Phase Transitions for Countable Markov Shifts

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13 February 2000

Abstract

We study the analyticity of the topological pressure for some one-parameter families of potentials on countable Markov shifts. We relate the non-analyticity of the pressure to changes in the recurrence properties of the system. We give sufficient conditions for such changes to exist and not to exist. We apply these results to the Pomeau-Manneville map, and use them to construct examples with different critical behavior.[1]

1 Introduction

A well known theorem of Ruelle [**Ru2**, **Ru1**] states that for every topologically mixing topological Markov shift X with a finite number of states, the topological pressure P_{top} is analytic on the space of Hölder continuous functions. That is, $\forall\phi,\psi\in C(X)$ Hölder continuous, $t\mapsto P_{\mathrm{top}}(\phi+t\psi)$ is real analytic in a neighborhood of $t=0$ (whence for every t). In ferromagnetism, this is sometimes interpreted as '*lack* of phase transitions' (see [**E**]).

If the number of states is *countable*, this theorem is no longer true. [**S3**] contains an example of a ϕ which depends on a finite number of coordinates ('finite range potential') for which $P_{top}(\phi+t\phi)$ has a positive Lebesgue measure set of critical points. Other finite range examples with critical behavior can be found in [**Hof**], [**Lo**], [**W1**], [**W2**]. Infinite range examples include the Pomeau-Manneville map (see e.g. [**PM**], [**Lo**]) and the Farey map [**PS**] (see also [**LSV**]).

The purpose of this paper is to study critical phenomena for some smooth one-parameter families of infinite range potentials on countable Markov shifts. The critical phenomena we consider are non-analyticity of the pressure, changes in the existence of an equilibrium measure, and changes in its finiteness, when it exists.

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*Part of a dissertation prepared in the Tel-Aviv University

[1] *2000 Mathematics Subject Classification*: 37D35, 37C30, 37B10. *Key Words*: thermodynamic formalism, phase transitions, countable Markov shifts, topological pressure, intermittency, Pomeau-Manneville map.

It was observed in [**S2**], that there are three modes of recurrence for potentials on countable Markov shifts: positive recurrence, null recurrence and transience. Positive recurrent potentials admit finite equilibrium measures. Null recurrent potentials admit conservative infinite equilibrium measures. Transient potentials do not have conservative conformal measures. A change in the mode of recurrence of a one-parameter family affects, therefore, the existence or finiteness of the equilibrium measure.

We show that a change in the mode of recurrence is also related with non-analyticity of the pressure, and give conditions governing the existence of such changes (theorems 2 and 3). We use these results to derive some of the properties of the Pomeau-Manneville map, and show that all systems with the same symbolic structure have similar properties. This explains why the Pomeau-Manneville map has the same critical behavior as that of the examples considered in [**PS**], [**Lo**], [**W1**] and [**W2**] (theorem 5). We also construct examples with different critical behavior, using the methods of [**S3**]. Among these examples is a potential which is 'intermittent' (i.e. admits infinite conservative equilibrium measure) for a whole interval of 'temperatures' (example 4). This is different than the Pomeau-Manneville example, which is intermittent only for a specific 'temperature'.

The structure of the paper is as follows. Section 2 contains a survey of relevant results on the thermodynamic formalism of countable Markov shifts. Section 3 contains the statement of our main results, theorems 2, 3 and 4. Section 4 contains an application of these results to the study of the renewal shift and the closely related Pomeau-Manneville map. Section 5 contains other examples. Section 6 contains the proof of theorem 2. Section 7 contains the proof of theorems 3 and 4.

2 Survey of the thermodynamic formalism for countable Markov shifts

In this section we survey some results from [**S1**, **S2**] concerning the thermodynamic formalism of some infinite range potentials on countable Markov shifts. For a survey on finite range potentials see [**GS**].

2.1 Basic Definitions and notational conventions

Let S be a countable set and $A = (t_{ij})_{S\times S}$ a matrix of zeroes and ones with no columns or rows which are all zeroes. Let X be the set

$$X := \left\{x \in S^{\mathbf{N}\cup\{\mathbf{0}\}} : t_{x_i x_{i+1}} = 1 \, , \, \forall i \geq 0\right\}$$

endowed with the relative product topology, which is also given by the base of *cylinders*

$$[a_0, \ldots, a_{n-1}] := \{x \in X : x_i = a_i \, , \, 0 \leq i \leq n-1\}$$

where $n \in \mathbf{N}$ and $a_0, \ldots, a_{n-1} \in S$. An admissible word is a $\underline{a} \in S^n$ such that $[\underline{a}] \neq \emptyset$. Its length is $|\underline{a}| = n$. Let $T : X \to X$ be the *left shift* $(Tx)_i := x_{i+1}$. The topological dynamical system (X, T) is called a *(one sided) topological Markov shift.* We say that X is topologically mixing if (X, T) is topologically mixing. The members of S are called the *states* of the shift, and the matrix A is called the *transition matrix*. The sets $[a]$ where $a \in S$ are called the *partition sets.*

Let $\phi : X \to \mathbf{R}$ be some real function (also called *potential*). The *variations* of ϕ are $V_n(\phi) := \sup\{|\phi(x) - \phi(y)| : x, y \in X\ ,\ x_i = y_i\ ,\ 0 \leq i \leq n-1\}$. ϕ is said to have *summable variations* if $\sum_{n \geq 2} V_n(\phi) < \infty$. ϕ is called *weakly Hölder continuous* (with parameter θ) if there exist $A > 0$ and $\theta \in (0,1)$ such that for all $n \geq 2$, $V_n(\phi) \leq A\theta^n$. Note that in both cases the quantification begins with $n = 2$ so ϕ may be unbounded or may satisfy $V_1(\phi) = \infty$.

For every ϕ with summable variations we associate the corresponding *Ruelle operator* [**Ru2**] $(L_\phi f)(x) := \sum_{Ty=x} e^{\phi(y)} f(y)$. If $\|L_\phi 1\|_\infty < \infty$ this is a bounded operator on the Banach space $C_B(X) = \{f \in C(X) : \|f\|_\infty < \infty\}$ (complex valued functions). One checks that $(L_\phi^n f)(x) = \sum_{T^n y = x} e^{\phi_n(y)} f(y)$ where $\phi_n := \sum_{k=0}^{n-1} \phi \circ T^k$.

We use the following notational conventions. All logarithms are natural logarithms. The indicator functions of sets $A \subseteq X$ are denoted by 1_A, and $1 := 1_X$. $a = B^{\pm n} b$ means that $B \geq 1$, $a, b > 0$ and $B^{-n} b \leq a \leq B^n b$; $a_n \asymp b_n$ means that $\exists B \forall n$, $a_n = B^{\pm 1} b_n$; $a_n \propto b_n$ means that $\exists c \neq 0$ such that $a_n / b_n \to c$; and $a_n \sim b_n$ means that $a_n / b_n \to 1$.

2.2 Pressure and Recurrence

Let $a \in S$ be some fixed state and set $\varphi_a(x) := 1_{[a]}(x) \inf\{n \geq 1 : T^n(x) \in [a]\}$ (where $\inf \emptyset := \infty$ and $0 \cdot \infty = 0$). Set

$$Z_n(\phi, a) := \sum_{T^n x = x} e^{\phi_n(x)} 1_{[a]}(x) \quad \text{and} \quad Z_n^*(\phi, a) := \sum_{T^n x = x} e^{\phi_n(x)} 1_{[\varphi_a = n]}(x)$$

It is known that if X is topologically mixing, ϕ has summable variations[2] , and $\|L_\phi 1\|_\infty$ is finite then the following limit exists, is finite and is independent of the choice of $a \in S$

$$P_G(\phi) := \lim_{n \to \infty} \frac{1}{n} \log Z_n(\phi, a) \tag{1}$$

$P_G(\phi)$ is called the *Gurevic pressure* of ϕ ([**S1**], [**G2**], [**G1**]) and satisfies the following variational principle

$$P_G(\phi) = \sup \left\{ h_\mu(T) + \int \phi d\mu : \ \mu \in \mathcal{P}_T(X) \quad ; \quad - \int \phi d\mu < \infty \right\}$$

[2] the following results, including theorem 1 below, were stated in [**S1**] and [**S2**] under stronger continuity assumptions on ϕ, but the proofs given there are also valid for ϕ with summable variations.

where $\mathcal{P}_T(X)$ is the set of T-invariant Borel probability measures.

Let $\lambda := \exp[P_G(\phi)]$. We say that ϕ is *recurrent* if for some $a \in S$, $\sum_{n\geq 1} \lambda^{-n} Z_n(\phi, a)$ diverges and *transient* if it converges. We say that ϕ is *positive recurrent* if it is recurrent and $\sum_{n\geq 1} n\lambda^{-n} Z_n^*(\phi, a) < \infty$ and *null recurrent* if it is recurrent and $\sum_{n\geq 1} n\lambda^{-n} Z_n^*(\phi, a) = \infty$. It turns out that these definitions do not depend on the choice of $a \in S$ and that [**S2**]:

Theorem 1 *Let X be a topologically mixing countable Markov shift, and let ϕ be some real function on X with summable variations. If ϕ has finite Gurevic pressure, then ϕ is recurrent if and only if there exist $\lambda > 0$, a conservative measure ν finite and positive on cylinders, and a positive continuous function h such that $L_\phi^* \nu = \lambda \nu$ and $L_\phi h = \lambda h$. In this case $\lambda = \exp P_G(\phi)$ and there exist $a_n \uparrow \infty$ such that for every cylinder $[\underline{a}]$ and $x \in X$*

$$\frac{1}{a_n} \sum_{k=1}^{n} \lambda^{-k} \left(L_\phi^k 1_{[\underline{a}]} \right)(x) \underset{n\to\infty}{\longrightarrow} h(x)\nu[\underline{a}]$$

The sequence $\{a_n\}_{n>0}$ satisfies $a_n \sim \left(\int_{[a]} h d\nu\right)^{-1} \sum_{k=1}^{n} \lambda^{-k} Z_k(\phi, a)$ for every $a \in S$. Furthermore,

1. *if ϕ is positive recurrent then $\nu(h) < \infty$, $a_n \propto n$, and for every $[\underline{a}]$ $\lambda^{-n} L_\phi^n 1_{[\underline{a}]} \underset{n\to\infty}{\longrightarrow} h\nu[\underline{a}]$ uniformly on compacts, where h is normalized so that $\nu(h) = 1$.*
2. *if ϕ is null recurrent then $\nu(h) = \infty$, $a_n = o(n)$, and for every $[\underline{a}]$ $\lambda^{-n} L_\phi^n 1_{[\underline{a}]}$ tends to zero uniformly on cylinders.*

It is easy to check that $hd\nu$ is T-invariant, that h is bounded away from zero and infinity on partition sets and that $V_n(\log h) \leq \sum_{k\geq n+1} V_k(\phi)$. It is also clear from the convergence part of the theorem that ν and h are unique up to a multiplicative constant. As a corollary we obtain,

Lemma 1 *Let X be topologically mixing and let ϕ be a function with summable variations and finite Gurevic pressure. Then there exist two continuous functions ϕ' and φ such that $\phi' \leq 0$, $P_G(\phi') = 0$ and $\phi' = \phi + \varphi - \varphi \circ T - P_G(\phi)$. The function φ is bounded on partition sets. If ϕ is recurrent then $L_{\phi'} 1 = 1$, and if ϕ is transient then $L_{\phi'} 1 \leq 1$. If ϕ is weakly Hölder continuous then so are ϕ' and φ.*

