N}} \cap X,\phi,a\right) + \varepsilon$. Adding a finite number of states to $\{1,\ldots,M\}$, one can construct a topologically mixing finite topological Markov shift $Y \subseteq X$ such that

$$\frac{1}{m}\log Z_m\left(\phi,a\right) < \frac{1}{m}\log Z_m\left(Y,\phi,a\right) + \varepsilon. \tag{5}$$

Set $a_n = \log Z_n(Y,\phi,a)$. Just like in lemma 3, $a_n + a_m \le a_{n+m} + 3\log B_1$, whence for $n = km + r \quad (r = 0,\ldots,k-1)$

$$\frac{ka_m + a_r}{km + r} \le \frac{a_{km+r} + 3(k+1)\log B_1}{km+r} \le \frac{a_n}{n} + \frac{k+1}{k}\varepsilon.$$

Setting $n \to \infty$, we have

$$\frac{1}{m}\log Z_m\left(Y,\phi,a\right) \le P_{top}\left(\phi|_Y\right) + \varepsilon. \tag{6}$$

By (4),(5),(6) , $P_G(\phi) \le P_{top}\left(\phi|_Y\right) + 3\varepsilon$. □

Corollary 1 *Let X be topologically mixing countable Markov shift and ϕ locally Hölder continuous. Then*

$$P_G\left(\phi\right) = \sup\left\{P_{top}\left(\phi|_K\right) : K \subseteq X \textit{ compact ; } T^{-1}K = K\right\}.$$

Denote by $\mathcal{M}_T(X)$ the set of all invariant Borel probability measures. The variational principle for finite topological Markov shifts states that the topological pressure of a continuous function ϕ is equal to $\sup\left\{h_\mu(T) + \int \phi \, d\mu : \mu \in \mathcal{M}_T(X)\right\}$. Such a theorem was proved for countable Markov shifts by Gurevic [7] for the case $\phi = 0$ (in other words, the Gurevic entropy is equal to the supremum of metric entropies). The following theorem extends this result. We consider only probabilities μ such that $\mu(-\phi) < \infty$ to avoid the situation $h_\mu(T) = \infty, \mu(\phi) = -\infty$.

Theorem 3 *Let X be a topologically mixing countable Markov shift and ϕ locally Hölder continuous such that $\|L_\phi 1\|_\infty < \infty$. Then*

$$P_G(\phi) = \sup\left\{ h_\mu(T) + \int \phi \, d\mu \right| \left. \mu \in \mathcal{M}_T(X) \ ; \ \mu(-\phi) < \infty \right\} < \infty.$$

Proof. The inequality $P_G(\phi) \leq \sup\{h_\mu(T) + \mu(\phi)\}$ follows from corollary 1 and the fact that each $T|_K$ has an equilibrium measure with respect to $\phi|_K$ (see Ruelle [16], Bowen [4]). We prove the reverse inequality.

Fix $\mu \in \mathcal{M}_T(X)$. Set $[\geq m] = \{x : x_0 \geq m\}$, $\alpha_m = \{[1], \ldots, [m-1], [\geq m]\}$ and $\mathcal{B}_m = \sigma(\alpha_m)$. Note that $\mathcal{B}_m \nearrow \bigcup_m \mathcal{B}_m \subseteq \sigma(\cup_m \mathcal{B}_m) = \mathcal{B}$ whence

$$h_\mu(T, \alpha_m) + \int \phi \, d\mu \underset{m\to\infty}{\longrightarrow} h_\mu(T) + \int \phi \, d\mu.$$

Fix m and set $\beta = \alpha_m$, $\mathcal{B}' = \mathcal{B}_m$. We use the cylinder notation for members of β_0^n:

$$\forall a_i \in \beta \quad [a_0, \ldots, a_n] = \bigcap_{k=0}^{n} T^{-k} a_i.$$

By the invariance of μ and lemma 1, for every n

$$\frac{1}{n} H_\mu(\beta_0^n) + \mu(\phi) = \frac{1}{n}\left(H_\mu(\beta_0^n) + \mu(\phi_n)\right)$$

$$\leq \frac{1}{n} \sum_{a,b\in\beta} \mu(a \cap T^{-n} b) \sum_{\substack{[\underline{a}] \subseteq a\cap T^{-n}b \\ [\underline{a}]\in\beta_0^n}} \mu([\underline{a}] | \, a \cap T^{-n} b) \log \frac{e^{\phi_n[\underline{a}]}}{\mu[\underline{a}]} + \frac{\log B_1}{n}$$

$$\leq \frac{1}{n} \sum_{a,b\in\beta} \mu(a \cap T^{-n} b) \log \left(\sum_{\substack{[\underline{a}] \subseteq a\cap T^{-n}b \\ [\underline{a}]\in\beta_0^n}} e^{\phi_n[\underline{a}]} \right) + \frac{1}{n} H_\mu(\beta \vee T^{-n}\beta) + \frac{\log B_1}{n}.$$

Where the last inequality follows from the Jensen inequality for sums. Set

$$P_n(a,b) = \frac{1}{n} \log \sum_{\substack{[\underline{a}] \subseteq a\cap T^{-n}b \\ [\underline{a}]\in\beta_0^n}} e^{\phi_n[\underline{a}]}.$$

Then, since $H_\mu(\beta \vee T^{-n}\beta) \leq 2H_\mu(\beta) < \infty$ (β is finite)

$$\frac{1}{n} H_\mu(\beta_0^n) + \mu(\phi) \leq \sum_{a,b\in\beta} \mu\left(a \cap T^{-n} b\right) P_n(a,b) + o(1) \quad (n \to \infty) \tag{7}$$

Lemma 4 *Under the assumptions of the theorem* $\sup\phi < \infty$, *and*

1. if $a, b \neq [\geq m]$ then $\overline{\lim_{n\to\infty}} P_n(a,b) \leq P_G(\phi)$;
2. if $a = [\geq m]$ or $b = [\geq m]$ then $\overline{\lim_{n\to\infty}} P_n(a,b) \leq \sup\phi + \log \|L_\phi 1\|_\infty$.

Before we prove this lemma, we use it to finish the proof of the theorem. Fix $\varepsilon > 0$. According to the lemma, $\forall a, b \in \beta\ \exists N_{ab}$ such that $\forall n > N_{ab}$

$$a, b \neq [\geq m] \quad \Rightarrow P_n(a,b) \leq P_G(\phi) + \varepsilon$$

$$a = [\geq m] \text{ or } b = [\geq m] \quad \Rightarrow P_n(a,b) \leq \sup\phi + \log\|L_\phi 1\|_\infty + \varepsilon.$$

Set $C = \sup\phi + \log\|L_\phi 1\|_\infty$ and $N = \max\{N_{ab} : a, b \in \beta\}$ (N is finite because β is finite). Then $\forall n > N$,

$$\begin{aligned}\sum_{a,b\in\beta} \mu\left(a \cap T^{-n}b\right) P_n(a,b) &\leq \mu\left([\geq m]^c \cap T^{-n}[\geq m]^c\right)\left(P_G(\phi)+\varepsilon\right) \\ &\quad + \mu\left([\geq m] \cup T^{-n}[\geq m]\right)(C+\varepsilon) \\ &\leq \left(1 - \mu[\geq m]\right) P_G(\phi) + 2C\mu[\geq m] + \varepsilon\end{aligned}$$

whence by (7) (recall that $\beta = \alpha_m$)

$$h_\mu(T, \alpha_m) + \int \phi\, d\mu \leq \left(1 - \mu[\geq m]\right) P_G(\phi) + 2C\mu[\geq m] + \varepsilon.$$

Thus,

$$h_\mu(T) + \int \phi\, d\mu = \lim_{m\to\infty}\left(h_\mu(T,\alpha_m) + \int \phi\, d\mu\right) \leq P_G(\phi) + \varepsilon$$

as required. It is therefore enough to prove the lemma.

Proof of the lemma. It is easy to see that $\|L_\phi 1\| < \infty$ implies that $\sup\phi < \infty$ (note, however, that $\inf\phi$ may be equal to $-\infty$). We begin by assuming that $a, b \neq [\geq m]$. Fix $x_b \in b$. Then $P_n(a,b) = \frac{1}{n}\log B_1^{\pm 1}\left(L_\phi^n 1_{[a]}\right)(x_b) \to P_G(\phi)$. (The convergence to $P_G(\phi)$ follows from the topological irreducibility of X and lemma 1). Next, suppose that $b = [\geq m]$. For every $[\underline{a}] \in \beta_0^n$ choose a point $x_{\underline{a}} \in [\underline{a}]$ such that $\phi_n[\underline{a}] < \phi_n\left(x_{\underline{a}}\right) + \log 2$. Write

$$\left\{[\underline{a}] \in \beta_0^n : [\underline{a}] \subseteq a \cap T^{-n}b\right\} = \bigcup_{k=0}^{n-1} A_k \tag{8}$$

where for $k = 0, \ldots, n-1$ A_k are the pairwise disjoint sets

$$A_k = \left\{[\underline{a}] \in \beta_0^n : \underline{a} = (a, a_1, \ldots, a_k, b, \ldots, b) \text{ where } a_i \in \beta \text{ and } a_k \neq b\right\}$$

Set

$$S_k = \sum_{\underline{a} \in A_k} e^{\phi_n\left(x_{\underline{a}}\right)}$$

and let $k_n \leq n$ be such that S_{k_n} is maximal. Note that

$$S_{k_n} = \sum_{\underline{a} \in A_{k_n}} e^{\phi_{k_n}\left(x_{\underline{a}}\right)+\phi_{n-k_n}\left(T^{k_n} x_{\underline{a}}\right)} \leq e^{(n-k_n)\sup\phi} \sum_{\underline{a} \in A_{k_n}} e^{\phi_{k_n}\left(x_{\underline{a}}\right)}$$

Choose arbitrary points $x_i \in [i]$. Since the k_n-th coordinate of each $x_{\underline{a}}$ belongs to $\{1, \ldots, m-1\}$ we have that

$$S_{k_n} \leq B_1 e^{(n-k_n)\sup\phi} \sum_{i=1}^{m-1} \left(L_\phi^{k_n} 1_{[a]}\right)(x_i) \leq B_1 (m-1)\ e^{(n-k_n)\sup\phi} \left\|L_\phi 1\right\|_\infty^{k_n}.$$

By (8) and the definition of S_{k_n} (recall that $\phi_n[\underline{a}] < \phi_n\left(x_{\underline{a}}\right) + \log 2$)

$$\begin{aligned}
P_n(a,b) &\leq \frac{1}{n} \log \sum_{k=0}^{n-1} 2S_k \\
&\leq \frac{1}{n} \log\left(2nS_{k_n}\right) \\
&\leq \sup\phi + \log\left\|L_\phi 1\right\|_\infty + o(1) \qquad (n \to \infty)
\end{aligned}$$

Thus, if $b = [\geq m]$ then $\overline{\lim_{n\to\infty}} P_n(a,b) \leq \sup\phi + \log\left\|L_\phi 1\right\|_\infty$.

