On an example with a topological pressure which is not analytic

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Abstract

We construct a function $\phi = \phi(x_0, x_1)$ on a topologically mixing countable Markov shift such that the set of values $\beta > 0$ for which the pressure function $P_{top}(\beta\phi)$ is not analytic in a neighbourhood of β has positive Lebesgue measure. The construction is based on renewal theoretic ideas.

1 Introduction

A well known theorem of Ruelle [4] states that Hölder continuous potentials on topologically mixing topological Markov shifts with a finite number of states, have no phase transitions in the sense that the function $\beta \mapsto P_{top}(\phi + \beta\psi)$ is real analytic for all ϕ, ψ Hölder. This is particularly interesting in the case $\phi = \psi$, because then the parameter β corrseponds to the inverse temperature of the system. If the number of states is infinite, Ruelle's result is no longer true. Wang [8] and Lopes [1] constructed examples of potentials $\phi = \phi(x_0, x_1)$ where the function $p(\beta) = P_{top}(\beta\phi)$ has one point of non analyticity. Gurevic and Savchenko [3] constructed an example of $\phi = \phi(x_0, x_1)$ such that $p(\beta) = P_{top}(\beta\phi)$ has an arbitrarily large but finite number of critical points. It remained unknown, however, whether there exists an example with an infinite number of critical points. In this note we prove the following theorem:

Theorem 1 *There exists a topologically mixing countable Markov shift X and a potential $\phi : X \to \mathbf{R}$ such that $\phi(x) = \phi(x_0, x_1)$ and such that the following set has positive Lebesgue measure:*

$$\{\beta > 0 : P_{top}(t\phi) \text{ is not real analytic in a neighbourhood of } t = \beta\}$$

2 Basic Definitions

Let S be some countable set, and let $\mathbf{A} = (t_{ij})_{S\times S}$ be a matrix of zeroes and ones with no rows or columns which are all made of zeroes. The *one sided topological Markov* *shift* corresponding to $\mathbf{A}$ and S is the topological dynamical system (X,T) where $X = \left\{(x_0, x_1, \ldots) \in S^{\mathbf{N}\cup\{0\}} : \forall i \;\; t_{x_i x_{i+1}} = 1\right\}$ is endowed with the relative product topology (S being discrete), and $T : X \to X$ is the left shift. This system is called *topologically mixing* if $\forall U, V$ open $\exists N = N_{U,V}$ such that $\forall n > N \; U \cap T^{-n}V$ is not empty. A base for its topology is given by the *cylinder sets* $[a_0, \ldots, a_{n-1}] = \{x : \forall k \;\; x_k = a_k\}$. Let $\phi : X \to \mathbf{R}$ be a function such that $\sum_{k\geq 2} V_k(\phi) < \infty$ where

$$V_k(\phi) = \sup\{\phi(x) - \phi(y) : x_i = y_i \;\; \forall 0 \leq i \leq k-1\}$$

Clearly, every function of the form $\phi(x) = \phi(x_0, x_1)$ satisfies this condition. For every n set $\phi_n = \phi + \phi \circ T + \ldots + \phi \circ T^{n-1}$. The *Gurevic pressure* of ϕ is defined by

$$P_{top}(\phi) = P_G(\phi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{T^n x = x} e^{\phi_n(x)} 1_{[a]}(x)$$

where a is some arbitrary state $a \in S$. It is known that if X is topologically mixing, $\sum_{k\geq 2} V_k(\phi) < \infty$, and $\| \sum_{Ty=x} e^{\phi(y)} \|_\infty < \infty$ then $P_G(\phi)$ is well defined, independent of the choice of $a \in S$, finite and satisfies $P_G(\phi) = \sup\{h_\mu + \int \phi \, d\mu\}$ where the supremum ranges over all T-invariant Borel probability measures μ such that $-\int \phi \, d\mu < \infty$ (see [5]). Thus, $P_G(\phi)$ is a natural extension of the notion of topological pressure for countable Markov shifts.