Proof. Set $\lambda := \exp P_G(\phi)$. Assume that ϕ is transient. Fix some state $a \in S$ and set $h := \sum_{n\geq 1} \lambda^{-n} L_\phi^n 1_{[a]}$. By transience, topological mixing and summable variations h is finite. Also, if ϕ is weakly Hölder, then so are $\log h$ and $\log h \circ T$. It is easy to check that $\lambda^{-1} L_\phi h \leq h$. Set $\varphi := \log h$ and $\phi' := \phi + \varphi - \varphi \circ T - P_G(\phi)$. Clearly, $L_{\phi'} 1 \leq 1$ whence $\phi' \leq 0$ as required. The case when ϕ is recurrent is handled by replacing h in the last argument by the h given by theorem 1 (see [**Wal**] for a similar normalization procedure). □

3 Statement of main results

We recall the well-known process of *inducing* in the context of topological Markov shifts (see [**S2**] and [**A**] section 1.5). Fix some state $a \in S$. Set $\overline{S} := \{[\underline{a}] : |\underline{a}| \geq 1 \ ; \ a_i = a \text{ iff } i = 0 \ ; \ [\underline{a}, a] \neq \emptyset\}$, $\overline{X} := \overline{S}^{\mathbf{N}\cup\{0\}}$ and let $\overline{T} : \overline{X} \to \overline{X}$ be the left shift. For every $\phi : X \to \mathbf{R}$ set

$$\overline{\phi} := \left(\sum_{k=0}^{\varphi_a - 1} \phi \circ T^k \right) \circ \pi$$

where $\pi : \overline{X} \to [a]$ is given by $\pi([\underline{a}_0], [\underline{a}_1], \ldots) := (\underline{a}_0, \underline{a}_1, \ldots)$. The pair $(\overline{X}, \overline{\phi})$ is called the *induced system* and $\overline{\phi}$ is called the *induced potential* (on $[a]$).

Induced systems are in many cases easier to handle than the original systems, as shown by the following example. A system (X, ϕ) is called *Bernoulli* if $X = S^{\mathbf{N}\cup\{0\}}$ and if $\phi(x) = \phi(x_0)$. A potential is called *Markov* if $\phi(x) = \phi(x_0, x_1)$. If ϕ is a Markov potential, then $(\overline{X}, \overline{\phi})$ is a Bernoulli system.[3]

If ϕ is weakly Hölder continuous, so is $\overline{\phi}$. Summable variations alone, however, is not enough: $\overline{\phi}$ may not have summable variations, even if ϕ does. The existence of pressure, however, is always guaranteed:

Lemma 2 *Let X be topologically mixing and let $\phi : X \to \mathbf{R}$ be some function with summable variations. Let $a \in S$ be some fixed state and let $\overline{X}$ and $\overline{\phi}$ be the induced system and induced potential. Then the following limit exists for all $[\underline{a}] \in \overline{S}$ (although it may be infinite) and is independent of the choice of $[\underline{a}]$:*

$$P_G(\overline{\phi}) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(\overline{\phi}, [\underline{a}])$$

Proof. Follows from the proof of theorem 1 in [**S1**] and the standard estimation $V_n[\overline{\phi} + \overline{\phi} \circ \overline{T} + \ldots + \overline{\phi} \circ \overline{T}^{n-1}] \leq \sum_{k=2}^{\infty} V_k[\phi]$. □

To state our main results, we need the following definition.

Definition 1 *Let X be topologically mixing and let $\phi : X \to \mathbf{R}$ have summable variations and finite Gurevic pressure. Fix $a \in S$ and let $(\overline{X}, \overline{\phi})$ be the induced system. Set $p_a^*[\phi] := \sup\{p : P_G(\overline{\phi + p}) < \infty\}$. The* a-discriminant *of ϕ is $\Delta_a[\phi] := \sup\{P_G(\overline{\phi + p}) : p < p_a^*[\phi]\} \leq \infty$.*

As we shall later see (section 6, proposition 3),

$$\Delta_a[\phi] = P_G(\overline{\phi + p_a^*[\phi]}) \tag{2}$$

[3] This is also true for the larger class of potentials ϕ for which $\exists a \in S$ such that $\phi(x) = \phi(x_0, \ldots, x_{\varphi_a(x)})$ as long as the inducing is done with respect to $[a]$. The state a can be viewed as a 'gap' between non-interacting clusters of interacting particles. Analogous potentials are studied in a different mathematical setting in [**FF**].

Both $\Delta_a[\phi]$ and $p_a^*[\phi]$ are determined by $\sum \xi^n Z_n^*(\phi, a)$ in the following way. Let R be the radius of convergence of this series. Then

$$\left|\Delta_a[\phi] - \log \sum_{k=1}^{\infty} R^k Z_k^*(\phi, a)\right| \leq \sum_{k=2}^{\infty} V_k(\phi) \tag{3}$$

$$p_a^*[\phi] = -\limsup_{n\to\infty} \frac{1}{n} \log Z_n^*(\phi, a) \tag{4}$$

Both relations follow from the stronger statement (section 6, proposition 3):

$$\left|P_G(\overline{\phi + p}) - \log \sum_{k=1}^{\infty} e^{kp} Z_k^*(\phi, a)\right| \leq \sum_{k=2}^{\infty} V_k(\phi) \tag{5}$$

Note that when ϕ is a Markov potential, both (5) and (3) are equalities (because for Markov potentials $\sum_{k\geq 2} V_k(\phi) = 0$). Our basic result is:

Theorem 2 (Discriminant theorem). *Let X be a topologically mixing countable Markov shift and let $\phi : X \to \mathbf{R}$ be some function with summable variations and finite Gurevic pressure. Let $a \in S$ be some arbitrary fixed state.*

1. *The equation $P_G(\overline{\phi + p}) = 0$ has a unique solution $p(\phi)$ if $\Delta_a[\phi] \geq 0$, and no solution if $\Delta_a[\phi] < 0$. The Gurevic pressure of ϕ is given by*

$$P_G(\phi) = \begin{cases} -p(\phi) & \Delta_a[\phi] \geq 0 \\ \\ -p_a^*[\phi] & \Delta_a[\phi] < 0 \end{cases} \tag{6}$$

2. *ϕ is positive recurrent if $\Delta_a[\phi] > 0$ and transient if $\Delta_a[\phi] < 0$. In the case $\Delta_a[\phi] = 0$, ϕ is either positive recurrent or null recurrent.*

Theorem 2 should be understood in the context of one-parameter families of potentials. Given such a family $\{\phi_\beta\}$, let $\{\Delta_\beta\}$ be the corresponding one-parameter family of discriminants. When Δ_β changes sign, $\{\phi_\beta\}$ changes its recurrence properties and the case in (6) changes. A change in the mode of recurrence implies, by theorem 1, a change in the qualitative properties of the equilibrium measure (existence and finiteness). A change of case in (6) may imply non-smoothness for $\beta \mapsto P_G(\phi_\beta)$. This suggests that the search for critical phenomena for one-parameter families may be done by studying the sign changes of the discriminant. This can sometimes be done with the aid of (3), as we shall see in sections 4 and 5. The proof of theorem 2 is given in section 6.

We now discuss the case when the discriminant does not change sign and remains positive. Let ϕ be some function with summable variations and finite pressure. We say that ϕ is *strongly positive recurrent* if for some state $a \in S$

$$\Delta_a[\phi] > 0$$

(This generalizes the notion of *stable positivity* for Markov potentials discussed in [**GS**].) The Discriminant Theorem implies that every strongly positive recurrent function is positive recurrent. The opposite statement is false (example 2 below).

We are interested in differentiability of the pressure functional, i.e. in the existence of directional derivatives $\frac{d}{dt}\big|_{t=0} P_G(\phi + t\psi)$. We restrict ourselves to the following set of directions:

$$Dir(\phi) := \left\{ \psi : \sum_{n=2}^{\infty} V_n(\psi) < \infty, \quad \exists \varepsilon > 0 \text{ s.t. } \forall |t| < \varepsilon,\ P_G(\phi + t\psi) < \infty \right\}$$

The following theorem completes theorem 2 by saying that if the discriminant is positive, then there is no critical phenomena of the sort that can be encountered when Δ changes sign. Its proof of is given in section 7.

Theorem 3 *Let X be a topologically mixing and ϕ be a weakly Hölder continuous function such that $P_G(\phi) < \infty$. If ϕ is strongly positive recurrent then $\forall \psi \in Dir(\phi)$ weakly Hölder continuous, $\exists \varepsilon > 0$ such that $\phi + t\psi$ is positive recurrent for all $|t| < \varepsilon$ and such that $t \mapsto P_G(\phi + t\psi)$ is real analytic in $(-\varepsilon, \varepsilon)$.*

The case $\psi = \phi$ is particularly interesting, as it appears in the study of the one-parameter family $\{\beta\phi\}_{\beta \geq \beta_0}$.[4] If $P_G(\beta_0 \phi) < \infty$, then $P_G(\beta\phi) < \infty$ for all $\beta > \beta_0$, because by lemma 1, ϕ is cohomologous to a non-positive function. Therefore, $\forall \beta > \beta_0$, $\phi \in Dir(\beta\phi)$. This may not be true for $\beta = \beta_0$:

Example 1 *Let $X = \mathbf{N}^{\mathbf{N} \cup \{0\}}$ and $\phi(x) := -\log\left(x_0 (\log 2x_0)^2\right)$. Then $P_G(\beta\phi) < \infty$ for $\beta \geq 1$, and $P_G(\phi) = \infty$ for $\beta < 1$.*

Proof. $P_G(\beta\phi) = \log \sum_{k \geq 1} 1/[k^\beta (\log 2k)^{2\beta}]$. □

Corollary 1 *Let X be a topologically mixing and ϕ be weakly Hölder continuous function such that $P_G(\phi) < \infty$ and $\phi \in Dir(\phi)$. The following conditions are equivalent:*

1. *ϕ is strongly positive recurrent.*
2. *for every weakly Hölder continuous $\psi \in Dir(\phi)$ there exists $\varepsilon > 0$ such that $\phi + t\psi$ is positive recurrent for every real t such that $|t| < \varepsilon$.*
3. *for every $a \in S$ $\Delta_a[\phi] > 0$.*

Proof. The first statement implies the second by theorem 3. The third statement trivially implies the first. It remains to show that the second statement implies the third. Assume that the second statement is true, but that the third statement is false. Then for some $a \in S$, $\Delta_a[\phi]$ is not positive. Since by our assumptions ϕ is positive recurrent, $\Delta_a[\phi]$ cannot be negative, so $\Delta_a[\phi] = 0$. Set $\psi := 1_{[a]}$. Since $\psi = 1$ on $\overline{X}$, $\Delta_a[\phi] = t$. This contradicts the second statement because if $t < 0$ then $\phi + t\psi$ is transient. □

We remark that the assumptions of theorem 3 can be weakened:

[4] Such families appear in models for systems whose inverse temperature β is changed [**E**].