It remains to treat the case when $a = [\geq m]$ and $b \neq [\geq m]$. Choose $x_b \in b$. Then

$$P_n(a,b) \leq \frac{1}{n} \log B_1 \left(L_\phi^n 1_a\right)(x_b) \leq \log\left\|L_\phi 1\right\|_\infty + \frac{1}{n} \log B_1$$

and again $\overline{\lim_{n\to\infty}} P_n(a,b) < \sup\phi + \log\left\|L_\phi 1\right\|_\infty$. □

4 Positive Recurrence and the Ruelle Perron Frobenius Theorem

Throughout this section X is a fixed topologically mixing countable Markov shift and $\phi : X \to \mathbf{R}$ a locally Hölder continuous function.

Definition 2 *Let X be topologically mixing and ϕ a locally Hölder continuous function. ϕ is positive recurrent if for some (hence all) $a \in S$ there is a constant M_a and an integer N_a such that $\forall n > N_a$ $Z_n(\phi, a)/\lambda^n \in [M_a^{-1}, M_a]$ for some $\lambda > 0$.*

This section is devoted to the proof of the following theorem.

Theorem 4 *Let X be a topologically mixing countable Markov shift, and let ϕ be a locally Hölder continuous function such that $P_G(\phi) < \infty$. Then ϕ is positive recurrent if and only if the following is true: for $\lambda = e^{P_G(\phi)}$ there exist a $\sigma-$finite measure ν and a function $h > 0$ such that $L_\phi^* \nu = \lambda \nu$, $L_\phi h = \lambda h$, $\nu(h) = 1$ and such that for every uniformly continuous function f such that $\|fh^{-1}\|_\infty < \infty$, $\lambda^{-n} L_\phi^n f \underset{n\to\infty}{\longrightarrow} \nu(f)\, h$ uniformly on compacts.*

Remark 1. This is a generalization of a similar theorem of Vere-Jones [20],[21] for positive countable matrices (a close inspection of the proof reveals that it only uses the continuity properties of ϕ on cylinders $[a,b]$, whence the proof also works for matrices). Our proof, however, is essentially different than his. Many works contain sufficient conditions for the existence of ν and h, which are stronger than positive recurrence (they include extra topological assumptions on X), but are more convenient to check. Among these we would particularly like to mention the work of Ruelle [16] on finite topological Markov shifts (where no additional assumptions are needed); the work of Aaronson, Denker and Urbanski [1],[2] on countable Markov shifts satisfying the big images condition (see below); and the works of Walters [24] and Yuri [25],[26] on models of certain transformations admitting Markov partitions. Yuri [27] contains a proof of the convergence of the Ruelle operator for some examples of systems which essentially satisfy the finite images property ($\{T[a] : a \in S\}$ is finite). For such examples h is bounded away from zero and infinity (see proposition 2) and the condition $\|fh^{-1}\|_\infty < \infty$ is equivalent to $\|f\|_\infty < \infty$

Remark 2. It follows from the topological mixing of X and the definition of h and ν that $\log h$ is locally Hölder continuous, and that ν gives finite positive mass to cylinders. For a discussion of when $\log h$ is bounded away from zero and infinity, see proposition 2 below.

Remark 3. It follows from the convergence property that h and ν are uniquely determined up to a multiplicative constant.

Remark 4. ν is a $e^{\phi - P_G(\phi)}$ *conformal measure*:

$$\frac{d\nu}{d\nu \circ T} = e^{\phi - P_G(\phi)}$$

(where $\nu \circ T$ is given by $(\nu \circ T)(A) = \sum_{a \in S} \nu(T([a] \cap A))$). This follows from the fact that $\lambda^{-1} L_\phi$ acts as the transfer operator of ν: $\nu(f\, g \circ T) = \nu((\lambda^{-1} L_\phi f)\, g)-$ see lemma 10 below.

Remark 5. $dm = h\, d\nu$ is an exact invariant measure. Exactness follows from the fact that the transfer operator of m is given by $\hat{T}_m f = \lambda^{-1} h^{-1} L_\phi(hf)$ whence for suitable functions $\hat{T}_m^n f \underset{n\to\infty}{\longrightarrow} m(f)$.

It follows from lemma 1 that for every $x \in [a]$

$$\lambda^{-n} Z_n(\phi, a) = B_1^{\pm 1} \lambda^{-n} \left(L_\phi^n 1_{[a]}\right)(x).$$

The *necessity* of the positive recurrence condition follows from the convergence of $\lambda^{-n}\left(L_\phi^n 1_{[a]}\right)(x)$ to a positive limit. We show the *sufficiency* of the positive recurrence condition. The proof is divided into three steps: the existence of h; the existence of ν; and the convergence of $\lambda^{-n}L_\phi^n$.

The reader should note that the assumption that $P_G(\phi)$ is finite implies that for every $a \in S$ and n $Z_n(\phi, a) < \infty$. It follows from the above estimation and from topological mixing that $L_\phi^n 1_{[a]} < \infty$ for every n. Thus, $L_\phi^n f < \infty$ for every bounded function f which is supported inside a finite union of cylinders.

4.1 Existence of h

Set $\lambda = e^{P_G(\phi)}$. We'll need the following property of positive recurrent functions.

Lemma 5 *Let X be topologically mixing countable Markov shift and ϕ a positive recurrent locally Hölder continuous function. For every continuous function $f \geq 0$ and $x \in X$*

$$\varliminf_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N-1} \lambda^{-k}\left(L_\phi^k f\right)(x) = 0 \Rightarrow f \equiv 0$$

Proof. Assume that $f(y) > 0$. f is continuous, so there is a cylinder $[\underline{a}] = [a_0, \ldots, a_{n-1}]$ such that $f \geq \varepsilon > 0$ in $[\underline{a}]$. Using topological mixing it possible to have $a_{n-1} = y_0$. Choose m large enough such that there exist $\underline{p} \in P^m(x)$ such that $p_0 = y_0$. Then $\lambda^{-k}\left(L_\phi^k f\right)(x) \geq \varepsilon\lambda^{-k}e^{\phi_m(\underline{p}x)}\left(L_\phi^{k-m}1_{[\underline{a}]}\right)\left(\underline{p}x\right)$. By lemma 1,

$$\exists C > 1\ \forall k > n \quad \lambda^{-k}\left(L_\phi^k f\right)(x) > \varepsilon C\lambda^{-(k-(n+m)+1)}Z_{k-(n+m)+1}(\phi, y_0).$$

By the positive recurrence of ϕ, $\varliminf_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N-1} \lambda^{-k}\left(L_\phi^k f\right)(x) > 0$. □

Lemma 6 *Let X be a topologically mixing countable Markov shift and ϕ a locally Hölder continuous function. If ϕ is positive recurrent and $P_G(\phi) < \infty$, then there exists a positive locally Hölder continuous function h such that $L_\phi h = e^{P_G(\phi)}h$.*

Proof. Fix $a \in S$ and set $\lambda = \exp P_G(\phi)$. We show that the sequence $\left\{\lambda^{-n}L_\phi^n 1_{[a]}\right\}_{n=1}^{\infty}$ is uniformly bounded and equicontinuous in partition sets. Fix $b \in S$. It follows from the topological mixing of X and the local Hölder continuity of ϕ that $\exists C_{ab}, c_{ab}$ such that for every $x \in [b]$

$$\forall n \quad \lambda^{-n}\left(L_\phi^n 1_{[a]}\right)(x) < C_{ab}\lambda^{-(n+c_{ab})}Z_{n+c_{ab}}(\phi, a)$$

whence, by positive recurrence, $K_1 = \sup_n \left\|\left(\lambda^{-n}L_\phi^n 1_{[a]}\right)1_{[b]}\right\|_\infty < \infty$.

We show equicontinuity on $[b]$. Fix two points $x, y \in [b]$. Then

$$\begin{aligned}\left|\lambda^{-n}\left(L_\phi^n 1_{[a]}\right)(x)-\lambda^{-n}\left(L_\phi^n 1_{[a]}\right)(y)\right| &\leq \lambda^{-n}\sum_{\underline{p}\in P^n(b)}\left|e^{\phi_n(\underline{p}x)}1_{[a]}\left(\underline{p}x\right)-e^{\phi_n(\underline{p}y)}1_{[a]}\left(\underline{p}y\right)\right| \\ &= \lambda^{-n}\sum_{\substack{\underline{p}\in P^n(b)\\ p_0=a}}\left|e^{\phi_n(\underline{p}x)}-e^{\phi_n(\underline{p}y)}\right| \\ &= \lambda^{-n}\sum_{\substack{\underline{p}\in P^n(b)\\ p_0=a}} e^{\phi_n(\underline{p}y)}\left|e^{\phi_n(\underline{p}x)-\phi_n(\underline{p}y)}-1\right|\end{aligned}$$

By the local Hölder continuity of ϕ, $\exists K_2 \forall n$ $\left|e^{\phi_n(\underline{p}x)-\phi_n(\underline{p}y)}-1\right| < K_2 r^{t(x,y)}$. Thus,

$$\forall n \quad \left|\lambda^{-n}\left(L_\phi^n 1_{[a]}\right)(x)-\lambda^{-n}\left(L_\phi^n 1_{[a]}\right)(y)\right| \leq K_1 K_2 r^{t(x,y)}$$

which proves equicontinuity.