We say that a one parameter family of functions $F_\beta(\xi)$ is an *exponent power series* if it is of the form $F_\beta(\xi) = \sum_{n,k\geq 0} a_{n,k}^\beta \xi^n$ where $a_{n,k} \geq 0$. Clearly, every function of the form $\sum_{n,k} c_k a_{n,k}^\beta \xi^n$ where c_k are non negative integers is an exponent power series, and if F_β, G_β are exponent power series then so are $F_\beta + G_\beta$, $F_\beta G_\beta$, and $F_\beta \circ G_\beta$. We say that a function $f(\beta)$ is an *exponent sum* if it of the form $f(\beta) = F_\beta(1)$ where F_β is an exponent power series convergent at 1.

3 Proof of theorem 1

3.1 Construction of a certain exponent power series

We construct an exponent power series $F_\beta(\xi)$ with radius of convergence $R(\beta)$ with the following properties, where m denotes the Lebesgue measure:

$$m\{\beta : R(\beta) \text{ is not analytic in a neighbourhood of } \beta\} > 0 \tag{1}$$

$$F_\beta(R(\beta)) < 1 \tag{2}$$

Let $\{I_n\}_{n=1}^\infty$ be disjoint open intervals in $[1,2]$ such that $\bigcup_{n\geq 1} I_n$ is dense in $[1,2]$ but $m\left(\bigcup_{n\geq 1} I_n\right) < 1$. Then $\overline{\bigcup_{n\geq 1} \partial I_n}$ has positive Lebesgue measure. Set $I_n = (a_n, b_n)$, $N_n = 2^n$ and $p_n(\beta) := -2N_n^{-3\beta}\left(N_n^{\beta/a_n} - N_n\right)\left(N_n^{\beta/b_n} - N_n\right)$. Expanding and collecting all terms with the same sign write $p_n(\beta) = A_n(\beta) - B_n(\beta)$ where

$A_n(\beta) = 2N_n\left(N_n^{\beta\left(a_n^{-1}-3\right)} + N_n^{\beta\left(b_n^{-1}-3\right)}\right)$ and $B_n(\beta) = 2N_n^{\beta\left(a_n^{-1}+b_n^{-1}-3\right)} + 2N_n^2 N_n^{-3\beta}$. Set

$$B(\beta) := \sum_{n=1}^{\infty} B_n(\beta) \quad \text{and} \quad A^{(n)}(\beta) := A_n(\beta) + B(\beta) - B_n(\beta).$$

Both $\frac{1}{2}B(\beta)$ and $\frac{1}{2}A^{(n)}(\beta)$ are convergent exponent sums for $\beta > \frac{2}{3}$. Also, $p_n(\beta) = A_n(\beta) - B_n(\beta) = A^{(n)}(\beta) - B(\beta)$. Thus $A^{(n)}(\beta) > B(\beta)$ if and only if $\beta \in I_n$, whence $M(\beta) := \max\left\{B(\beta), A^{(1)}(\beta), A^{(2)}(\beta), \ldots\right\}$ is given by

$$M(\beta) = \begin{cases} A^{(n)}(\beta) & \text{if } \beta \in I_n \\ B(\beta) & \text{if } \beta \notin \bigcup_{n\geq 1} I_n \end{cases} \tag{3}$$

Now set $c_n := \lfloor 2^n/n^3 \rfloor$ and

$$F_\beta(\xi) := \sum_{n=2}^{\infty} c_n \left(\left(\frac{1}{2}B(\beta)\xi\right)^n + \sum_{j=1}^{n-1}\left(\frac{1}{2}A^{(j)}(\beta)\xi\right)^n\right)$$

By the preceding discussion, this is an exponent power series. Let $R(\beta)$ denote its radius of convergence. Then $R(\beta) = 1/M(\beta)$ where $M(\beta)$ is given by (3). $M(\beta)$ is not analytic in the neighbourhood of each of the points in $\overline{\bigcup_{n\geq 1}\partial I_n}$, because $\bigcup_{n\geq 1}\partial I_n$ are points of non differentiability. Thus (1) is satisfied. Also,

$$F_\beta(R(\beta)) \leq \sum_{n=2}^{\infty} \frac{1}{n^3}\frac{1}{M(\beta)^n}\left(B(\beta)^n + \sum_{j=1}^{n-1} A^{(k)}(\beta)^n\right) \leq \sum_{n=2}^{\infty}\frac{1}{n^2}$$

and (2) follows as well.