Theorem 4 *Let X be a topologically mixing and ϕ be a function with summable variations, such that $P_G(\phi) < \infty$, $\Delta_a[\phi] > 0$ and such that the induced potential on a, $\overline{\phi}$, is weakly Hölder. Then $\forall \psi \in Dir(\phi)$ such that $\overline{\psi}$ is weakly Hölder continuous, $\exists \varepsilon > 0$ such that $\phi + t\psi$ is positive recurrent $\forall |t| < \varepsilon$, and such that $t \mapsto P_G(\phi + t\psi)$ is real analytic in $(-\varepsilon, \varepsilon)$.*

In section 7 we prove this stronger version.

4 The renewal shift

The examples studied in [**PM**],[**Hof**],[**GW**], [**W1**],[**W2**], [**PS**] and [**Lo**] share the same critical behavior: for some potential ϕ, the function $\beta \mapsto P_G(\beta\phi)$ has one point of non-differentiability β_c, and is constant for $\beta > \beta_c$. A close look at these examples shows that they can be represented as different potentials on the same countable Markov shift, the *renewal shift*. This is the shift with set of states $S := \mathbf{N} \cup \{0\}$ and transition matrix $(t_{ij})_{S\times S}$ whose 1 entries are t_{00}, t_{0i} and $t_{i,i-1}$ $(i = 1, 2, 3, \ldots)$. The main result of this section is

Theorem 5 *Let X be the renewal shift and let $\phi : X \to \mathbf{R}$ be a function with summable variations such that $\sup \phi < \infty$ and such that $\overline{\phi}$ is weakly Hölder continuous. Then there exists $0 < \beta_c \leq \infty$ such that:*

1. *$\beta\phi$ is strongly positive recurrent for $0 < \beta < \beta_c$ and transient for $\beta > \beta_c$.*
2. *$P_G(\beta\phi)$ is real analytic in $(0, \beta_c)$ and linear in (β_c, ∞). It is continuous but not analytic at β_c (in case $\beta_c < \infty$).*
3. *Set $A_n := e^{\sup\{\phi_n(x): x \in [0,n-1,\ldots,0]\}}$ and let $R(\beta)$ be the radius of convergence of $F_\beta(\xi) := \sum_{n\geq 1} A_n^\beta \xi^n$. If $F_\beta(R(\beta))$ is infinite for every β then $\beta_c = \infty$. If $\exists \beta > 0$ such that $F_\beta(R(\beta)) < 1$ then $\beta_c < \infty$.*

Proof. It is easy to check that X it topologically mixing. Also, $\beta\phi$ has finite pressure for all $\beta \geq 0$, since $P_G(\beta\phi) \leq \log \|L_{\beta\phi} 1\| \leq \log(2e^{\beta \sup \phi})$. One can easily check that for every function f, $n \in \mathbf{N}$ and $\beta > 0$

$$Z_n^*(\beta f, 0) = Z_n^*(f, 0)^\beta \quad \text{and} \quad p_0^*[\beta f] = \beta p_0^*[f] \tag{7}$$

Henceforth $(\overline{X}, \overline{\phi})$ denotes the induced system on $[0]$, $P(\beta) := P_G(\beta\phi)$ and $\Delta[\beta] := \Delta_0[\beta\phi]$.

If $p_0^*[\phi] = \infty$ then $\Delta[\beta] = \sup\{P_G(\overline{\phi + p}) : p < \infty\} = \infty$ because $P_G(\overline{\phi + p}) \geq P_G(\overline{\phi}) + p$. In this case parts 1 and 2 follow with $\beta_c = \infty$ from theorem 4 and the discussion after theorem 3. We therefore restrict ourselves to the case $p_a^*[\phi] < \infty$.

Without loss of generality, assume that $p_0^*[\phi] = 0$ (else pass to $\phi + p_0^*[\phi]$ and use (4)). By (7), $p_0^*[\beta\phi] = 0$ for all $\beta > 0$, whence by (2)

$$\Delta[\beta] = P_G(\overline{\beta\phi}) \tag{8}$$

As before, if $\Delta[\beta] > 0$ for every β, parts 1 and 2 follow with $\beta_c = \infty$. Assume $\exists \beta > 0$ such that $\Delta[\beta] \le 0$ and set $\beta_c := \inf\{\beta > 0 : \Delta[\beta] \le 0\}$. Note that $\beta_c > 0$ because according to (3) and (7)

$$\Delta[\beta] \ge \log \sum_{n=1}^{\infty} \left(e^{np_0^*[\phi]} Z_n^*(\phi, 0)\right)^{\beta} - \beta \sum_{n=2}^{\infty} V_n(\phi) \underset{\beta \to 0^+}{\longrightarrow} +\infty$$

We claim that $\Delta[\beta] \to -\infty$ as $\beta \uparrow +\infty$. Fix some β_0 such that $\Delta[\beta_0] \le 0$. By (8) $\overline{\beta_0 \phi}$ has finite pressure, whence by lemma 1, $\overline{\beta_0 \phi}$ is cohomologous to $\overline{\psi} + P_G(\overline{\beta_0 \phi})$ where $\overline{\psi}$ is weakly Hölder continuous (in $\overline{X}$) such that $L_{\overline{\psi}} 1 \le 1$. Since ϕ has summable variations, $V_1(\overline{\phi}) < \infty$. It follows from lemma 1 that $V_1(\overline{\psi}) < \infty$ as well. By (8), for all $t > 1$,

$$\Delta[t\beta_0] = P_G(\overline{t\beta_0 \phi}) = P_G(\overline{t\psi}) + tP_G(\overline{\beta_0 \phi})$$

Since $P_G(\overline{\beta_0 \phi}) = \Delta[\beta_0] \le 0$, we have for all $t > 1$,

$$\Delta[t\beta_0] \le P_G(\overline{t\psi}) \le \log ||L_{\overline{t\psi}} 1||_\infty$$

By construction, $L_{\overline{\psi}} 1 \le 1$. Therefore, since every $x \in \overline{X}$ has more than one pre-image, $\overline{\psi}$ is strictly negative. It follows from this and $V_1(\overline{\psi}) < \infty$ that $||L_{\overline{t\psi}} 1||_\infty \to 0$ as $t \uparrow \infty$. This implies that $\Delta[\beta] \to -\infty$ as $\beta \uparrow \infty$.

We show that $\Delta[\beta] < 0$ in $(\beta_c, +\infty)$. By the definition of β_c there are $\beta_n \downarrow \beta_c$ such that $\Delta[\beta_n] \le 0$. By what we just showed there are $\beta_n' \uparrow \infty$ such that $\Delta[\beta_n'] < 0$. By (8) $\Delta[\beta] = P_G(\overline{\beta \phi})$, so $\Delta[\beta]$ is convex in (β_n, β_n'). By convexity, $\Delta[\beta] < 0$ in $(\beta_n, \beta_n']$. Since $\beta_n \downarrow \beta_c$ and $\beta_n' \uparrow \infty$, $\Delta[\beta]$ is strictly negative in $(\beta_c, +\infty)$.

We have shown that $\Delta[\beta] < 0$ in (β_c, ∞). It is obvious that $\Delta[\beta] > 0$ in $(0, \beta_c)$. Part 1 now follows from the discriminant theorem.

We prove part 2. The analyticity of $P(\beta)$ in $(0, \beta_c)$ follows from theorem 4 and that fact that $P(\beta) < \infty$. The discriminant theorem and (7) imply that $\forall \beta > \beta_c$, $P_G(\beta\phi) = p_0^*[\beta\phi] = \beta p_0^*[\phi]$ and $\forall \beta \in (0, \beta_c)$, $P_G(\beta\phi) > p_0^*[\beta\phi] = \beta p_0^*[\phi]$. Thus $P_G(\beta\phi)$ is linear in (β_c, ∞), but not in $(0, \infty)$. This implies that β_c is a point of non-analyticity. The continuity of $P(\beta)$ in β_c follows from the convexity of this function.

To prove part 3, recall that $\Delta[\beta] > 0$ for $\beta > 0$ small, and note that by (8) that $\log F_\beta(R(\beta)) - \beta \sum_{n=2}^{\infty} V_n(\phi) \le \Delta[\beta] \le \log F_\beta(R(\beta))$. □

Example 2 *$\beta_c \phi$ can be positive recurrent, null recurrent or transient.*

Proof. Let $\{f_n\}_{n \ge 1}$ be a sequence such that $f_n > 0$ and $\log f_n = o(n)$. Set $\phi := \sum_{n \ge 1} 1_{[0, n-1]} \log f_n$. Then, $Z_n^*(\phi, 0) = f_n$, $p_0^*[\phi] = 0$ and $\sum_{n \ge 2} V_n(\phi) = 0$, whence by (2), $\Delta_0[\beta\phi] = \log \sum_{n \ge 1} f_n^{\beta}$. Let $\zeta(s) := \sum_{n \ge 1} n^{-s}$.

1. *Positive recurrence.* Set $f_n := \frac{1}{\zeta(3)n^3}$. Then $\Delta_0[\beta\phi] = \log[\zeta(3\beta)/\zeta(3)^\beta]$ whence $\beta_c = 1$. Note that $\Delta_0[\beta_c\phi] = 0$ whence $P_G(\beta_c\phi) = -p_0^*[\beta_c\phi] = 0$. It also follows that $\beta_c\phi$ is recurrent. Positive recurrence follows from $\sum_{n\geq 1} ne^{-nP_G(\phi)} Z_n^*(\phi, 0) = \sum_{n\geq 1} n/(\zeta(3)n^3) < \infty$. Note that $\beta_c\phi$ is positive recurrent but not strongly positive recurrent.

2. *Null recurrence.* The same calculations with $f_n = 1/(\zeta(2)n^2)$ then $\beta_c\phi$.

3. *Transience.* Set $f_n := C/n[\log(2n)]^2$ where C is a constant such that $\sum_{n\geq 1} f_n = \frac{1}{2}$. Similar calculations show that $\Delta_0[\beta\phi]$ is infinite for $\beta < 1$ and $\Delta_0[\beta\phi] \leq -\log 2$ for $\beta \geq 1$. Thus $\beta_c = 1$ and $\beta_c\phi$ is transient. □

For an example of the possible applications of theorem 5, consider the Pomeau-Manneville map $T : [0,1] \to [0,1]$ given by $T(x) = x + x^{1+s} (\mathrm{mod}\, 1)$ where the value of T at its discontinuity is 0, $T(1) = 1$ and $s > 0$ [**PM**]. The following proposition, is a generalization of results which are known for $f = -\log|T'|$ (see [**PM**] and [**Lo**]) to other potentials, whose equilibrium measure is not necessarily equivalent to Lebesgue's measure.

Proposition 1 (The Pomeau–Manneville Model). *Let T be the Pomeau-Manneville map and let $f : [0,1] \to \mathbf{R}$ be $C[0,1] \cap C^1(0,1]$ such that $f'(x) \sim c\alpha x^{\alpha-1}$ as $x \searrow 0$, where $c \neq 0$ and $\alpha > 0$. Set*

$$P(\beta) := \sup\left\{h_m(T) + \beta\int f dm : m \in \mathcal{P}_T([0,1]);\ -\int f dm < \infty;\ m\{0\} = 0\right\}$$

1. *There exists $0 < \beta_c \leq \infty$ such that $P(\beta)$ is real analytic in $(0, \beta_c)$ and linear in (β_c, ∞). It is continuous but not real-analytic at β_c.*

2. *β_c is finite if and only if $\alpha \leq s$ and $c < 0$. In particular, it is finite for $f := -\log T'$.*

Proof. It is common knowledge that T can be described symbolically as a renewal shift. We check that the symbolic representation of f has summable variations and apply theorem 5. To do this we recall some facts on the natural Markov partition of T (see [**I**], lemma 4.8.6 in [**A**] and [**T1**]).