It follows that the family $\left\{\frac{1}{n}\sum_{i=0}^{n-1}\lambda^{-i}L_\phi^i 1_{[a]}\right\}_{n=0}^{\infty}$ is also uniformly bounded and equicontinuous in each partition set. Since X is separable, it follows from the Arzela-Ascoli theorem that there exists a subsequence $n_k \nearrow \infty$ and a locally Hölder continuous function h such that for every x

$$\frac{1}{n_k}\sum_{i=0}^{n_k-1}\lambda^{-i}\left(L_\phi^i 1_{[a]}\right)(x) \underset{k\to\infty}{\longrightarrow} h(x). \tag{9}$$

Using positive recurrence, it is not difficult to show that $h > 0$. It is not difficult to verify that $\log h$ is also locally Hölder continuous (perhaps with a different parameter). An argument based on the Fatou lemma for sums shows that $L_\phi h \leq \lambda h$ (use the fact that $\lambda^{-n_k}\left(L_\phi^{n_k}1_{[a]}\right)(x)$ is uniformly bounded for every x). Set $f = h - \lambda^{-1}L_\phi h$. This is a non negative continuous function. For every $x \in X$

$$0 \leq \frac{1}{N}\sum_{k=0}^{N-1}\lambda^{-k}\left(L_\phi^k f\right)(x) = \frac{1}{N}\left(h(x)-\lambda^{-N}\left(L_\phi^N h\right)(x)\right) \leq \frac{h(x)}{N} \underset{N\to\infty}{\longrightarrow} 0$$

which implies by lemma 5 that $f = 0$. Thus, $L_\phi h = \lambda h$. □

4.2 Existence of ν

Fix $a \in S$ and set

$$\forall n > 0 \;\; Z_n^*(\phi, a) = \sum\left\{e^{\phi_n(x)} : x_0 = a \;;\; T^n x = x \;;\; \forall 1 \leq i \leq n-1 \; x_i \neq a\right\}.$$

Lemma 7 *Let X be topologically mixing and ϕ locally Hölder continuous such that $P_G(\phi) < \infty$. If ϕ is positive recurrent then $\forall a \in S$ $\sum_{n=1}^{\infty}\lambda^{-n}Z_n^*(\phi, a) < \infty$.*

Proof. Fix $N \in \mathbf{N}$. Let N_a and M_a be such that $\forall n \geq N_a$ $Z_n(\phi, a) = M_a^{\pm 1}\lambda^n$. Apply the concatenation part of lemma 1 to see that

$$\sum_{n=1}^{N-N_a} Z_n^*(\phi, a)\, Z_{N-n}(\phi, a) < B_1^3 Z_N(\phi, a).$$

The result follows by setting $Z_{N-n}(\phi, a) = M_a^{\pm 1}\lambda^{N-n}$ and $Z_N(\phi, a) = M_a^{\pm 1}\lambda^N$. □

Denote the atomic probability measure supported on $\{x\}$ by δ_x.

Lemma 8 *Let X be topologically mixing and ϕ locally Hölder continuous such that $P_G(\phi) < \infty$. Fix $z \in X$ and $a \in S$. If ϕ is positive recurrent then the following sequence of measures is tight*

$$\mu_n = \lambda^{-n} \sum_{\substack{T^n x = z \\ x_0 = a}} e^{\phi_n(x)}\delta_x$$

Proof. We assume for simplicity that $S = \mathbf{N}$. We prove this in the case $z_0 = a$ (the modifications for the case $z_0 \neq a$ are standard and are left to the reader). Fix $\varepsilon > 0$. Let $\varphi_{[a]}(x) = \inf\{n > 0 : T^n x \in [a]\}$. If $\varphi_{[a]}(x) < \infty$ let $T_{[a]}x = T^{\varphi_{[a]}(x)}x$. Set

$$\tau_k(x) = \varphi_{[a]}\left(T_{[a]}^{k-1}x\right)$$

(if $\tau_i(x) = \infty$ set $\tau_j(x) = \infty$ $\forall j > i$). For every sequence of natural numbers $\{n_k\}_{k=1}^{\infty}$ set $R(n_1, n_2, \ldots) = \{x \in [a] : \tau_i(x) \leq n_i\}$.We show that $\exists \{n_k\}_{k=1}^{\infty}$ such that $\forall n$ $\mu_n(R(n_1, n_2, \ldots)^c) < \frac{\varepsilon}{2}$ and use this to construct a compact set $F = F_\varepsilon$ such that $\forall n$ $\mu_n(F^c) < \varepsilon$.

Indeed,

$$\mu_n(R(n_1, n_2, \ldots)^c) = \sum_{i=1}^{\infty} \mu_n\{x \in [a] : \forall j \leq i-1 \;\; \tau_j(x) \leq n_j \; ; \; \tau_i(x) > n_i\}$$

Set $Z_n = Z_n(\phi, a)$ and $Z_n^* = Z_n^*(\phi, a)$. Define $Z_0 = Z_0^* = 1$ and $Z_n = Z_n^* = 0$ for $n < 0$. By positive recurrence, $\exists M = M_a$ such that $Z_n = M^{\pm 1}\lambda^n$ for n large enough. Repeated use of the concatenation part in lemma 1 yields that for $B = B_1^3$

$$\begin{aligned}
\mu_n(R(n_1, n_2, \ldots)^c) &\leq \lambda^{-n} \sum_{i=1}^{\infty} B^i \sum_{\substack{k_j \leq n_j \\ j \leq i-1}} \prod_{j=1}^{i-1} Z_{k_j}^* \left(\sum_{k_i > n_i} Z_{k_i}^* Z_{n-(k_1+\ldots+k_i)}\right) \\
&\leq \lambda^{-n} \sum_{i=1}^{\infty} B^{2i} \sum_{\substack{k_j \leq n_j \\ j \leq i-1}} \left(Z_{(k_1+\ldots+k_{i-1})} \sum_{k_i > n_i} Z_{k_i}^* Z_{n-(k_1+\ldots+k_i)}\right) \\
&\leq M^2 \sum_{i=1}^{\infty} B^{2i} \sum_{\substack{k_j \leq n_j \\ j \leq i-1}} \sum_{k_i > n_i} \lambda^{-k_i} Z_{k_i}^* \\
&= M^2 \sum_{i=1}^{\infty} B^{2i} \left(\prod_{j \leq i-1} n_j\right) \sum_{k_i > n_i} \lambda^{-k_i} Z_{k_i}^*
\end{aligned}$$

It is therefore enough to choose (by induction) n_i to satisfy

$$\sum_{k_i>n_i} \lambda^{-k_i} Z^*_{k_i} < \frac{\varepsilon}{2^{i+1}B^{2i}M^2 \prod\limits_{j\le i-1} n_j}.$$

Henceforth, we assume that $\{n_k\}_{k=1}^{\infty}$ is such that

$$\forall n\ \mu_n \left(R\left(n_1, n_2, \ldots\right)^c\right) < \frac{\varepsilon}{2} \tag{10}$$

Set

$$Z^*_n(N) = \sum \left\{e^{\phi_n(x)} : x \in [a]\ ;\ T^n x = x\ ;\ 1 \le i \le n-1\ x_i \ne a\ ;\ \exists i\ x_i > N\right\}.$$

For every i choose $N_i \in \mathbf{N}$ in an increasing way such that $N_i > \max\{z_k : k \le i\}$ and

$$\forall n \le n_i\ \ Z^*_n(N_i) < \frac{\varepsilon}{2^{i+1}B^4 M} Z^*_n(\phi, a) \tag{11}$$

Set $F = \{x \in [a] : \forall i\ x_i \le N_i\}$. F is a compact set. We show that $\forall n\ \mu_n(F^c) < \varepsilon$, proving the lemma.

For every sequence of natural numbers $\{k_i\}_i$ set

$$S\left(k_1, k_2, \ldots\right) = \{x \in [a] : \forall i\ \ \tau_i(x) = k_i\}.$$

Then $R(n_1, n_2, \ldots) = \bigcup\{S(k_1, k_2, \ldots) : \{k_i\}_i$ s.t. $\forall i\ k_i \le n_i\}$. We show that for every sequence $\{k_i\}_i$ such that $k_i \le n_i$, for every n

$$\mu_n\left(F^c \cap S\left(k_1, k_2, \ldots\right)\right) \le \frac{\varepsilon}{2M}\mu_n\left(S\left(k_1, k_2, \ldots\right)\right) \tag{12}$$

Fix a sequence $\{k_i\}_i$ such that $\forall i\ k_i \le n_i$. Suppose $\mu_n(S(k_1, k_2, \ldots)) > 0$ (else there is nothing to prove). Then $\exists N$ such that $k_1 + \ldots + k_N = n$. Set

$$Z^*_{k_1,\ldots,k_i} = \sum \left\{e^{\phi_n(x)} : x \in [a]\ ;\ T^{k_1+\ldots+k_i}x = x\ ;\ \tau_j(x) = k_j\ ,\ \ j = 1, \ldots, i\right\}.$$

Since $\forall i\ \max\{z_k : k \le i\} < N_i$, (in particular $a < N_i$)

$$\mu_n\left(F^c \cap S\left(k_1, k_2, \ldots\right)\right) \le \sum_{i=1}^{N} \mu_n \left\{x \in S\left(k_1, k_2, \ldots\right) : \exists j \in \left(\sum_{m=1}^{i-1} k_m, \sum_{m=1}^{i} k_m\right)\ \ x_j > N_j\right\}$$

evidently $j \ge i$ whence

$$\begin{aligned}
\mu_n\left(F^c \cap S\left(k_1, k_2, \ldots\right)\right) &\le \sum_{i=1}^{N} \mu_n \left\{x \in S\left(k_1, k_2, \ldots\right) : \exists j \in \left(\sum_{m=1}^{i-1} k_m, \sum_{m=1}^{i} k_m\right)\ \ x_j > N_i\right\} \\
&\le \lambda^{-n} B^2 \sum_{i=1}^{N} Z^*_{k_1,\ldots,k_{i-1}} Z^*_{k_i}(N_i)\, Z^*_{k_{i+1},\ldots,k_N} \\
&\le \lambda^{-n} B^2 \sum_{i=1}^{N} \tfrac{\varepsilon}{2^{i+1}B^4M} Z^*_{k_1,\ldots,k_{i-1}} Z^*_{k_i} Z^*_{k_{i+1},\ldots,k_N} \\
&\le B^2 \sum_{i=1}^{N} \tfrac{\varepsilon B^2}{2^{i+1}B^4M} \mu_n\left(S\left(k_1, k_2, \ldots\right)\right) \\
&= \tfrac{\varepsilon}{2M} \mu_n\left(S\left(k_1, k_2, \ldots\right)\right)
\end{aligned}$$

and (12) is proved.