3.2 Construction of X and ϕ

Let $F_\beta(\xi)$ be the exponent power series which was constructed in the previous section, and set

$$F_\beta(\xi) = \sum_{n=2}^{\infty} \xi^n \sum_{k=0}^{\infty} a_{n,k}^{\beta}$$

where $a_{n,k}$ are non negative. We construct a topological Markov shift and a potential $\phi = \phi(x_0, x_1)$ such that $\forall n \in \mathbf{N}\ \forall \beta > \frac{2}{3}\ \ Z_n^*(\beta\phi, a) = \sum_{k\geq 0} a_{nk}^{\beta}$ for some fixed state a where $\varphi_a(x) := 1_{[a]}(x)\inf\{n > 0 : T^n x \in [a]\}$ and

$$Z_n^*(\phi, a) = \sum_{T^n x = x} e^{\phi_n(x)} 1_{[\varphi_a(x)=n]}(x)$$

Let S be some countable set enumerated in the following way:

$$S := \{a\} \cup \bigcup_{n=2}^{\infty}\bigcup_{k=0}^{\infty}\{b_{nk}(1), \ldots, b_{nk}(n-1)\}$$

Let X be the one sided topological Markov shift on this set of states given by the transition matrix $\mathbf{A} = (t_{ij})_{S\times S}$ whose non zero entries are exactly $t_{ab_{nk}(1)}$, $t_{b_{nk}(i)b_{nk}(i+1)}$, $t_{b_{nk}(n-1)a}$ for all $n, k \geq 1$ and $i = 1, \ldots, n-1$. Then every $x \in X$ such that $T^n x = x$ and $\varphi_a(x) = n$ is of the form

$$x = (a, b_{nk}(1), \ldots, b_{nk}(n-1); a, b_{nk}(1), \ldots, b_{nk}(n-1); \ldots).$$

Now define $\phi = \phi(x_0, x_1)$ by

$$\phi(x) = \begin{cases} \log a_{nk} & x \in [a, b_{nk}(1)] \\ 0 & \text{else} \end{cases}$$

It is easy to verify that X is topologically mixing and that for every $n \in \mathbf{N}$, $Z_n^*(\beta\phi, a) = \sum_{k\geq 1} a_{nk}^{\beta}$ (for $n = 1$ we have $Z_1^*(\phi, a) = 0$ since there is no $x \in [a]$ such that $Tx = x$.)

We show that $P_G(\beta\phi)$ has the required analyticity properties. Set $Z_n(\beta\phi, a) = \sum_{T^n x = x} e^{\beta\phi_n(x)} 1_{[a]}$. Then $Z_n(\beta\phi, a) = Z_n^*(\beta\phi, a) + \sum_{k=1}^{n-1} Z_{n-k}^*(\phi, a) Z_k(\phi, a)$ whence

$$1 + \sum_{n=1}^{\infty} \xi^n Z_n(\beta\phi, a) = \frac{1}{1 - F_\beta(\xi)}$$

By the definition of the Gurevic pressure, the radius of convergence of the series on the left side is $e^{-P_G(\beta\phi)}$. Thus, by (2), $P_G(\beta\phi) = -\log R(\beta)$ and the theorem follows from (1). $\square$

Remark. The set of points for which the pressure function constructed above is not differentiable is countable (it contains each of the points a_n and b_n). This cannot be improved, since the pressure function is by definition convex in β.

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