Define by induction $c_0 := 1$ and $c_n = c_{n+1} + c_{n+1}^{1+s}$. Rewriting this as $c_{n+1}^{-s} = c_n^{-s}(1 + c_{n+1}^s)^s = c_n^{-s}(1 + sc_{n+1}^s + o(c_{n+1}^s))$ we see that $c_{n+1}^{-s} - c_n^{-s} \sim s$ whence $c_n \sim (sn)^{-1/s}$. It follows from the recursive relation which defines $\{c_n\}$ that

$$c_n - c_{n+1} \sim \frac{1}{(sn)^{1+1/s}}$$

Set $I[n] := (c_{n+1}, c_n]$ and $I[a_0, \ldots, a_{n-1}] := \bigcap_{k=0}^{n-1} T^{-k} I[a_k]$. One checks that $TI[0] = (0,1]$ and $TI[n+1] = I[n]$, whence $I[a_0, \ldots, a_{n-1}]$ is not empty if and only if $(a_0, \ldots, a_{n-1})$ is an admissible word of the renewal shift.

Claim 1. *The diameter of* $I[a_0,\dots,a_{n-1}]$ *satisfies for every* $\varepsilon>0$

$$|I[a_0,\dots,a_{n-1}]| = O\left(\frac{1}{n^{1+1/s-\varepsilon}}\right) \tag{9}$$

Proof. By the previous discussion,

$$I[a_0,\dots,a_{n-1}] = I[a_0,\dots,a_{n-1};a_{n-1}-1,\dots,0]$$

so we may assume that $a_{n-1}=0$. Set $M:=1+\sup\{a_k\}$ and $N:=|\{k : a_k=0\}|$. Since $(a_0,\dots,a_{n-1})$ is admissible with respect to the transition matrix of the renewal shift, $MN\ge n$. Thus, for every $\beta\in(0,1)$ either $M\ge n^{\beta}$ or $N\ge n^{1-\beta}$.

Set $m_n:=\lfloor n^{\beta}\rfloor+1$. If $M\ge n^{\beta}$ then for some power k, $T^k I[a_0,\dots,a_{n-1}]\subseteq I[m_n]$ whence since $T'\ge 1$ $|I[a_0,\dots,a_{n-1}]|\le|I[m_n]|=c_{m_n}-c_{m_n+1}=O(n^{-\beta(1+1/s)})$. If $N\ge n^{1-\beta}$ then for every $x\in I[a_0,\dots,a_{n-1}]$

$$(T^n)'(x)=\prod_{i=0}^{n-1}T'(T^ix)\ge\left(\inf_{x\in I[0]}T'(x)\right)^N$$

whence for $\theta:=1/\inf_{x\in I[0]}T'(x)$

$$|I[a_0,\dots,a_{n-1}]|\le\theta^N \tag{10}$$

Since $\theta<1$ and $N\ge n^{1-\beta}$ we have again that $I[a_0,\dots,a_{n-1}]=O(n^{-\beta(1+1/s)})$. Since $\beta\in(0,1)$ was arbitrary, the claim is proved.

Let (X,σ) be the renewal shift and $\pi_0: X\to[0,1]$ be the map defined by the equation $\{\pi_0(x)\}=\bigcap_{n\ge0}I[x_0,\dots,x_{n-1}]=\bigcap_{n\ge0}\overline{I[x_0,\dots,x_{n-1}]}$. By (9) π_0 is well defined. It is easy to check that $\pi_0\circ\sigma=T\circ\pi_0$, that π_0 is 1-1 and that $\pi_0(X)=[0,1]\setminus\cup_{n\ge0}T^{-n}\{0\}$.

Claim 2. *Let* f *be* $C[0,1]\cap C^1(0,1]$ *in* $[0,1]$ *such that* $f'(x)\sim c\alpha x^{\alpha-1}$ *as* $x\downarrow 0$, *where* $c\ne0$ *and* $\alpha>0$. *Then* $\phi:=f\circ\pi_0$ *has summable variations and* $\overline{\phi}$, *the induced potential on* $[0]$, *is weakly Hölder continuous.*

Proof. Fix $x,y\in[a_0,\dots,a_{n-1}]$ where without loss of generality $a_{n-1}=0$. Fix $\varepsilon>0$ (to be determined later). Then there exists $\xi\in I[a_0,\dots,a_{n-1}]$ such that

$$|\phi(x)-\phi(y)|=|f'(\xi)|\cdot|I[a_0,\dots,a_{n-1}]|=O\left(\frac{\xi^{\alpha-1}}{n^{1+1/s-\varepsilon}}\right)$$

Since $\xi\in I[a_0,\dots,a_{n-1}]\subseteq(c_{a_0+1},c_{a_0})$, and since by the structure of the renewal shift $a_0\le n-1$, we have that $\xi^{\alpha-1}=O(1+c_n^{\alpha-1})$ whence

$$V_n(\phi)=O\left(\frac{1+n^{-(\alpha-1)/s}}{n^{1+1/s-\varepsilon}}\right)$$

If $\alpha \geq 1$ the nominator is bounded and choosing $\varepsilon < 1/(2s)$ we see that $V_n(\phi)$ is summable. If $\alpha < 1$ then the nominator is $O(n^{(1-\alpha)/s})$ and we have that $V_n(\phi) = O(n^{-(1+\alpha/s-\varepsilon)})$. Choosing $\varepsilon < \alpha/s$ we see the $\sum V_n(\phi)$ is summable. In any case, ϕ has summable variations. The weak Hölder continuity of $\overline{\phi}$ can be proved in a similar way.

Claim 3. $P(\beta) = P_G(\beta\phi)$ *where* $\phi := f \circ \pi_0$.

Proof. Since $\sup f < \infty$ and $\forall x\ |T^{-1}x| = 2$, $\|L_{\beta\phi}1\|_\infty < \infty$. It follows as in ([**S1**], theorem 3) that

$$P_G(\beta\phi) = \sup\{h_\mu(\sigma) + \beta\int \phi d\mu : \quad \mu \in \mathcal{P}_\sigma(X) \quad ; \quad -\int \phi d\mu < \infty\}$$

(the argument there works also for functions with summable variations).

The claim follows because $m \leftrightarrow m \circ \pi_0$ is a 1-1 onto correspondence between the sets of measures which define $P(\beta)$ and $P_G(\beta\phi)$.

Claims 2 and 3 show that we can apply theorem 5 to $\phi = f \circ \pi_0$ and deduce the existence of β_c. We check the conditions for the finiteness of β_c. Let A_n be as in theorem 5. Since ϕ has summable variations, $A_n \asymp \exp f_n(d_n)$ where $d_n \in I[0]$ are defined by $T(d_n) = c_n$. It is easy to check that $d_n \downarrow c_1$, whence $A_n \asymp \exp\sum_{k=1}^n f(c_k)$. Without loss of generality, $f(0) = 0$ (addition of constants does not affect the finiteness of β_c). Then by the assumptions on f, $f(x) \sim cx^\alpha$. Since $c_k \sim (sk)^{-1/s}$, $f(c_k) \sim c(sk)^{-\alpha/s}$. Thus $\sum_{k=1}^n f(c_k) \asymp c\int_1^n x^{-\alpha/s}dx$. It follows that there exist constants K_1, K_2, K_3, K_4 such that

$$K_1 \exp\left(K_2\beta c\int_1^n \frac{1}{x^{\alpha/s}}dx\right) \leq A_n^\beta \leq K_3 \exp\left(K_4\beta c\int_1^n \frac{1}{x^{\alpha/s}}dx\right)$$

Let $F_\beta(\xi)$ and $R(\beta)$ be as in theorem 5. Using the above,

1. If $\alpha > s$ then $A_n^\beta \asymp 1$ for every $\beta > 0$. In this case $F_\beta(R(\beta)) = \infty$ for every β, so $\beta_c = \infty$.

2. If $\alpha = s$ then $K_1 n^{K_2\beta c} \leq A_n^\beta \leq K_3 n^{K_4\beta c}$. It follows that $R(\beta) = 1$ and that $F(R(\beta))$ is infinite for every β if $c > 0$, and $F(R(\beta)) \underset{\beta\to\infty}{\longrightarrow} 0$ if $c < 0$. Thus for $\alpha = s$, if $c > 0$ then β_c is infinite, and if $c < 0$ then $\beta_c < \infty$.

3. If $\alpha \in (0, s)$ and $a := 1 - (\alpha/s)$ then for some constants C_1, C_2, C_3, C_4, $C_1 e^{C_2\beta c n^a} \leq A_n^\beta \leq C_3 e^{C_4\beta c n^a}$. Since $a < 1$, $R(\beta) = 1$ for every β. It follows that if $c > 0$ then $F(R(\beta)) = \infty$ for every β, and if $c < 0$ then $F(R(\beta)) \underset{\beta\to\infty}{\longrightarrow} 0$. Thus for $0 < \alpha < s$, if $c > 0$ then β_c is infinite and if $c < 0$ then β_c is finite.

Thus $\beta_c < \infty$ if and only if $0 < \alpha \leq s$ and $c < 0$. □

5 Other Examples

In this section we construct examples whose critical behavior is different than that of potentials on the renewal shift. Our constructions are based on the tools of [**S3**] which we now explain. We say that a one parameter family of functions $F_\beta(\xi)$ is an *exponent power series* if it is of the form $F_\beta(\xi) = \sum_{n,k\geq 0} a_{nk}^\beta \xi^n$ where $a_{nk} \geq 0$. Clearly, if F_β and G_β are exponent power series, then so are $F_\beta G_\beta$, $F_\beta \circ G_\beta$ and $c_1 F_\beta + c_2 G_\beta$ where c_1, c_2 are positive integers. We say that an exponent power series F_β is *aperiodic* if the power expansion of F_β contains two co-prime powers of ξ. We say that F_β is *adequate* if it is of the form $c^\beta \xi + \xi^2 G_\beta(\xi)$ where $c \geq 0$ and G_β is an exponent power series.

The following theorem was essentially proved in [**S3**]. We include its proof for completeness.