By (10)

$$\begin{aligned}\mu_n(F^c) &\le \mu_n(F^c\cap R(n_1,n_2,\ldots))+\tfrac{\varepsilon}{2}\\ &= \tfrac{\varepsilon}{2}+\sum\{\mu_n(F^c\cap S(k_1,k_2,\ldots)) : \forall i\ k_i\le n_i\}\end{aligned}$$

(this is a finite sum, since only a finite number of $S(k_1,k_2,\ldots)$ are not empty mod μ_n). By (12),

$$\mu_n(F^c)\le\frac{\varepsilon}{2}+\frac{\varepsilon}{2M}\sum\{\mu_n(S(k_1,k_2,\ldots)) : \forall i\ k_i\le n_i\}<\frac{\varepsilon}{2}+\frac{\varepsilon}{2M}\mu_n(X)$$

By positive recurrence $\mu_n(X)<M$ whence $\forall n\ \mu_n(F^c)<\varepsilon$. □

Lemma 9 *Let X be topologically mixing and $\phi : X\to\mathbf{R}$ be a locally Hölder continuous such that $P_G(\phi)<\infty$. If ϕ is positive recurrent then there exists a $\sigma-$finite measure ν such that $\forall f\in L^1(\nu)\quad \nu(L_\phi f)=e^{P_G(\phi)}\nu(f)$ and such that $\nu(h)\in(0,\infty)$ where h is given by lemma 6.*

Proof. Assume for simplicity that $S=\mathbf{N}$. Fix some $z\in[a]$. For every $a,i\in\mathbf{N}$

$$\mu_{a,i}=\lambda^{-i}\sum_{\substack{T^ix=z\\ x_0=a}}e^{\phi_i(x)}\delta_x\quad;\quad \nu_n=\frac{1}{n}\sum_{i=0}^{n-1}\sum_{a=1}^{\infty}\mu_{a,i}.$$

By lemma 8, $\left\{\nu_n|_{\{x:x_0<N\}}\right\}_{n=1}^{\infty}$ is tight for every N. Thus, there exists a subsequence $n_k\nearrow\infty$ and a σ-finite measure ν, finite on cylinders such that for every bounded continuous function with a bounded support (i.e. $\exists N\ \mathrm{supp}\, f\subseteq\{x : x_0\le N\}$) $\nu_{n_k}(f)\underset{k\to\infty}{\longrightarrow}\nu(f)$. The positive recurrence of ϕ implies that $\nu\ne 0$.

ϕ is positive recurrent, so there exists a function $h>0$ such that $L_\phi h=\lambda h$. Evidently, $\nu(h)>0$. We show that $\nu(h)<\infty$. Indeed, for every N

$$\begin{aligned}\nu\left(h1_{[x_0<N]}\right) &= \lim_{k\to\infty}\frac{1}{n_k}\sum_{i=0}^{n_k-1}\sum_{a=1}^{\infty}\mu_{a,i}\left(h1_{[x_0<N]}\right)\\ &\le \lim_{k\to\infty}\frac{1}{n_k}\sum_{i=0}^{n_k-1}\sum_{a=1}^{\infty}\mu_{a,i}(h)\\ &= \lim_{k\to\infty}\frac{1}{n_k}\sum_{i=0}^{n_k-1}\lambda^{-i}\sum_{T^ix=z}e^{\phi_i(x)}h(x)\\ &= h(z)\end{aligned}$$

whence $\nu(h)<\infty$.

We show that $L_\phi^*\nu=\lambda\nu$. It is enough to check this for indicator functions. Fix some cylinder $[\underline{a}]$. h is continuous so $\exists C>0$ such that $\forall x\ 1_{[\underline{a}]}(x)<Ch(x)$. Since

L_ϕ is positive, $L_\phi 1_{[\underline{a}]} < Ch$. It follows that $\nu\left(L_\phi 1_{[\underline{a}]}\right) < \infty$. A standard argument (that uses the fact that $\forall k\ \lambda^{-k} L_\phi^k 1_{[\underline{a}]} < Ch$) shows that $\nu\left(L_\phi 1_{[\underline{a}]}\right) \le \lambda\nu\,[\underline{a}]$ for every cylinder $[\underline{a}]$. Since the cylinders generate the $\sigma-$algebra, for every $f \in L^1(\nu)$ such that $f \ge 0$, $\nu(L_\phi f) \le \lambda\nu(f)$. In particular,

$$\nu\left(L_\phi\left(Ch - 1_{[\underline{a}]}\right)\right) \le \lambda\nu\left(Ch - 1_{[\underline{a}]}\right)$$

whence $\nu\left(L_\phi 1_{[\underline{a}]}\right) \ge \lambda\nu\,[\underline{a}]$. Thus, $\nu\left(L_\phi 1_{[\underline{a}]}\right) = \lambda\nu\,[\underline{a}]$ for every cylinder $[\underline{a}]$ whence

$$\forall f \in L^1(\nu) \quad \nu(L_\phi f) = \lambda\nu(f).$$

□

4.3 Convergence of $\lambda^{-k} L_\phi^k$

The proof of the following lemma appears in [1] and [19].

Lemma 10 *Let X be a topologically mixing Markov shift and μ a nonsingular $\sigma-$finite measure (i.e. $\mu \circ T^{-1} \sim \mu$). Let $\mu \circ T$ be the measure given by $(\mu \circ T)(A) = \sum_{a \in S} \mu(T([a] \cap A))$. Then,*

1. *$\mu \ll \mu \circ T$ and $g = \frac{d\mu}{d\mu \circ T}$ mod $\mu \circ T$ iff $L_{\log g}$ acts as the transfer operator of μ.*
2. *Set $g^{(n)} = \prod_{k=0}^{n-1} g \circ T^k$. If μ is conservative, supported on X and $\exists C\ \forall x, y \in T^n[\underline{a}]$, $[\underline{a}] \in \alpha_0^{n-1}$*
$$g^{(n)}(\underline{a}x) = C^{\pm 1} g^{(n)}(\underline{a}y)$$
then μ is exact. This happens, in particular, when $\log g$ is locally Hölder continuous.

Lemma 11 *Let ν, h, λ be as the above. Then $dm = h\,d\nu$ is an exact invariant (finite) measure and for every uniformly continuous f such that $\|f\,h^{-1}\|_\infty < \infty$*

$$\lambda^{-n}\left(L_\phi^n f\right)(x) \underset{n\to\infty}{\longrightarrow} h(x)\int f\,d\nu$$

uniformly on compacts.

Proof. $dm = h\,d\nu$ is invariant because

$$m(f \circ T) = \nu(h\,f \circ T) = \lambda^{-1}\nu(L_\phi(hf \circ T)) = \lambda^{-1}\nu(f L_\phi h) = m(f).$$

Since $L_\phi^* \nu = \lambda\nu$, ν is non singular and its transfer operator is $\lambda^{-1} L_\phi$:

$$\nu(\varphi\psi \circ T) = \lambda^{-1}\nu(L_\phi(\varphi\psi \circ T)) = \nu\left(\left(\lambda^{-1} L_\phi \varphi\right)\psi\right).$$

Thus,

$$\frac{d\nu}{d\nu \circ T} = \lambda^{-1} e^{\phi}.$$

By lemma 10, ν is exact. Thus, so is m.

We prove the convergence result. Fix some uniformly continuous function f such that $|f| < Ch$. Set $\psi = f + Ch > 0$. For every $x \in [a]$, $0 \leq \lambda^{-n} \left(L_\phi^n \psi\right)(x) \leq 2Ch(x) \leq 2Ch[a]$. It follows, along the lines of the proof of lemma 6 that the family $\left\{\lambda^{-n} L_\phi^n \psi\right\}_{n=1}^{\infty}$ is uniformly bounded and equicontinuous on partition sets, hence every sequence has a subsequence that converges uniformly on compact sets. Just like in the proof of lemma 6 every limit point is an eigenfunction of L_ϕ, hence an invariant density of ν. Since ν is ergodic, it has only one invariant density (up to a constant), h. Thus $\lambda^{-n} L_\phi^n \psi \to \nu(\psi) h$ and the convergence is uniform on compacts. Since $\lambda^{-n} L_\phi^n \psi = \lambda^{-n} L_\phi^n (f + Ch) = \lambda^{-n} L_\phi^n f + Ch$ we also have that $\lambda^{-n} L_\phi^n f \to \nu(f) h$ where the convergence is uniform on compacts. □

5 The rate of convergence in the RPF theorem

A topological Markov shift X is said to have the *big images property* if

$$\exists b_1, \ldots, b_n \in S \; \forall a \in S \; \exists i \;\; [a, b_i] \neq \emptyset \tag{13}$$

The reason why (13) is called the big images property is that if m is some finite measure supported on X, then (13) is equivalent to the condition $\inf \{m(T[a]) : a \in S\} > 0$. Probability measures dm with locally Hölder continuous derivatives $dm/dm \circ T$ (see lemma 10) on countable Markov shifts satisfying (13), were studied by Aaronson and Denker in [2] (see also [1] and [3]). In this section we apply their results to determine the rate of convergence in the RPF theorem under certain assumptions.

Let β be the partition generated by the image sets $\beta = \sigma \{T[a] : a \in S\}$. β may be infinite. Set

$$D_\beta f = \sup_{b \in \beta} \sup_{x, y \in b} \frac{|f(x) - f(y)|}{r^{t(x,y)}}$$

and $\mathcal{L} = \{f \in C(X) : \|f\|_{\mathcal{L}} := \|f\|_\infty + D_\beta f < \infty\}$. The following theorem follows from results in [2].

Theorem 5 *Let X be a topologically mixing topological Markov shift that satisfies (13). Let ϕ be a locally Hölder continuous function such that $P_G(\phi) < \infty$. If ϕ is positive recurrent and h its corresponding eigenfunction is bounded away from zero and infinity, then $\exists K > 0$, $\exists \theta \in (0,1)$ such that for every $f \in \mathcal{L}$ $\left\|\lambda^{-n} L_\phi^n f - h\nu(f)\right\|_{\mathcal{L}} < K\theta^n \|f\|_{\mathcal{L}}$ where $\lambda = e^{P_G(\phi)}$ and ν is the eigenmeasure such that $\nu(h) = 1$.*

Proof. Let $\lambda = e^{P_G(\phi)}$ and v, h satisfy $L_\phi^* \nu = \lambda \nu$, $L_\phi h = \lambda h$ and $\nu(h) = 1$. Set $dm = h\, d\nu$ and $g^{(n)} = dm/dm \circ T^n$. Then $g^{(n)} = \lambda^{-n} e^{\phi_n} h / h \circ T^n$. Fix an admissible

$\underline{p}$ of length $\left|\underline{p}\right| = n$. Then for every $x, y \in T^n\left[\underline{p}\right]$

$$\left|\log\frac{g^{(n)}\left(\underline{p}x\right)}{g^{(n)}\left(\underline{p}y\right)}\right| = \left|\phi_n\left(\underline{p}x\right) - \phi_n\left(\underline{p}y\right)\right| + \left|\log h\left(\underline{p}x\right) - \log h\left(\underline{p}y\right)\right| + \left|\log h\left(y\right) - \log h\left(x\right)\right|$$

By the local Hölder continuity of ϕ and $\log h$ and the assumption that h is bounded away from zero and infinity, there exists some C independent of $\underline{p}, n$ such that for every $x, y \in T^n\left[\underline{p}\right]$

$$\left|\log\frac{g^{(n)}\left(\underline{p}x\right)}{g^{(n)}\left(\underline{p}y\right)}\right| < Cr^{t(x,y)}$$

(the assumption that h is bounded away from zero and infinity is needed for the case $x, y \in T^n\left[\underline{p}\right]$, $x_0 \neq y_0$.) It follows from this and from (13) that m is a Gibbs-Markov measure in the terminology of [2]. One of the properties of a Gibbs-Markov measure is that its transfer operator $\hat{T}$ satisfies that $\exists K > 0\ \exists\theta \in (0,1)$ such that $\forall\psi \in \mathcal{L}$ $\left\|\hat{T}^n\psi - m\left(\psi\right)\right\|_{\mathcal{L}} < K\theta^n\left\|\psi\right\|_{\mathcal{L}}$ (see [2]). For $dm = h\,d\nu$, $\hat{T}\psi = \lambda^{-1}h^{-1}L_\phi\left(h\psi\right)$, so $\forall\psi \in \mathcal{L}$

$$\left\|h^{-1}\left(\lambda^{-n}L_\phi^n\left(\psi h\right) - h\nu\left(\psi h\right)\right)\right\|_{\mathcal{L}} < K\theta^n\left\|h^{-1}\left(\psi h\right)\right\|_{\mathcal{L}} \tag{14}$$

We finish the proof by proving the existence of $H > 1$ such that $\left\|f\right\|_{\mathcal{L}} = H^{\pm 1}\left\|fh^{-1}\right\|_{\mathcal{L}}$ and then setting $\psi = fh^{-1}$ in (14). The existence of H follows from the identity

$$\frac{f\left(x\right)}{h\left(x\right)} - \frac{f\left(y\right)}{h\left(y\right)} = \left(\frac{f\left(x\right)}{h\left(x\right)}\right)\left(1 - \frac{h\left(x\right)}{h\left(y\right)}\right) + \frac{f\left(x\right) - f\left(y\right)}{h\left(y\right)}$$

which implies (since $\log h$ is bounded away from zero and infinity and locally Hölder continuous) that $\exists H_1 > 1$ such that $D_\beta\left(fh^{-1}\right) = H_1^{\pm 1}\left(D_\beta f + \left\|f\right\|_\infty\right)$. Clearly $\exists H_2 > 1$ such that $\left\|fh^{-1}\right\|_\infty = H_2^{\pm 1}\left\|f\right\|_\infty$ and the result follows by setting $H = \max\left\{H_1, H_2\right\}$. □

Example 1. We show that exponential convergence may fail if X does not satisfy (13). Fix a renewal sequence $\left\{u_n\right\}_{n=1}^\infty$ that tends to a positive limit u_∞ in a subexponential rate and such that $u_1 > 0$. The following possible construction of such a $\left\{u_n\right\}_{n=1}^\infty$ was shown to me by J. Aaronson: let $\beta_k = e^{-1/k^3}$. Set $u_n = \prod_{k=1}^\infty \beta_k^{k\wedge n}$. Note that $u_n \to u_\infty = \exp\left(-\sum_{k=1}^\infty k^{-2}\right) > 0$. $\left\{u_n\right\}_n$ is a renewal sequence, by the Kaluza theorem (see theorem 5.3.2 in [3]). Also

$$\begin{aligned}\left|u_n - u_\infty\right| &= u_\infty\left|\tfrac{u_n}{u_\infty} - 1\right| \\ &= u_\infty\left|1 - \prod_{k=1}^\infty \beta_k^{k\wedge n - k}\right| \\ &= u_\infty\left|1 - \exp\left(\sum_{k=1}^\infty k\left(k+n\right)^{-3}\right)\right| \\ &= u_\infty\left[\sum_{k=1}^\infty k\left(k+n\right)^{-3} + o\left(\sum_{k=1}^\infty k\left(k+n\right)^{-3}\right)\right]\end{aligned}$$

so u_n converges to its limit in a subexponential way.

We now use $\{u_n\}_n$ to construct a positive recurrent locally Hölder continuous potential ϕ for which $\lambda^{-n}L_\phi^n$ does not converge exponentially to its limit. Since $\{u_n\}_n$ is a renewal sequence, there exist $f_k \geq 0$ such that $u_n = f_1 u_{n-1} + \ldots + f_{n-1}u_1 + f_n$. Note that $\sum f_k = 1$, because u_n tends to a positive limit. Let $F = \sup\{k : f_k > 0\}$ (it will later become clear that $F = \infty$). Let X be the topological Markov shift on the set of states $S = \{n \in \mathbf{N} : n \leq F\}$ given by the transition matrix $(t_{ij})_{S\times S}$ where $t_{ij} = 1$ iff $i > 1$ and $j = i - 1$ or $i = 1$ and $f_j > 0$. This is a topologically mixing Markov shift (it is irreducible and $t_{11} = 1$, because $f_1 = u_1 > 0$). If $F = \infty$, X does not have the big images property. Now, set $\phi(x) = \log(p_{x_0}p_{x_0x_1}/p_{x_1})$ where $p_{1n} = f_n, p_{n\,n-1} = 1$ and

$$p_n = \begin{cases} f_n & f_n > 0 \\ 1 & f_n = 0 \end{cases}$$

ϕ is locally Hölder continuous, because it is constant on partition sets: $\phi(x) = \log f_1$ if $x_0 = 1$ and $\phi(x) = \log(p_{x_0}/p_{x_0-1})$ if $x_0 > 1$. Note that for every $x \in [1]$ $\left(L_\phi^n 1_{[a]}\right)(x) = Z_n(\phi, 1) = u_n$. It immediatly follows that ϕ is positive recurrent with pressure zero because $u_n \to u_\infty > 0$. By the construction of $\{u_n\}_n$, for every $x \in [1]$ $\left(L_\phi^n 1_{[a]}\right)(x) = u_n$ tends to its limit in a subexponential rate.

The pair of conditions that X satisfy (13) and h be bounded away from zero and infinity proves to be particularly natural in the context of the thermodynamic formalism (see, e.g., theorem 8 below). While the big images condition (13) is easy to check, the condition that h is bounded away from zero and infinity is usually rather difficult to verify. The following pair of propositions give a necessary condition and a sufficient condition for h to be bounded away from zero and infinity.

A topological Markov shift is said to have *finitely many images* if

$$|\{T[a] : a \in S\}| < \infty \tag{15}$$

(see Aaronson, Denker and Urbanski [1] and Yuri [25]). This is reflected in the transition matrix in that the number of different rows is finite. An easy combinatorial argument shows that in this case, the number of different columns must also be finite, whence $\forall n$ the set $\{P^n(x) : x \in X\}$ is finite. A basic property of shifts with finitely many images is the following ([1], [25]):

Proposition 2 *Let X be topologically mixing with finitely many images, and ϕ a positive recurrent locally Hölder continuous function such that $P_G(\phi) < \infty$. Then X satisfies (13) and the eigenfunction h is uniformly bounded away from zero and infinity.*

Proof. The fact that (15) implies (13) is trivial. We show that h is bounded away from zero and infinity. Fix x and assume that $P^1(x) = P^1(y)$. This implies that $\forall i\ P^i(x) = P^i(y)$. By lemma 1

$$\sum_{\underline{p}\in P^i(x)} e^{\phi_i(\underline{p}x)} 1_{[a]}\left(\underline{p}x\right) = B_0^{\pm 1} \sum_{\underline{p}\in P^i(y)} e^{\phi_i(\underline{p}y)} 1_{[a]}\left(\underline{p}y\right)$$

Taking limits as $i \to \infty$ yields $h(x) = B_0^{\pm 1} h(y)$. Thus, h is bounded away from zero and infinity on $\{y : P^1(y) = P^1(x)\}$. Since the collection $\{P^1(x) : x \in X\}$ is finite, h is bounded away from zero and infinity on X. □

Proposition 3 *Let X be topologically mixing with (13) and ϕ locally Hölder continuous, positive recurrent such that $P_G(\phi) < \infty$. Let h be the eigenfunction $L_\phi h = \lambda h$. If h is bounded away from zero and infinity then $\exists M\ \forall a \in S\ T^M[a] = X$.*

Proof. We prove that $\forall a\ \exists M = M_a$ such that $T^M[a] = X$. The fact that M can be chosen to be independent of a follows from the big images property (13). Assumeϕ that $h > C > 0$ everywhere. Fix $a \in S$. Since $1_{[a]} \in \mathcal{L}$, and since $\|\cdot\|_\infty \le \|\cdot\|_{\mathcal{L}}$,

$$\left\|\lambda^{-n} L_\phi^n 1_{[a]} - h\nu[a]\right\|_\infty \underset{n\to\infty}{\longrightarrow} 0$$

Choose $M = M_a$ large enough such that for every x $\left|\lambda^{-M}\left(L_\phi^M 1_{[a]}\right)(x) - h\nu[a]\right| < C\nu[a]$. Then for every $x \in X$

$$\lambda^{-M} \sum_{T^M y = x} e^{\phi_M(y)} 1_{[a]}(y) > (h(x) - C)\,\nu[a] > 0$$

and in particular, $\forall x\ \exists y \in [a]$ such that $T^M y = x$. □

Example 2. We construct a topologically mixing countable Markov shift and a locally Hölder continuous function positive recurrent ϕ such that X has big images, but h is not bounded away from zero and $\nu(X) = \infty$. Let X be the topological Markov shift with the transition matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & \cdots \\ 1 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and $\phi(x) = \log\left(p_{x_0} p_{x_0 x_1} / p_{x_1}\right)$ where

$$p_n = \frac{1}{n} \quad ; \quad p_{ij} = \begin{cases} 1 - \frac{1}{i} & j = 1, i > 1 \\ \frac{1}{i} & j = i + 1 \\ 0 & else \end{cases}$$

To see that ϕ is locally Hölder continuous it is enough to verify that $V_1(\phi) < \infty$. To see this check that for every $a \in S$ and $x, y \in T[a]$

$$|\phi(ax) - \phi(ay)| = \left|\log \frac{p_{ax} p_y}{p_x p_{ay}}\right| \le \log 3$$

Let ν be the Markov measure given by $\nu[a_0, \ldots, a_n] = p_{a_0} p_{a_0 a_1} \cdot \ldots \cdot p_{a_{n-1} a_n}$. This is an infinite measure since $\sum p_n = \infty$. One checks that $d\nu / d\nu \circ T = \exp \phi$ whence L_ϕ acts as the transfer operator of ν. If follows that $L_\phi^* \nu = \nu$. Let h be given by $h(x) = x_0 / (x_0 - 1)!$. It is easy to check that $L_\phi h = h$. Since $\nu(h) < \infty$, ϕ is positive recurrent. We see that h is not bounded away from zero and that $\nu(X) = \infty$. Note that there is no M for which $T^M[1] = X$, in accordance with proposition 3.