Theorem 6 *For every adequate exponent power series F_β there exists an irreducible topological Markov shift X and a Markov potential $\phi = \phi(x_0, x_1)$ such that for all β, $P_G(\overline{\beta\phi + p}) = \log F_\beta(e^p)$. If F_β is aperiodic, X is topologically mixing.*

Proof. Write

$$F_\beta(\xi) = c^\beta \xi + \sum_{n=2}^{\infty} \xi^n \sum_{k=1}^{N_n} a_{nk}^\beta$$

where $0 \leq N_n \leq \infty$. Let S be a countable set indexed in the following way

$$S := \{a\} \cup \bigcup_{n=2}^{\infty} \bigcup_{k=1}^{N_n} \{b_{nk}(1), \ldots, b_{nk}(n-1)\}$$

Let $(t_{ij})_{i,j\in S}$ be the transition matrix whose non zero entries are exactly $t_{a,b_{nk}(1)}$, $t_{b_{nk}(i)b_{nk}(i+1)}, t_{b_{nk}(n-1)a}$ for all $n, k \geq 1$ and $i = 1, \ldots, n-1$ with the addition of t_{aa} if and only if $c \neq 0$. Let X be the corresponding topological Markov shift. Define $\phi(x)$ by $\phi(x) := \log a_{nk}$ if $x \in [a, b_{nk}(1)]$, $\phi(x) := \log c$ if $x \in [a, a]$ and $\phi(x) := 0$ otherwise. One checks that

$$F_\beta(\xi) = \sum_{n=1}^{\infty} \xi^n Z_n^*(\beta\phi, a)$$

whence by (5) and the fact that $\forall k \geq 2\ V_k(\phi) = 0$, $P_G(\overline{\beta\phi + p}) = \log F_\beta(e^p)$. Note that X is irreducible, because all states connect to a and a connects to all states. It is topologically mixing if and only if there are two words of co-prime lengths which connect a to a. This can be easily seen to be equivalent to the aperiodicity of $\sum \xi^n Z_n^*(\beta\phi, a)$, hence to that of F_β. □

The following example shows that $\{\beta\phi\}_{\beta>0}$ can change from recurrent to transient an infinite number of times. (This is different than the example with infinite number of non differentiability points in [**S3**], which is always transient.)

Example 3 *There exists X topologically mixing and $\phi = \phi(x_0, x_1)$ such that for some $\beta_n \downarrow 0$, $\beta\phi$ is recurrent in (β_{i+1}, β_i) for i even and transient for i odd.*

Proof. Consider the following sequence of numbers

$$N_n := \frac{2}{n} \binom{2n-2}{n-1}$$

A calculation with Stirling's formula shows that $N_n \sim \pi^{-1/2} n^{-3/2} 2^{2n-1}$. Another calculation shows that

$$4^{-n} N_n = \frac{1}{4^{n-1}} \binom{2n-2}{n-1} - \frac{1}{4^n} \binom{2n}{n} \tag{11}$$

Multiplying both sides of (11) by 4^n we see that N_n are all natural numbers. Summing both sides of (11) over n we see that $\sum_{n \geq 2} N_n 4^{-n} = \frac{1}{2}$.

Fix some $\beta_n \downarrow 0$ with the property that $\sum \theta^{1/\beta_n} < \infty$ for all $\theta \in (0,1)$ (e.g. $\beta_n := 1/n$). Set $\alpha_n(\beta) := -2(\frac{1}{2})^{\beta/\beta_n}$ and $p(\beta) := \prod_{n \geq 1}(1 + \alpha_n(\beta))$. Then for all $\beta > 0$, $p(\beta)$ is well defined, non zero for $\beta \notin \{\beta_n\}_n$ and satisfies

$$p(\beta) = 1 + \alpha_1(\beta) + [1 + \alpha_1(\beta)]\, \alpha_2(\beta) + \ldots$$

where the convergence on the right is absolute. Collecting summands with the same sign write $p(\beta) = A(\beta) - B(\beta)$ where $A(\beta) = \sum a_n^\beta$ and $B(\beta) = \sum b_n^\beta$ for some $a_n, b_n \geq 0$. If $\beta \in (\beta_{i+1}, \beta_i)$ then $\beta > \beta_n$ iff $n \geq i+1$ whence

$$sgn\,(A(\beta) - B(\beta)) = sgn \left(\prod_{n=1}^{i} (1 - 2^{1-\beta/\beta_n}) \prod_{n=i+1}^{\infty} (1 - 2^{1-\beta/\beta_n}) \right) = (-1)^i$$

Thus, $A(\beta) > B(\beta)$ iff i is even.

Now construct X and $\phi = \phi(x_0, x_1)$ such that $P_G(\overline{\beta\phi + p}) = \log F_\beta(e^p)$ where $F_\beta(\xi)$ is the exponent power series

$$F_\beta(\xi) = 8A(\beta)B(\beta)\xi^2 + \sum_{n=2}^{\infty} N_n B(\beta)^n \xi^n$$

Since $N_n \asymp \frac{4^n}{n^{3/2}}$ the radius of convergence of $F_\beta(\xi)$ is $R(\beta) = 1/4B(\beta)$ whence

$$\Delta_a[\beta\phi] = \log F_\beta(R(\beta)) = \log \left(8 \frac{A(\beta)B(\beta)}{16B(\beta)^2} + \sum_{n=2}^{\infty} N_n 4^{-n} \right)$$

whence $\Delta_a[\beta\phi] = \log \frac{1}{2}(1 + A(\beta)/B(\beta))$. This is positive iff $A(\beta) > B(\beta)$. Thus $\beta\phi$ is recurrent for $\beta \in (\beta_{i+1}, \beta_i)$ and i even, and transient for $\beta \in (\beta_{i+1}, \beta_i)$ and i odd. $\square$

We have seen that for potentials ϕ on the renewal shift, $\beta\phi$ can be null recurrence for at most one value of β (the critical point). Our next example shows that for other topological Markov shifts null recurrence can occur in an entire interval. A trivial example would be a Markov shift for which the potential $\phi \equiv 0$ is null recurrent. We therefore restrict ourselves to examples where ϕ is not cohomologous to a constant.

Example 4 *There exist a topologically mixing topological Markov shift X and a function $\phi = \phi(x_0, x_1)$ not cohomologous to a constant such that $\beta\phi$ is null recurrent for every β.*

Proof. Let N_n be as in example 3 and set $f_\beta(p) := 2^\beta(e^p + e^{2p})$. Construct X topologically mixing and $\phi = \phi(x_0, x_1)$ such that

$$P_G(\overline{\beta\phi + p}) = \log\left(2\sum_{n=2}^{\infty} N_n f_\beta(p)^n\right)$$

Since $N_n \asymp 4^n n^{-3/2}$, $p_a^*[\beta\phi]$ is determined by the equation $f_\beta(p_a^*[\beta\phi]) = 1/4$. It follows from this that $\Delta_a[\beta\phi] = 0$. By the Discriminant theorem, for all β $\beta\phi$ is recurrent and $P_G(\beta\phi) = -p_a^*[\beta\phi]$. It also follows that ϕ is not cohomologous to a constant, since $P_G(\beta\phi)$ is not a linear function of β (it is given by the equation $f_\beta[-P_G(\beta\phi)] = 1/4$).

We show that $\beta\phi$ is null recurrent for all β. Since $V_2(\phi) = 0$, $P_G(\overline{\beta\phi + p}) = \log\sum_{n\geq 1} e^{np} Z_n^*(\beta\phi, a)$ whence

$$\sum_{n=1}^{\infty} n e^{-nP_G(\beta\phi)} Z_n^*(\beta\phi, a) = \frac{d}{dp}\bigg|_{p=p_a^*} e^{P_G(\overline{\beta\phi+p})} = 2f_\beta'(p_a^*[\beta\phi])\sum_{n=2}^{\infty} nN_n 4^{-(n-1)}$$

and this diverges, because $N_n \asymp 4^n n^{-3/2}$. □

Our last example shows that all modes of recurrence can co-exist for interval ranges of inverse temperatures.

Example 5 *There exist X topologically mixing and $\phi = \phi(x_0, x_1)$ such that for some $1 < \beta_1 < \beta_2 < \infty$, $\beta\phi$ is null recurrent for $\beta \in (1, \beta_1)$, positive recurrent for $\beta \in (\beta_1, \beta_2)$ and transient for $\beta \in (\beta_2, \infty)$.*

Proof. Fix some positive $a_n \sim 1/[2n(\log n)^2]$ such that $a_1 = \frac{3}{4}$, $\sum_{n\geq 1} a_n = 1$ and set $A(\beta) = \sum_{n\geq 1} a_n^\beta$, $F_\beta(\xi) := \sum_{n\geq 1} a_n^\beta A(\beta)^n \xi^{n+1}$ and

$$G_\beta(\xi) = F_\beta(2F_\beta(\xi))$$

This is an adequate aperiodic exponent power series. Let X and ϕ be the corresponding shift and potential.

Let $R_F(\beta)$ and $R_G(\beta)$ denote the radii of convergence of $F_\beta(\xi)$ and $G_\beta(\xi)$. Note that $R_F(\beta) = 1/A(\beta)$ and $F_\beta(R_F(\beta)) = 1$. Let β_2 be the solution of $R_F(\beta_2) = 2$. Clearly, $R_F(\beta) < 2$ for $\beta < \beta_2$ and $R_F(\beta) > 2$ for $\beta > \beta_2$. Thus,

1. if $\beta \in (1, \beta_2)$ then $2F_\beta(R_F(\beta)) = 2 > R_F(\beta)$ so $R_G(\beta) = F_\beta^{-1}(\frac{1}{2}R_F(\beta))$. In this case $G_\beta(R_G(\beta)) = F_\beta(2 \cdot \frac{1}{2}R_F(\beta)) = F_\beta(R_F(\beta)) = 1$.
2. if $\beta > \beta_2$ then $2F_\beta(R_F(\beta)) = 2 < R_F(\beta)$ so $R_G(\beta) = R_F(\beta)$. In this case $G_\beta(2F_\beta(R_F(\beta)) < F_\beta(R_F(\beta)) = 1$.

Since $\Delta_a[\beta\phi] = \log G_\beta(R_G(\beta))$, $\beta\phi$ is transient for $\beta > \beta_2$ and recurrent for $\beta \in (1, \beta_2)$.

We check positive recurrence and null recurrence for $\beta \in (1, \beta_2)$. Fix some $\beta \in (1, \beta_2)$. Since $G_\beta(R_G(\beta)) = 1$, $\Delta_a[\beta\phi] = 0$ whence $e^{-P_G(\beta\phi)} = e^{-p_a^*[\beta\phi]} = R_G(\beta)$. Thus $\sum_{n\geq 1} ne^{-nP_G(\beta\phi)} Z_n^*(\phi, a) = R_G(\beta) \left.\frac{d}{d\xi}\right|_{\xi = R_G(\beta)} G_\beta(\xi)$. Now, since $R_G(\beta) = F_\beta^{-1}(\frac{1}{2}R_F(\beta))$,

$$\left.\frac{d}{d\xi}\right|_{\xi = R_G(\beta)} F_\beta(2F_\beta(\xi)) = F'_\beta(R_F(\beta)) \cdot 2F'_\beta(R_G(\beta))$$

Since $R_G(\beta) < R_F(\beta)$, this is finite iff $F'_\beta(R_F(\beta)) < \infty$, which is comparable to

$$\frac{1}{A(\beta)} \sum_{n=2}^{\infty} \frac{n}{2^\beta n^\beta (\log n)^{2\beta}}$$

This sum is infinite for $\beta \in (1, 2)$ and finite for $\beta \in (2, \beta_2)$. (Note that $\beta_2 > 2$ since $a_1^2 > \frac{1}{2}$ whereas $a_1^{\beta_2} < A(\beta_2) = \frac{1}{2}$.) $\square$

6 Proof of theorem 2

The proof of theorem 2 is based on a generalization of certain renewal theoretic ideas. These are presented in the following subsection. The proof of theorem 2 is given in the subsection following it.