6 A decomposition theorem for positive recurrent functions

Throughout this section λ, h and ν are as in theorem 4. In this section we prove the following decomposition theorem for positive recurrent potentials.

Theorem 6 *Let X be topologically mixing and ϕ a locally Hölder continuous function such that $P_G(\phi) < \infty$. ϕ is positive recurrent iff there exist a locally Hölder continuous function ψ, an invariant probability measure m supported on X and a constant c such that*

$$\phi = \log \frac{dm}{dm \circ T} + \psi \circ T - \psi + c. \tag{16}$$

If this is the case, then $\psi = \log h + const$, $dm = h\, d\nu$ and $c = P_G(\phi)$.

Proof. It ϕ is positive recurrent then the result follows from theorem 4, by setting $\psi = \log h$ and $dm = h\, d\nu$. Assume that (16) holds. The transfer operator of m is given by $L_{\log g}$ where $g = \frac{dm}{dm \circ T}$. Let ν' be the σ-finite measure $d\nu' = e^{-\psi} dm$. By (16) $L_\phi f = e^c e^\psi L_{\log g}\left(e^{-\psi} f\right)$ whence for every $f \in L^1(\nu')$, $\varphi \in L^\infty(\nu')$

$$\nu'(\varphi L_\phi f) = e^c m\left(\varphi L_{\log g}\left(e^{-\psi} f\right)\right) = e^c \nu'(f \varphi \circ T). \tag{17}$$

This shows that $e^{-c} L_\phi$ acts as the transfer operator of ν' whence by lemma 10 $\log \frac{d\nu'}{d\nu' \circ T} = \phi - c$ and ν' is exact. Since $m \sim \nu'$, m is also exact hence, being invariant and finite, strong mixing.

We show that ϕ must be positive recurrent. A calculation shows that for some $C > 1$ and for every $x \in [a]$

$$\begin{aligned} Z_n(\phi, a) &= e^{nc} Z_n(\log g, a) \\ &= C^{\pm 1} e^{nc} L_{\log g}^n 1_{[a]}(x) \\ &= C^{\pm 1} e^{nc} \tfrac{1}{m[a]} \int 1_{[a]} L_{\log g}^n 1_{[a]}\, dm \\ &= C^{\pm 1} e^{nc} m\left(T^{-n}[a] \mid [a]\right). \end{aligned}$$

(the third inequality follows from the integration of the second). The result follows from the strong mixing of m. A by product of the proof is that $c = P_G(\phi)$.

By (17) $L_\phi^* \nu' = e^{P_G(\phi)} \nu'$. By the invariance of m, $L_{\log g} 1 = 1$ whence $L_\phi e^\psi = e^{P_G(\phi)} e^\psi$. Thus, by theorem 4 ν' is proportional to ν and e^ψ to h where ν and h are the corresponding eigenvectors of L_ϕ (which exist since ϕ is positive recurrent). Since $m(1) = 1$, $dm = h\, d\nu$. □

7 Equilibrium measures

In this section we describe $dm = h\, d\nu$ as an equilibrium measure in a certain sense. We have already remarked that for countable Markov shifts the quantity $h_\mu(T) + \mu(\phi)$ is not generally well defined because of the possibility that $h_\mu(T) = \infty, \mu(\phi) = -\infty$. As in Walters [23] and Yuri [25], we write formally $h_\mu(T) + \mu(\phi) = \mu(I_\mu + \phi)$ where

$$I_\mu = I_\mu\left(\alpha \middle| T^{-1}\mathcal{B}\right) = -\sum_{a \in S} 1_{[a]} \log \mu\left([a] \middle| T^{-1}\mathcal{B}\right)$$

and $\alpha = \{[a] : a \in S\}$ (this is true when $H_\mu(\alpha) < \infty$). Our approach in this section is based on the fact that it may happen that $\mu(I_\mu + \phi)$ makes sense even when $h_\mu(T) + \mu(\phi)$ does not.

Lemma 12 *Let X be a topologically mixing and μ an invariant probability measure. Let $g_\mu = d\mu / d\mu \circ T$ (see lemma 10). Then $L_{\log g_\mu} 1 = 1$ and*

1. $E_\mu(f | T^{-1}\mathcal{B}) = \left(L_{\log g_\mu} f\right) \circ T$
2. $I_\mu(\alpha | T^{-1}\mathcal{B}) = -\log g_\mu$.

Proof. Ledrappier [13]. □

Theorem 7 *Let X be topologically mixing and ϕ a positive recurrent locally Hölder continuous function such that $P_G(\phi) < \infty$. Then ϕ is cohomologous via a locally Hölder continuous transfer function to a function ϕ' such that $\forall \mu \in \mathcal{M}_T(X)$ $I_\mu + \phi'$ is one sided integrable with respect to μ and*

$$\mu\left(I_\mu\left(\alpha \middle| T^{-1}\mathcal{B}\right) + \phi'\right) \leq P_G(\phi') = P_G(\phi)$$

The only $\mu \sim \mu \circ T$ for which this is an equality is $d\mu = h\, d\nu$.

Proof. Assume that ϕ is positive recurrent. We follow an idea of Ledrappier (see [13], [23], [25]). By the positive recurrence of ϕ,

$$\phi = \log \frac{dm}{dm \circ T} + \psi \circ T - \psi + P_G(\phi)$$

where m is an invariant probability and $\psi = \log h$. Set $g = \frac{dm}{dm \circ T} = e^{\phi + \psi - \psi \circ T - P_G(\phi)}$ and $\phi' = \log g + P_G(\phi)$. ϕ and ϕ' are cohomologous, because $\phi' = \phi + \psi - \psi \circ T$. Fix an invariant probability μ. We show that $I_\mu(\alpha|T^{-1}\mathcal{B})+\phi'$ is one sided integrable and has integral not larger than $P_G(\phi)$. Set $g_\mu=\frac{d\mu}{d\mu\circ T}$ and $A_{s,t}=\left[s<\frac{g}{g_\mu}<t\right]$. By lemma 12, and the inequality $\log x\le x-1$

$$\begin{aligned}\int_{A_{s,t}}\left(I_\mu\left(\alpha|T^{-1}\mathcal{B}\right)+\phi'\right)d\mu &\le \int_{A_{s,t}}\left(\frac{g}{g_\mu}-1\right)d\mu+P_G(\phi)\\ &=\int E_\mu\left(1_{A_{s,t}}\left(\frac{g}{g_\mu}-1\right)\middle|T^{-1}\mathcal{B}\right)d\mu+P_G(\phi)\\ &=\int\sum_{Ty=Tx}1_{A_{s,t}}(y)\left[g(y)-g_\mu(y)\right]d\mu(x)+P_G(\phi).\end{aligned}$$

Since $\sum_{Ty=z}g(y)=\sum_{Ty=z}g_\mu(y)=1$, $I_\mu+\phi'$ is μ one sided integrable and

$$-\infty\le\int\left(I_\mu\left(\alpha|T^{-1}\mathcal{B}\right)+\phi'\right)d\mu\le P_G(\phi). \tag{18}$$

Since $\log x=x-1$ iff $x=1$, the right hand of (18) is an equality iff $g_\mu=g$ $\mu-$a.e. By the definition of g and theorem 6, this happens when $d\mu=h\,d\nu$. If $\mu\sim\mu\circ T$ then this is the only case. Indeed, in this case $g_\mu=g$ $\mu\circ T$ a.e. whence $L_{\log g}$ acts as the transfer operator of μ. It is easy to verify that by theorem 4 for every $[\underline{a}]$, $L^n_{\log g}1_{[\underline{a}]}=\lambda^{-n}h^{-1}L_\phi\left(h1_{[\underline{a}]}\right)\underset{n\to\infty}{\longrightarrow}\int_{[\underline{a}]}h\,d\nu$ uniformly on compacts whence $\mu[\underline{a}]=\mu\left(L^n_{\log g}1_{[\underline{a}]}\right)\underset{n\to\infty}{\longrightarrow}\int_{[\underline{a}]}h\,d\nu$. □

Corollary 2 *Let X be topologically mixing and ϕ locally Hölder continuous positive recurrent function. If the eigenfunction h is bounded away from zero and infinity, then for every $\mu\in\mathcal{M}_T(X)$ $I_\mu+\phi$ is one sided integrable with respect to μ and $\mu(I_\mu+\phi)\le P_G(\phi)$. The only measure $\mu\sim\mu\circ T$ for which this is an equality is $h\,d\nu$.*

Proof. Use the previous proof together with the fact that $\psi-\psi\circ T$ is μ integrable for every μ and $\mu(\psi-\psi\circ T)=0$. □

Hofbauer [9] (see also [24]) gives examples where apart from the measure $hd\nu$, there is another 'equilibrium measure' δ which is supported on a single fixed point. Evidently $\delta\not\sim\delta\circ T$.

8 Gibbs measures

An invariant measure m is a *Gibbs measure* for ϕ in the sense of Bowen [4] if there exist $M>1$ and P_m such that for every $x\in[a_0,\dots,a_{n-1}]$

$$m[a_0,\dots,a_{n-1}]=M^{\pm1}e^{\phi_n(x)-nP_m}. \tag{19}$$

It is known (Ruelle [16], Bowen [4]) that for finite topological Markov shifts and Hölder continuous functions ϕ Gibbs measures exist, coincide with the equilibrium measures and satisfy that P_m is equal to the topological pressure of ϕ. The following theorem treats this topic for countable shifts.