6.1 A renewal sequence of operators

Let $a \in S$ be some fixed state. Let $C_B[a]$ be the Banach space $C_B[a] := \{f \in C_B(X) : f(x) = 0 \text{ for } x \notin [a]\}$ equipped with the supremum norm. Let $1, 0 : C_B[a] \to C_B[a]$ be the operators defined by $1f = f$, $0f = 0$ $\forall f \in C_B[a]$. Consider the operators $T_n,\ R_n : C_B[a] \to C_B[a]$ given by $T_0 := 1$, $R_0 := 0$ and

$$\begin{aligned} T_n f \quad &:= 1_{[a]} L_\phi^n f \\ R_n f \quad &:= 1_{[a]} L_\phi^n (f 1_{[\varphi_a = n]}) \end{aligned}$$

(see also [**FL**] and [**PS**].) A direct calculation shows that these operators satisfy the following 'renewal equation' for $n \geq 1$,

$$\begin{aligned} T_n \quad &= R_1 T_{n-1} + R_2 T_{n-2} + \ldots + R_n T_0 \\ T_n \quad &= T_{n-1} R_1 + T_{n-2} R_2 + \ldots + T_0 R_n \end{aligned} \tag{12}$$

Set

$$T_a[\phi](z) := 1 + \sum_{n=1}^{\infty} z^n T_n \quad , \quad R_a[\phi](z) := \sum_{n=1}^{\infty} z^n R_n$$

These are well defined bounded linear operators on $C_B[a]$ for $|z| < \lambda^{-1}$. To see this use the summable variations property to prove that $\|T_a[\phi](z)\| = \left\|T_a[\phi](z)1_{[a]}\right\|_\infty \leq B \sum_{n\geq 0} |z|^n Z_n(\phi, a)$ where $B := \exp \sum_{n\geq 2} V_n(\phi)$, and note that the radius of convergence of the series $\sum z^n Z_n(\phi, a)$ is λ^{-1} by (1). In terms of these generating functions, we can restate (12) in the following form $\forall |z| < \lambda^{-1}$,

$$T_a[\phi](z) = [1 - R_a[\phi](z)]^{-1}$$

It also follows from (12) that for all $|z| < \lambda^{-1}$,

$$T_a[\phi](z) = 1 + \sum_{n=1}^{\infty} R_a[\phi](z)^n \tag{13}$$

Note that (13) is also valid for all z real such that $z \geq \lambda^{-1}$, as long as both sides are applied to positive functions.

For every bounded linear operator S on $C_B[a]$ let $\rho(S)$ denote the spectral radius of S (with respect to the supremum norm), with the convention that the 'operator' $Sf = \infty 1_{[a]}$ has an infinite spectral norm. The following two propositions relate the renewal sequence to the discriminant.

Proposition 2 *Let $\overline{\phi}$ denote the induced potential on $[a]$. Then*

$$P_G(\overline{\phi + p}) = \log \rho\left(R_a[\phi](e^p)\right)$$

Proof. Let $\pi : \overline{X} \to [a]$ be the natural embedding. A calculation shows that for every p and $f \in C_B[a]$

$$(R_a[\phi](e^p)f) \circ \pi = L_{\overline{\phi+p}}(f \circ \pi) \tag{14}$$

Fix some $[\underline{a}] \in \overline{S}$. Since $\overline{X} = \overline{S}^{\mathbf{N}\cup\{0\}}$, $Z_n(\overline{\phi + p}, [\underline{a}]) \asymp \|L^n_{\overline{\phi+p}} 1\|_\infty$. The proposition follows from this and (14). □

Proposition 3 *Let X be topologically mixing countable Markov shift, let ϕ be a function with summable variations and finite Gurevic pressure. Let $\overline{X}$ and $\overline{\phi}$ denote the induced pair with respect to $a \in S$. Then $P_G(\overline{\phi + p})$ is convex, strictly increasing and continuous in $(-\infty, p_a^*[\phi]]$. Also, (2)-(5) hold.*

Proof. Fix $a \in S$ and set $\gamma(p) := P_G(\overline{\phi + p})$, $p^* := p_a^*[\phi]$, $z := e^p$ and $R(z) := R_a[\phi](z)$. Proposition 2 and summable variations imply that for all $x \in [a]$,

$$\begin{aligned} \gamma(p) &= \lim_{n\to\infty} \frac{1}{n} \log \|R(e^p)^n 1_{[a]}\| \\ &= \lim_{n\to\infty} \frac{1}{n} \log R(e^p)^n 1_{[a]}(x) \end{aligned}$$

We use this formula to prove that γ is convex, strictly increasing and continuous in $(-\infty, p^*)$ and that (5) and (4) hold. The expressions $R(e^p)^n 1_{[a]}(x)$ are convex in p, since they are of the form $\sum a_i e^{b_i p}$ where $a_i \geq 0$. Thus $\gamma(p)$ is convex in p, being a limit of convex functions. Clearly, $\forall p_1 < p_2 < p^*$ and all $0 \leq f \in C_B[a]$, $R(e^{p_2})f \geq e^{p_2 - p_1} R(e^{p_1})f$. Iterating this and using the above formula for $\gamma(p)$ we have that $\gamma(p)$ is strictly increasing. It follows that γ is finite in $(-\infty, p^*)$, whence by convexity it is continuous there. Standard estimations show that for every $0 \leq f \in C_B[a]$, $R(e^p)f = B^{\pm 1} \sum_{n \geq 1} Z_n^*(\phi, a) f$ where $B = \exp \sum_{k \geq 2} V_k(\phi)$. Iterating this, and using the above formula for $\gamma(p)$, we have (5). Clearly, (5) implies (4), so (4) is also proved.

It remains to prove that $\gamma(p)$ is continuous on the left in p^*. We prove this under the assumption that $\gamma(p^*) < \infty$ (the proof for the infinite case is essentially the same). By monotonicity, it is enough to prove that for every $\varepsilon > 0$ there exists $p < p^*$ such that $\gamma(p) > \gamma(p^*) - \varepsilon$. Fix $x \in [a]$. Setting $R(e^p) = \sum_{n \geq 1} e^{np} R_n$ in the above formula for γ, we have

$$\gamma(p) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{k_1, \ldots, k_n = 1}^{\infty} e^{p(k_1 + \ldots + k_n)} R_{k_1} \cdot \ldots \cdot R_{k_n} 1_{[a]}(x) \tag{15}$$

By the definition of p^*, there are N and p and $n > \log B / \varepsilon$ such that

$$a_n := \log \sum_{k_1, \ldots, k_n}^{N} e^{p(k_1 + \ldots + k_n)} R_{k_1} \cdot \ldots \cdot R_{k_n} 1_{[a]} \geq n(\gamma(p^*) - \varepsilon)$$

By the summable variations property, $a_{m_1} + a_{m_2} \leq a_{m_1 + m_2} + \log B$ for every m_1, m_2. Write $m = kn + r$ where $0 \leq r \leq n - 1$. Then

$$\frac{a_m}{m} \geq \frac{k a_n + a_r - (k+1) \log B}{kn + r} \underset{m \to \infty}{\longrightarrow} \frac{a_n}{n} - \frac{\log B}{n} \geq \gamma(p^*) - 2\varepsilon$$

whence $\gamma(p) \geq \gamma(p^*) - 2\varepsilon$. This proves that γ is continuous in $(-\infty, p^*]$. This also implies that (2). This and (5) imply (3). □

We will need the following version of the Kac formula, whose proof follows in a standard way from theorem 1 and general results for Markov operators (see theorem VI.C in [**F**]).

Lemma 3 *Let X be a topologically mixing Markov shift, ϕ some function on X and $a \in S$ some fixed state. Let $(\overline{X}, \overline{\phi})$ be the induced system on $[a]$, and assume that both ϕ and $\overline{\phi}$ have summable variations. Then ϕ is recurrent with pressure zero if and only if $\overline{\phi}$ is positive recurrent with pressure zero. In this case, if $L_\phi^* \nu = \nu, L_\phi h = h, L_{\overline{\phi}}^* \overline{\nu} = \overline{\nu}, L_{\overline{\phi}} \overline{h} = \overline{h}$ then up to a multiplicative constant $\overline{\nu} = \nu \circ \pi$, $\overline{h} = h \circ \pi$, $\nu(A) = \int (\sum_{n=0}^{\infty} \ell^n 1_A) \circ \pi \, d\overline{\nu}$ and $h = \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} L_\phi^k (h_a 1_{[\varphi_a = n]}) \mod \nu$ where ℓ is the operator $\ell(f) := L_\phi(f \cdot 1_{[a]^c})$ and $h_a := 1_{[a]} \overline{h} \circ \pi^{-1}$.*

6.2 Proof of theorem 2.

Throughout the proof let a and ϕ be fixed. Let $T(z) := T_a[\phi](z), R(z) := R_a[\phi](z)$. Let $B := \exp \sum_{k\geq 2} V_k[\phi]$ and set $\Delta := \Delta_a[\phi]$, $\gamma(p) := P_G(\phi + p)$, and $p^* := p_a^*[\phi]$.

Part 1. *proof of (6).*

Assume $\Delta \geq 0$. According to proposition 3, γ is continuous and strictly increasing in $(-\infty, p^*]$, $\gamma(p^*) \geq 0$ and $\gamma(p) \to -\infty$. Therefore, there exists a unique $p_a(\phi)$ for which $\gamma(p_a(\phi)) = 0$.

We claim that $p_a(\phi) = -P_G(\phi)$. Fix some $p < p_a(\phi)$. Since γ is strictly increasing, $\gamma(p) < 0$, whence $\rho(R(e^p)) < 1$. By (13), $\|T(e^p)1_{[a]}\|_\infty \leq 1 + \sum \|R(e^p)^n\| < \infty$ whence, by summable variations, $\sum e^{np} Z_n(\phi, a)$ converges. The radius of convergence of this series is $\exp[-P_G(\phi)]$. Therefore $p \leq -P_G(\phi)$. Taking $p \uparrow p_a(\phi)$ we have $p_a(\phi) \leq -P_G(\phi)$. Assume by way of contradiction that $\exists p_a(\phi) < p' < -P_G(\phi)$. Then $\|T(e^{p'})\| \leq B \sum e^{np'} Z_n(\phi, a) < \infty$ whence by (13), the series $1 + \sum R(e^{p'})^n 1_{[a]}(x)$ converges for every x. Summable variations imply that $\forall x \in [a]$ and $\forall n \geq 1$, $\|R(e^{p'})^n\| \leq BR(e^{p'})^n 1_{[a]}(x)$. Thus $\sum \|R(e^{p'})^n\| < \infty$ whence $\rho[R(e^{p'})] \leq 1$, or equivalently, $\gamma(p') \leq 0$. This, however, is impossible because γ is strictly increasing, $\gamma(p_a(\phi)) = 0$ and $p' > p_a(\phi)$. This proves that $p_a(\phi) = -P_G(\phi)$ and settles (6) for the case $\Delta \geq 0$.

Assume now that $\Delta < 0$. In this case there is no solution for the equation $\gamma(p) = 0$, because for $p \leq p^*$ $\gamma(p) \leq \Delta < 0$ and for $p > p^*$ $\gamma(p) = \infty$. We show that in this case $P_G(\phi) = -p^*$. By (4) and the inequality $Z_n^*(\phi, a) \leq Z_n(\phi, a)$, $p^* \geq -P_G(\phi)$. Assume by way of contradiction that $p^* > -P_G(\phi)$. Then for every $x \in [a]$, $T(e^{p^*})1_{[a]}(x) \geq B^{-1} \sum e^{np^*} Z_n(\phi, a) = \infty$, whence by (13), $1 + \sum R(e^{p^*})^n$ diverges everywhere on $[a]$. Thus $\rho[R(e^{p^*})] \geq 1$, whence $\Delta = \gamma(p^*) \geq 0$ in contradiction to our assumptions. This settles the case $\Delta < 0$.

Part 2. *proof that recurrence is equivalent to* $\Delta \geq 0$.

Assume that $\Delta \geq 0$. By the first part of the theorem $\gamma(-P_G(\phi)) = 0$ so the spectral norm of $R(e^{-P_G(\phi)})$ is equal to 1. Therefore, there exists $\xi \in [0, 2\pi)$ such that $1 - e^{i\xi} R(e^{-P_G(\phi)})$ does not have a bounded inverse operator. In particular, the series $1 + \sum_{k\geq 1} e^{ik\xi} R(e^{-P_G(\phi)})^k$ does not converge in the strong norm. It follows that there exists some $\varepsilon > 0$ such that for every N there exists $n = n(N) > N$ and $g_n \in C_B[a]$ such that $\|g_n\|_\infty = 1$ and $\|\sum_{k\geq n} e^{ik\xi} R(e^{-P_G(\phi)})^k g_n\|_\infty > \varepsilon$. Now, on $[a]$

$$\sum_{k=n}^{\infty} R(e^{-P_G(\phi)})^k 1_{[a]} \geq \sum_{k=n}^{\infty} R(e^{-P_G(\phi)})^k |g_n| \geq \left| \sum_{k=n}^{\infty} e^{ik\xi} R(e^{-P_G(\phi)})^k g_n \right| \geq \varepsilon$$

whence $\|\sum_{k\geq n} R(e^{-P_G(\phi)})^k 1_{[a]}\|_\infty \geq \varepsilon$ for every n. By the summable variations property this is only possible if $\sum R(e^{-P_G(\phi)})^k 1_{[a]}$ diverges on $[a]$ whence $\|T(e^{-P_G(\phi)})1_{[a]}\|_\infty = \infty$. This is equivalent to $\sum e^{-kP_G(\phi)} Z_k(\phi, a) = \infty$, so ϕ is recurrent.

Assume that $\Delta < 0$. Set $\rho := \rho[R(e^{p^*})]$. Then $\rho = \exp\Delta < 1$. By the definition of the spectral radius, there exists some C and $\rho_0 \in (\rho, 1)$ such that for every n, $\|R(e^{p^*})^n\|_\infty < C\rho_0^n$. The renewal equation implies that $\|T(e^{p^*})\| \leq C/(1-\rho_0)$. It follows that $T(e^{p^*})$ is bounded, whence $\sum e^{kp^*} Z_k(\phi, a)$ is convergent. By the first part of the theorem, and since $\Delta < 0$, $p^* = -P_G(\phi)$. It follows that ϕ is transient.

Part 3. *proof that* $\Delta > 0$ *implies positive recurrence.*

Assume that $\Delta > 0$. By what we have just proved ϕ is recurrent. Let ν and h be the eigenmeasure and eigenfunction given by theorem 1, and set $dm = hdm$. Recall that m an invariant measure, and that $m(X) < \infty$ if and only if ϕ is positive recurrent. We will prove positive recurrence by showing that $m(X) < \infty$.

Let ν_a, m_a be the measures $\nu_a(A) := \nu(A \cap [a])/\nu[a]$ and $m_a(A) := m(A \cap [a])/m[a]$. Let $T_a := T^{\varphi_a}$ be the induced transformation. Since m is T-invariant, m_a is T_a-invariant. Note that the transfer operator of T_a with respect to ν_a is $R(\lambda^{-1})$. To see this note that $\forall g \in L^\infty(\nu_a), \forall f \in L^1(\nu_a)$, $\nu_a[gR(\lambda^{-1})f] = \sum_{n=1}^\infty \nu\left[\lambda^{-n} L_\phi^n \left(g \circ T^n \cdot f 1_{[\varphi_a = n]}\right)\right] = \nu_a(g \circ T_a \cdot f)$.

Set $A(x) := \left[R(\lambda^{-1})\varphi_a\right](x)$. By Kac's formula, the fact the $R(\lambda^{-1})$ acts as the transfer operator of ν_a and the boundness of $h = \frac{dm}{d\nu}$ away from zero and infinity on partition sets, $\exists C_1 >$ such that

$$m(X) = \int \varphi_a \, dm_a = C_1^{\pm 1} \int \varphi_a \, d\nu_a = \int_{[a]} A(x) d\nu(x)$$

Clearly,

$$A(x) = \sum_{n=1}^\infty n\lambda^{-n} \sum_{T^n y = x} e^{\phi_n(y)} 1_{[\varphi_a = n]}(y) = B^{\pm 1} \sum_{n=1}^\infty n\lambda^{-n} Z_n^*(\phi, a)$$

Since $\Delta > 0$, $p^* > -P_G(\phi)$ so λ^{-1} is smaller than the radius of convergence of the series $\sum z^n Z_n^*(\phi, a)$. It follows that $\sum n\lambda^{-n} Z_n^*(\phi, a) < \infty$ whence $\|A(x)\|_\infty < \infty$. Since $\nu[a] < \infty$, $m(X) < \infty$ as required. $\square$

7 Proof of theorem 4

In this section we prove theorem 4, a strengthened version of theorem 3.

7.1 Preparatory lemmas

Let X be topologically mixing and let $a \in S$ be some fixed state. Let ϕ be some function with summable variations and finite pressure and let $(\overline{X}, \overline{\phi})$ be the induced system on $[a]$.

For every $\overline{x},\overline{y}\in\overline{X}$ set $\overline{t}(\overline{x},\overline{y}):=\inf\{n\geq 0:\overline{x}_n\neq\overline{y}_n\}$. Fix $\theta\in(0,1)$ and set for every function $f:\overline{X}\to\mathbf{C}$, $Df:=\sup\{|f(\overline{x})-f(\overline{y})|/\theta^{\overline{t}(\overline{x},\overline{y})}:\overline{x}\neq\overline{y}\}$. Let $\overline{\mathcal{L}}=\overline{\mathcal{L}}(\theta,a)$ be the space

$$\overline{\mathcal{L}}(\theta,a):=\{f\in C_B(\overline{X}):\|f\|_{\overline{\mathcal{L}}}:=\|f\|_\infty+Df<\infty\} \tag{16}$$

A standard argument shows that $\overline{\mathcal{L}}$ is a Banach space and that if $\|L_{\overline{\phi}}1\|_\infty<\infty$ then $L_{\overline{\phi}}(\overline{\mathcal{L}})\subset\overline{\mathcal{L}}$ and $\|L_{\overline{\phi}}\|_{\mathcal{B}(\overline{\mathcal{L}})}<\infty$ where $\mathcal{B}(\overline{\mathcal{L}})$ is the space of bounded operators on $\overline{L}$ equipped with the strong operator norm. The following lemma says that the induced system has a spectral gap, and is similar to well-known results in the theory of interval maps with indifferent fixed points ([**T1**],[**T2**], [**A**], [**ADU**], [**B**], [**PS**]).

Lemma 4 *Let X be topologically mixing, ϕ some function on X and $a\in S$ a fixed state. Let $(\overline{X},\overline{\phi})$ be the induced system on $[a]$ and assume that $\overline{\phi}$ is weakly Hölder continuous with exponent $\theta\in(0,1)$ and that $\|L_{\overline{\phi}}1\|_\infty<\infty$. Then $\overline{\phi}$ is positive recurrent if and only if the spectrum of $L_{\overline{\phi}}:\overline{\mathcal{L}}(\theta,a)\to\overline{\mathcal{L}}(\theta,a)$ consists of a simple eigenvalue $\overline{\lambda}$ and a subset of $\{z:|z|<\tau\overline{\lambda}\}$ where $\tau<1$. In this case, $\overline{\lambda}=e^{P_G(\overline{\phi})}$.*

Proof. Assume that $\overline{\phi}$ is positive recurrent with finite pressure and set $\overline{\lambda}:=\exp P_G(\overline{\phi})$. $\overline{X}$ has the structure of a full shift. It is known ([**S1**] section 5, see also [**A**] theorem 4.7.7 and [**Yu**]) that for such systems there exists $K>0$ and $\tau\in(0,1)$ such that for every $f\in\overline{\mathcal{L}}$

$$\left\|\overline{\lambda}^{-n}(L_{\overline{\phi}})^n f-\overline{h}\int f d\overline{\nu}\right\|_{\overline{\mathcal{L}}}<K\tau^n$$

where $L_{\overline{\phi}}\overline{h}=\overline{\lambda}\,\overline{h}$, $L_{\overline{\phi}}^*\overline{\nu}=\overline{\lambda}\overline{\nu}$ and $\overline{\nu}(\overline{h})=1$. This implies the required spectral property. The opposite direction is trivial, since the spectral property implies that $\overline{\lambda}^{-n}L_{\overline{\phi}}^n$ has a non trivial limit (the eigenprojection of $\overline{\lambda}$), and this is only possible if $\overline{\phi}$ is positive recurrent with pressure $\log\overline{\lambda}$. □

Lemma 5 *Let X be topologically mixing and let ϕ be a function with summable variations such that $P_G(\phi)<\infty$ and $\Delta_a[\phi]>0$. For every $\psi\in Dir(\phi)$, $\exists\varepsilon>0$ $\exists r>\exp[-P_G(\phi)]$ such that $\sum_{n\geq 1}nr^nZ_n^*(\phi+\varepsilon|\psi|,a)<\infty$.*

Proof. Without loss of generality, $P_G(\phi)=0$ (else pass to $\phi-P_G(\phi)$). Since $\Delta_a[\phi]>0$, $\exists r>\exp[-P_G(\phi)]=1$ such that $\|R_a[\phi](r)1_{[a]}\|_\infty$ is finite, or equivalently, $\sum_{n\geq 1}r^nZ_n^*(\phi,a)<\infty$. Without loss of generality

$$\limsup_{n\to\infty}\frac{1}{n}\log Z_n^*(\phi,a)<-\log r$$

Set $f_n(t) := (1/n)\log Z_n^*(\phi + t\psi, a)$ and $f(t) := \limsup_{n\to\infty} f_n(t)$. By Hölder's inequality, f_n are convex, whence so is f. Since $\psi \in Dir(\phi)$, there is some $\varepsilon > 0$ such that $\forall |t| < 2\varepsilon$, $-\infty \le f(t) \le P_G(\phi + t\psi) < \infty$.