Theorem 8 *Let X be topologically mixing and ϕ locally Hölder continuous such that $P_G(\phi) < \infty$. ϕ has a Gibbs measure iff ϕ is positive recurrent, the corresponding h is uniformly bounded away from zero and infinity, and X satisfies (13). If m is this measure then $P_m = P_G(\phi)$ and $dm = h\,d\nu$.*

Proof. Assume first that ϕ is positive recurrent, that $h = B^{\pm 1}$ and that X satisfies (13). Set $dm = h\,d\nu$. We show that this is a Gibbs measure. Since $\nu(h) = 1$ and $h = B^{\pm 1}$, we have that $\nu(X) < \infty$. Without loss of generality, assume that $\nu(X) = 1$. For every $[\underline{a}] = [a_0, \ldots, a_{n-1}]$

$$m[a_0, \ldots, a_{n-1}] = \nu\left(h\,1_{[\underline{a}]}\right) = \lambda^{-n}\nu\left(L_\phi^n\left(h\,1_{[\underline{a}]}\right)\right) = B^{\pm 1}\lambda^{-n}\nu\left(L_\phi^n 1_{[\underline{a}]}\right).$$

For every $x \in [\underline{a}]$ $\nu\left(L_\phi^n 1_{[\underline{a}]}\right) = \nu\left(1_{T^n[\underline{a}]}(y)\, e^{\phi_n(\underline{a}y)}\right) = B_0^{\pm 1} e^{\phi_n(x)}\nu\left(T[a_{n-1}]\right)$. According to (13), there are $b_1, \ldots, b_N \in S$ such that $\nu\left(T[a_{n-1}]\right) \geq \min\{\nu[b_i] : i = 1, \ldots, N\} > 0$. Thus, for

$$M = \frac{BB_0}{\min\{\nu[b_i] : i = 1, \ldots, N\}}$$

$m[a_0, \ldots, a_{n-1}] = M^{\pm 1} e^{\phi_n(x) - n\log\lambda}$ whence m is a Gibbs measure. Note that $P_m = \log\lambda = P_G(\phi)$.

We show the reverse implication. Assume that m is a Gibbs measure. It follows from the Gibbs property that $m \ll m \circ T$. A calculation shows that for $n > k$

$$E_{m\circ T^k}\left(\frac{dm}{dm \circ T^k}\middle|\, \alpha_0^{n-1}\right) = \sum_{[\underline{a}] \in \alpha_0^{n-1}} \frac{m[\underline{a}]}{m\left(T^k[\underline{a}]\right)} 1_{[\underline{a}]} = M^{\pm 2} e^{-kP_m} e^{\phi_k}$$

whence, by the martingale convergence theorem, $\frac{dm}{dm\circ T^k} = M^{\pm 2} e^{-kP_m} e^{\phi_k}$ $m \circ T^k$ a.e. By lemma 10 m is exact hence strong mixing. Positive recurrence follows from the estimation

$$e^{-nP_m} Z_n(\phi, a) = M^{\pm 1} m\left([a] \cap T^{-n}[a]\right).$$

Note that this implies $P_m = P_G(\phi)$. Set $\lambda = \exp P_G(\phi)$. We show (13). For every $[a]$ and $x_a \in [a]$

$$m[a] = \int_{[a]} \frac{dm}{dm \circ T}\, dm \circ T = M^{\pm 2}\lambda^{-1} e^{\phi(x_a)} m\left(T[a]\right) = M^{\pm 3} m[a]\, m\left(T[a]\right).$$

Dividing by $m[a]$ (which is not zero by the definition of m) we have

$$\inf\{m\left(T[a]\right) : [a] \in \alpha\} > M^{-3}.$$

This is equivalent to (13).

It remains to show that $dm = h\,d\nu$ and that $\|\log h\|_\infty < \infty$. By definition, for every $[\underline{a}] = [a_0, \ldots, a_{n-1}]$, $\nu[\underline{a}] = \lambda^{-n}\nu\left(L_\phi^n 1_{[\underline{a}]}\right) = \lambda^{-n}\nu\left(1_{T^n[\underline{a}]}(y)\, e^{\phi_n(\underline{a}y)}\right)$ whence by the big images property and the local Hölder continuity of ϕ, $\exists M_1 > 1$ such that $\forall\, [\underline{a}] = [a_0, \ldots, a_{n-1}]$ and $x \in [\underline{a}]$

$$\nu[\underline{a}] = M_1^{\pm 1}\lambda^{-n}e^{\phi_n(x)} = (MM_1)^{\pm 1}\, m[\underline{a}] \tag{20}$$

whence $\nu \sim m$. Thus, $h\,d\nu \sim dm$, whence by the ergodicity of $h\,d\nu$ and the invariance of m, $dm = hd\nu$. The fact that h is bounded away from zero and infinity follows from (20). □

Remark 1. According to the last theorem, if m is an invariant Gibbs measure then $\|\log h\|_\infty < \infty$ whence $\forall \mu \in \mathcal{M}_T(X)\quad I_\mu + \phi$ is $\mu-$one sided integrable and

$$m(I_m + \phi) = \sup\{\mu(I_\mu + \phi) : \mu \in \mathcal{M}_T(X)\} = P_G(\phi).$$

Remark 2. A topological Markov shift with the finite images property (15) always satisfies the big images property. For such a shift, the eigenfunction h is always bounded away from zero and infinity (proposition 2). The following proposition follows:

Proposition 4 *Let X be topologically mixing with finitely many images, and ϕ a locally Hölder continuous function such that $P_G(\phi) < \infty$. If ϕ is positive recurrent then the measure $dm = h\,d\nu$ is a Gibbs measure for ϕ and $m(I_m + \phi) = \sup\{\mu(I_\mu + \phi) : \mu \in \mathcal{M}_T(X)\} = P_G(\phi)$.*

9 Examples: intrinsically compact shifts

In this section we present a class of countable Markov shifts X for which it is easy to construct non trivial potentials ϕ with strong 'thermodynamic' properties. Our treatment is connected to the analysis of Walters in [24], its main novelty being the consideration of countable Markov shifts other than the full shift ($X = \mathbf{N}^{\mathbf{N}\cup\{0\}}$). Throughout this section, we assume that $\mathbf{A} = (t_{ij})_{S\times S}$ is a fixed transition matrix and that $S = \mathbf{N}$. We denote the powers of $\mathbf{A}$ by $\mathbf{A}^n = \left(t_{ij}^{(n)}\right)_{S\times S}$. Introduce the following notation for the rows and columns of $\mathbf{A}$:

$$R_{\mathbf{A}}(a) = (t_{a1}, t_{a2}, \ldots) \quad ; \quad C_{\mathbf{A}}(a) = \begin{pmatrix} t_{1a} \\ t_{2a} \\ \vdots \end{pmatrix}$$

We say that two states $a, b \in \mathbf{N}$ are $\mathbf{A}-$*equivalent*, and write $a \underset{\mathbf{A}}{\sim} b$, if $R_{\mathbf{A}}(a) = R_{\mathbf{A}}(b)$ and $C_{\mathbf{A}}(a) = C_{\mathbf{A}}(b)$. When there is no risk of ambiguity, we write R, C and $\sim$ instead of $R_{\mathbf{A}}, C_{\mathbf{A}}$ and $\underset{\mathbf{A}}{\sim}$. Denote by $\widehat{a}$ the equivalence class of a, with respect to $\sim$ and set $\widehat{\mathbf{N}} = \{\widehat{a} : a \in \mathbf{N}\}$. Note that the following functions are well defined, the last being a metric on $\widehat{\mathbf{N}}$: $\left(\widehat{a}, \widehat{b}\right) \mapsto t_{ab}$ and $d\left(\widehat{a}, \widehat{b}\right) = \sum_{n=1}^{\infty} 2^{-n} |t_{an} - t_{bn}| + \sum_{m=1}^{\infty} 2^{-m} |t_{ma} - t_{mb}|$.

Definition 3 *A transition matrix* $\mathbf{A}$ *is intrinsically compact if, with respect to the metric* d*,* $\widehat{\mathbf{N}}$ *is compact and* $\left(\widehat{a}, \widehat{b}\right) \mapsto t_{ab}$ *is upper semicontinuous.*

Lemma 13 *In order to check intrinsic compactness it is enough to check that the collection of pairs* $\{(R(a), C(a))\}$ *is closed with respect to convergence in each coordinate and that for every pair of* $d-$*accumulation points* $\widehat{a}, \widehat{b}$*,* $t_{\widehat{a}\widehat{b}} \geq \limsup_{\left(\widehat{a}_n, \widehat{b}_n\right) \to \left(\widehat{a}, \widehat{b}\right)} t_{\widehat{a}_n \widehat{b}_n}$.

Lemma 14 *The compactness of* $\left(\widehat{\mathbf{N}}, d\right)$ *implies that* $\mathbf{A}$ *has big images, namely*

$$\exists b_1, \ldots, b_n \;\; \forall a \, \exists i \; t_{ab_i} = 1. \tag{21}$$

Proof. If (21) were false, then $\forall n \; \exists a_n \; (\{b : t_{a_n b} = 1\} \cap \{1, \ldots, n\} = \emptyset)$ whence

$$\forall n \;\; \{b : t_{a_n b} = 1\} \subseteq (n, \infty).$$

Such a sequence $\{a_n\}_n$ must satisfy $R(a_n) \to (0, 0, \ldots)$. By compactness, it is possible to assume without loss of generality that $\widehat{a}_n$ is $d-$convergent. Denote its limit by $\widehat{a}$. By the definition of d, $\forall a \in \widehat{a} \; R(a) = (0, 0, 0, \ldots)$. This is a contradiction because $\mathbf{A}$ is a transition matrix, hence doesn't contain zero rows. □

Example 1. *The Full Shift Matrix.* This is the matrix $\mathbf{A} = (t_{ij})$, $\forall i, j \; t_{ij} = 1$. For this matrix, $\widehat{\mathbf{N}}$ is a singleton and intrinsic compactness is trivial.

Example 2. *Finite Images.* The finite images property (15) is equivalent to saying that the transition matrix $\mathbf{A}$ has a finite number of different rows (see Aaronson, Denker and Urbanski [1] and Yuri [25]). As mentioned before, $\mathbf{A}$ must also have only a finite number of columns. Thus, if $\mathbf{A}$ has finite images, then $\hat{\mathbf{N}}$ is a finite discrete space, whence $\mathbf{A}$ is intrinsically compact.

Example 3. Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots \\ 1 & 0 & 1 & 1 & \cdots \\ 1 & 1 & 0 & 1 & \cdots \\ 1 & 1 & 1 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

In this case, $\widehat{\mathbf{N}}$ contains one accumulation point, $\widehat{1}$, and $\widehat{n} \to \widehat{1}$. Thus, $\widehat{\mathbf{N}}$ is compact. Upper semicontinuity follows from lemma 13 and the fact that $t_{11} = 1$ (note that since $t_{nn} = 0$ for every $n > 1$, we have no continuity).