By convexity and since $f < \infty$, either $f(t) = -\infty$ everywhere in $(-2\varepsilon, 2\varepsilon)$, or $|f(t)| < \infty$ everywhere in $(-2\varepsilon, 2\varepsilon)$. In the first case the radius of convergence of $\sum_{k\ge 1} x^k Z_k^*(\phi + t\psi, a)$ is infinite for $t = \pm\varepsilon$ and we are done. In the second case, by convexity and finiteness, $f(t)$ is continuous in $(-2\varepsilon, 2\varepsilon)$. Thus, since r was chosen so that $f(0) < -\log r$, there exists $\varepsilon' < \varepsilon$ such that $\forall |t| < 2\varepsilon'$ $f(t) < -\log r$. It follows that r is strictly smaller than the radius of convergence of $\sum_{k\ge 1} x^k Z_k^*(\phi + t\psi, a)$ for $t = \pm\varepsilon'$ and again, we are done. □

Recall that function $F : \mathbf{C} \times \mathbf{C} \to \mathcal{B}(\overline{\mathcal{L}})$ is called *analytic* in a neighborhood of (z_0, w_0) if $\exists F_{nk} \in \mathcal{B}(\overline{\mathcal{L}})$ such that $F(z,w) = \sum_{n,k\ge 0}(w-w_0)^k(z-z_0)^k F_{nk}$ and the series converges in the strong operator norm in a neighborhood of (z_0, w_0).

Lemma 6 *Let X be topologically mixing, let ϕ be some function with summable variations such that $P_G(\phi) < \infty$ and let $\psi \in Dir(\phi)$. Let $a \in S$ be some state such that $\Delta_a[\phi] > 0$ and assume that $\overline{\phi}$ and $\overline{\psi}$, the induced potentials on $[a]$, are weakly Hölder continuous with parameter θ. Then $F : \mathbf{C} \times \mathbf{C} \to \mathcal{B}(\overline{\mathcal{L}})$ given by $F(z,w) = L_{\overline{\phi + z\psi + \log w}}$ is analytic in a neighborhood of $(z,w) = (0, e^{-P_G(\phi)})$.*

Proof. Throughout this proof $\|\cdot\|$ denotes the strong operator norm in $\mathcal{B}(\overline{\mathcal{L}})$. We assume, without loss of generality, that $P_G(\phi) = 0$ and prove analyticity in $(0,1)$. For every function $g : \overline{X} \to \mathbf{C}$ let M_g be the operator $M_g f = gf$. Set $\Lambda_n := \{x \in \overline{X} : \varphi_a(\pi(x)) = n\}$. This is a union of partition sets in $\overline{X}$. Set $\overline{R}_n := L_{\overline{\phi}} M_{1_{\Lambda_n}}$. Then,

$$\begin{aligned} F(z,w) &= \sum_{n=1}^{\infty} w^n \overline{R}_n M_{e^{z\overline{\psi}}} \\ &= \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} \frac{w^n z^k}{k!} \overline{R}_n M_{\overline{\psi}^k} \end{aligned}$$

We show that this converges in $\mathcal{B}(\overline{L})$ in some open ball containing $(z,w) = (0,1)$. Fix some N and set $A_N(x) := e^x - (1 + x + \frac{x^2}{2!} + \ldots, \frac{x^N}{N!})$. Then $\left\|\sum_{n,k>N} \frac{1}{k!} w^n z^k \overline{R}_n M_{\overline{\psi}^k}\right\| \le \sum_{n>N} |w|^n \left\|\overline{R}_n M_{A_N(z\overline{\psi})}\right\|$. We estimate the summands of the last series.

For every $\overline{p} \in \overline{S}$ set $Q_{\overline{p}} f(\overline{x}) := f(\overline{p}\,\overline{x})$. Let $\Lambda_n' := \{\overline{p} \in \overline{S} : [\overline{p}] \subseteq \Lambda_n\}$. By definition, $\overline{R}_n M_{A_N(z\overline{\psi})} = \sum_{\overline{p}\in\Lambda_n'} Q_{\overline{p}} M_{A_N(z\overline{\psi})e^{\overline{\phi}}}$. Since for every $f, g \in \overline{\mathcal{L}}$, $\|fg\|_{\overline{\mathcal{L}}} \le \|f\|_{\overline{\mathcal{L}}}\|g\|_{\overline{\mathcal{L}}}$,

$$\left\|Q_{\overline{p}} M_{A_N(z\overline{\psi})\exp\overline{\phi}}\right\| \le \left\|Q_{\overline{p}} e^{\overline{\phi}}\right\|_{\overline{\mathcal{L}}} \|Q_{\overline{p}} A_N(z\overline{\psi})\|_{\overline{\mathcal{L}}}$$

It is standard to check that $\forall x, y \in \mathbf{C}$, $|A_N(x) - A_N(y)| \le |x-y|(e^{|x|} + e^{|y|})$ and that $\forall x, y \in \mathbf{R}$, $|e^x - e^y| \le |x-y|(e^x + e^y)$. Using this and the inequality

$|A_N(x)| \leq e^{|x|}$, it is easy to show that there is some constant K_1 (independent of n and N) such that $\forall |z| < 1$,

$$\left\| Q_{\overline{p}} M_{A_N(z\overline{\psi}) \exp \overline{\phi}} \right\| \leq K_1 \left\| e^{\overline{\phi}} 1_{[\overline{p}]} \right\|_\infty \left\| e^{|z\overline{\psi}|} 1_{[\overline{p}]} \right\|_\infty$$

Summing over all $\overline{p} \in \Lambda'_n$ and using weak Hölder continuity, we have that for some K independent of n and N,

$$\left\| \overline{R}_n M_{A_N(z\overline{\psi})} \right\| \leq K Z_n^*(\phi + |z| \cdot |\psi|, a)$$

Let $\varepsilon > 0$ and $r > 1$ be as in lemma 5. Without out loss of generality $re^{-\varepsilon} > 1$ and $\varepsilon < 1$. Then for all $|z| < \varepsilon$ and $|w| < r$,

$$\left\| \sum_{n=N+1}^{\infty} \sum_{k=N+1}^{\infty} \frac{|w|^n |z|^k}{k!} R_n M_{\psi_n^k} \right\| \leq K \sum_{n=N+1}^{\infty} r^n Z_n^*(\phi + \varepsilon|\psi|, a) \underset{N\to\infty}{\longrightarrow} 0$$

whence $F(z, w)$ is analytic in a neighborhood of $(0, 1)$. □

7.2 Proof of theorem 4

Let ϕ be a function with summable variations and finite pressure and let $\psi \in Dir(\phi)$. Assume that $\exists a \in S$ such that $\Delta_a[\phi] > 0$ and such that the induced potentials on $[a]$, $\overline{\phi}, \overline{\psi}$ are weakly Hölder continuous with exponent $\theta \in (0, 1)$. Without loss of generality, assume that $P_G(\phi) = 0$. Set

$$\Gamma(z, w) := P_G(\overline{\phi + z\psi - w})$$

By the discriminant theorem, $\forall z \in \mathbf{R}$, if $\exists w \in \mathbf{R}$ such that $\Gamma(z, w) = 0$, then $w = P_G(\overline{\phi + z\psi})$. Thus, $P_G(\phi + z\psi)$ is given implicitly by

$$\Gamma(z, P_G(\phi + z\psi)) = 0 \tag{17}$$

We will show that Γ has a complex holomorphic extension to a neighborhood of $(z, w) = (0, 0)$ in $\mathbf{C} \times \mathbf{C}$, and apply the complex implicit function theorem ([**Boch**], page 39) to deduce that (17) defines $P_G(\phi + z\psi)$ real analytically in a neighborhood of $z = 0$. (This theorem applies since $\forall h > 0$, $\Gamma(0, h) \leq P_G(\overline{\phi} - h) = P_G(\overline{\phi}) - h$ whence $\Gamma_w(0, 0) \neq 0$.)

By theorem 2 and lemma 3, since $\Delta_a[\phi] > 0$ and $P_G(\phi) = 0$, $\overline{\phi}$ is positive recurrent with pressure zero. By (5), $\sum Z_n^*(\phi, a) < \infty$ whence $\|L_{\overline{\phi}}\| < \infty$. By lemma 4 the spectrum of $L_{\overline{\phi}} : \overline{\mathcal{L}} \to \overline{\mathcal{L}}$ consists of the simple isolated eigenvalue 1 and a compact subset of $\{z : |z| < \tau\}$ for some $\tau < 1$. By standard analytic perturbation theory [**Ka**], there exists $\delta > 0$ such that if $\|L - L_{\overline{\phi}}\|_{\mathcal{B}(\overline{\mathcal{L}})} < \delta$ then L has a (unique) simple eigenvalue $\lambda(L)$ of maximal magnitude, this eigenvalue is simple, has magnitude larger than $(1 + \tau)/2$, and the rest of the spectrum is contained in $\{z : |z| < (1+\tau)/2\}$. Furthermore, the map $L \mapsto \lambda(L)$ is holomorphic in $\{L \in \mathcal{B}(\overline{\mathcal{L}}) : \|L - L_{\overline{\phi}}\|_{\mathcal{B}(\overline{\mathcal{L}})} < \delta\}$. By lemma 6, $\exists \varepsilon > 0$ such that $(z,w) \mapsto L_{\overline{\phi + z\psi - w}}$ is holomorphic in $U := \{(z,w) \in \mathbf{C}^2 : |z|, |w| < \varepsilon\}$ and such that $\|L_{\overline{\phi+z\psi-w}} - L_{\overline{\phi}}\|_{\mathcal{B}(\overline{\mathcal{L}})} < \delta$ for all $|z|, |w| < \varepsilon$. In this neighborhood we define

$$\widehat{\Gamma}(z,w) := \log \lambda \left(L_{\overline{\phi+z\psi-w}} \right)$$

$\widehat{\Gamma}$ is holomorphic in U. For every z, w real such that $(z,w) \in U$, the spectrum of $L_{\overline{\phi+z\psi-w}}$ consists of a simple eigenvalue $\lambda(z,w)$ and a compact subset of $\{\lambda : |\lambda| < |\lambda(z,w)|\}$. By lemma 4, $\overline{\phi + z\psi - w}$ is positive recurrent with pressure $\log \lambda(z,w) = \widehat{\Gamma}(z,w)$. It follows that $\widehat{\Gamma}$ is a holomorphic extension of Γ. This proves that $t \mapsto P_G(\phi + t\psi)$ is real analytic in $(-\varepsilon, \varepsilon)$.

We show that $\phi + t\psi$ is positive recurrent for $|t|$ small. Real analyticity implies continuity, so $\exists \delta' > 0$ such that $\forall |t| < \delta'$, $P_G(\phi + t\psi) \in (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$. Set $w := -P_G(\phi + t\psi)$. Then $|w - \frac{\varepsilon}{3}| < \varepsilon$ whence $P_G(\overline{\phi + t\psi - w + \frac{\varepsilon}{3}}) = \Gamma(t, w - \frac{\varepsilon}{3}) < \infty$. Since $P_G(\overline{\phi + t\psi + p})$ is increasing in p, $P_G(\overline{\phi + t\psi + (\frac{\varepsilon}{3} - w)}) > \Gamma(t,w) = 0$ whence $\Delta_a[\phi + t\psi] > 0$. $\square$

Acknowledgments. This paper is part of a dissertation prepared in the Tel-Aviv university under the supervision of Jon Aaronson. I wish to express my deep gratitude to Jon Aaronson for his support and countless useful suggestions.

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