We add to every class $\hat{a} \in \widehat{\mathbf{N}}$ a state $\infty_{\hat{a}}$. Set $\mathbf{N}_\infty = \widehat{\mathbf{N}} \cup \left\{\infty_{\widehat{a}} : \hat{a} \in \widehat{\mathbf{N}}\right\}$. Let $\mathbf{A}_\infty$ be the extension of $\mathbf{A}$ given by $\forall a \in \widehat{\mathbf{N}} \ \ \infty_{\widehat{a}} \underset{\mathbf{A}_\infty}{\sim} a$. It easily follows that $\mathbf{A}_\infty$ is intrinsically compact iff $\mathbf{A}$ is. Set

$$X_\infty = \Sigma^+_{\mathbf{A}_\infty}. \tag{22}$$

For every $a \in \mathbf{N}$ set

$$a_{\min} = \min\{a' : a' \sim a\} = \min \hat{a} \tag{23}$$

and define $\psi : \mathbf{N} \to [0,1]$ by

$$\psi(a) = \begin{cases} 0 & a = \infty_{\widehat{a}} \\ \dfrac{1}{a + a_{\min}} & a \neq \infty_{\widehat{a}} \end{cases}.$$

ψ is 1-1 on each equivalence class and $\psi(a) \underset{a\to\infty}{\longrightarrow} 0$. We use the same notation for d and its natural extension to $\widehat{\mathbf{N}}_\infty$. Set

$$\begin{aligned} &\forall a, b \in \mathbf{N}_\infty \quad \delta(a,b) = d(\hat{a}, \hat{b}) + |\psi(a) - \psi(b)| \\ &\forall x, y \in X_\infty \quad \rho(x,y) = \sum_{n=0}^{\infty} \frac{1}{2^n} \delta(x_n, y_n) \end{aligned} \tag{24}$$

Lemma 15 *ρ is a compact metric.*

Proof. It is easy to see that ρ is a metric (note that ψ is 1-1 on equivalence classes whence $\rho(x,y) = 0$ implies $x = y$). It is not difficult to show that $(\mathbf{N}_\infty, \delta)$ is compact. The upper semicontinuity property of intrinsically compact matrices implies that X_∞ is closed in $\Omega = \prod_{n=1}^{\infty} (\mathbf{N}_\infty, \delta)$. Compactness easily follows. □

Lemma 16 *$\mathcal{B}_\rho = \mathcal{B}(X_\infty, \rho)$, the Borel σ-algebra with respect to ρ, is equal to $\mathcal{B} = \mathcal{B}(X_\infty)$, the σ-algebra generated by the cylinders.*

Proof. trivial. □

Let $\phi : X \to [-\infty, \infty)$ be a locally Hölder continuous function (in particular, if $\phi(x) = -\infty$ then $\phi = -\infty$ in $[x_0]$). Set $\phi_a = \sup_{x\in[a]} \phi(x)$.

Lemma 17 *If $\mathbf{A}$ is intrinsically compact, ϕ is locally Hölder continuous and $\|L_\phi 1\|_\infty < \infty$ then*

$$\sum_{a=1}^{\infty} e^{\phi[a]} < \infty.$$

Proof. By the big images property of intrinsically compact matrices, there exist $b_1,\dots,b_N$ such that $\forall a\ \exists i\ t_{ab_i}=1$. Fix $x^{(i)}\in[b_i]$. For every a there is an $1\le i_a\le N$ such that $ax^{(i_a)}\in X$ and so

$$\begin{aligned}\sum_{a=1}^{\infty} e^{\phi[a]} &\le e^{V_1(\phi)}\sum_{a=1}^{\infty} e^{\phi\left(ax^{(i_a)}\right)}\\ &\le e^{V_1(\phi)}\sum_{k=1}^{N}\sum_{a:\, i_a=k} e^{\phi\left(ax^{(k)}\right)}\\ &\le e^{V_1(\phi)}\sum_{k=1}^{N}\sum_{a\in P^1(x^{(k)})} e^{\phi\left(ax^{(k)}\right)}\\ &= e^{V_1(\phi)}\sum_{k=1}^{N}(L_\phi 1)\left(x^{(k)}\right)\end{aligned}$$

whence $\sum_{a=1}^{\infty} e^{\phi[a]}<\infty$. □

Set $C_\rho(X_\infty)=C(X_\infty,\rho)$, the space of ρ-continuous functions. By ρ-compactness, every $\rho-$continuous function is bounded, hence $\left(C_\rho(X),\|\cdot\|_\infty\right)$ is a well defined Banach space. The following lemma is the raison d'être of the construction:

Proposition 5 *Let* $\mathbf{A}$ *be an intrinsically compact matrix,* X_∞ *and* ρ *given by (22) and (24). For every locally Hölder continuous* $\phi: X_\infty\to[-\infty,\infty)$ *which is* $\rho-$*continuous on partition sets and for which* $\|L_\phi 1\|_\infty<\infty$, $L_\phi\left(C_\rho(X_\infty)\right)\subseteq C_\rho(X_\infty)$.

Proof. Assume that $\phi\not\equiv-\infty$ (else L_ϕ sends every function to 0 and there is nothing to prove). Fix $f\in C_\rho(X_\infty)$ and $\varepsilon>0$. Then,

$$|(L_\phi f)(x)-(L_\phi f)(y)|=\left|\sum_{a\in P^1(x)} e^{\phi(ax)}f(ax)-\sum_{a\in P^1(y)} e^{\phi(ay)}f(ay)\right|$$

$$\le\underbrace{\|f\|_\infty\sum_{a\in P^1(x)\triangle P^1(y)} e^{\phi[a]}}_{\mathrm{I}}+\underbrace{\left|\sum_{a\in P^1(x)\cap P^1(y)}\left(e^{\phi(ax)}f(ax)-e^{\phi(ay)}f(ay)\right)\right|}_{\mathrm{II}}.$$

By lemma 17 there exists an N large enough such that

$$\sum_{a=N+1}^{\infty} e^{\phi[a]}<\frac{\varepsilon}{\|f\|_\infty}.$$

There exists a $\delta>0$ small enough, such that if $d(\widehat{x_0},\widehat{y_0})<\delta$ then $P^1(x)\triangle P^1(y)\subseteq(N,\infty)$. Since $d(\widehat{x_0},\widehat{y_0})\le\rho(x,y)$,

$$\rho(x,y)<\delta\Rightarrow \mathrm{I}<\varepsilon.$$

The estimation of II is standard and is left to the reader. □

Theorem 9 *Let X be a topologically mixing countable Markov shift given by an intrinsically compact transition matrix, and let (X_∞, ρ) be given by (22),(24). Let $\phi : X \to \mathbf{R}$ be a locally Hölder continuous function such that $\|L_\phi 1\|_\infty < \infty$. Assume that ϕ can be extended to a function $\phi : X_\infty \to \mathbf{R} \cup \{-\infty\}$ which is $\rho-$continuous on partition sets and equal to $-\infty$ outside of $\bigcup_{a=1}^{\infty}[a]$. Then ϕ is positive recurrent.*

Proof. Construct d, δ and ρ as before. Set $\mathcal{P} = \{\mu \in (C_\rho(X_\infty))^* : \mu(1) = 1, \mu \geq 0\}$. We follow Ruelle [15] and introduce the function $G : \mathcal{P} \to \mathcal{P}$ defined by

$$Gm = \frac{L_\phi^* m}{m(L_\phi 1)}.$$

By proposition 5 $L_\phi(C_\rho(X_\infty)) \subseteq C_\rho(X_\infty)$ whence G is w^*-continuous (note that $\phi \neq -\infty$ on $\bigcup_{a\in\mathbf{N}}[a]$ whence $L_\phi 1 > 0$). By the Schauder-Tychonoff theorem (see Dunford and Schwartz [6]), G has a fixed point $\nu \in (C_\rho(X'))^*$, $G\nu = \nu$. Since (X_∞, ρ) is Hausdorff and compact, ν determines a unique $\rho-$Borel probability on X_∞. By lemma 16 ν is a Borel probability. By its definition, it is clear that

$$\forall f \in C_\rho(X_\infty) \quad \nu(L_\phi f) = \lambda\, \nu(f) \tag{25}$$

where $\lambda = \nu(L_\phi 1)$. A standard regularity argument shows that (25) is valid for every $L^1(\nu)$ function. It is not difficult to deduce from (25) that ν is supported on X, because

$$X = \bigcap_{n=0}^{\infty} \left\{x \in X_\infty : e^{\phi_n(x)} > 0\right\}.$$

It follows from (25) that $\lambda^{-1} L_\phi$ operates as the transfer operator of ν, whence by lemma 10, $\phi - \log\lambda = \log\frac{d\nu}{d\nu\circ T}$. In particular, $\log\frac{d\nu}{d\nu\circ T}$ is locally Hölder continuous. It follows from (21) and the fact that supp$\nu = X$, that $\inf\{\nu(Ta) : a \in \mathbf{N}\} > 0$. By lemma 2.1 in [1], $\exists h > 0$ such that $h\,d\nu$ is invariant, whence ν is conservative. By lemma 10, ν is exact whence $h\,d\nu$ is exact and strong mixing. The positive recurrence of ϕ follows just like in the proof of theorem 8. □

Example 1 *Let $X = \sum_{\mathbf{A}}^{+}$ where $\mathbf{A}$ is intrinsic compact. Let $f : [0,1] \to [0,\infty)$ be C^1 function such that*

$$\sup_x \left|\frac{f'(x)}{f(x)}\right| < \infty \ ; \ \sum_{a=1}^{\infty} f\left(\frac{1}{a+x}\right) \in L^\infty[0,1]$$

The function $\phi : X \to \mathbf{R}$ given by $\phi(x_0, x_1, \ldots) = \log f\left(\frac{1}{x_0 + \frac{1}{x_1 + \cdots}}\right)$ is positive recurrent.

Proof. For every $x_0, x_1, \ldots \in \mathbf{N} \cup \{\infty_{\widehat{a}} : a \in \mathbf{N}\}$ set $n = \inf\{k : x_k \in \mathbf{N}_\infty \backslash \mathbf{N}\}$ and

$$[x_0, x_1, \ldots] = \begin{cases} 0 & n = 0 \\ \frac{1}{x_0 + \frac{1}{\cdots \frac{1}{x_{n-1}}}} & 0 < n < \infty \\ \frac{1}{x_0 + \frac{1}{x_1 + \cdots}} & n = \infty \end{cases}$$

Use $\log f\,[x_0, x_1, \ldots]$ to extend ϕ. □

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