UNIQUE ERGODICITY FOR NON-UNIQUELY ERGODIC HOROCYCLE FLOWS

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Dedicated to A. Katok on the occasion of his 60th birthday

Abstract. We consider the horocycle flow associated to a $\mathbb{Z}^d$–cover of a compact hyperbolic surface. Such flows have no finite invariant measures, and infinitely many infinite ergodic invariant Radon measures. We prove that, up to normalization, only one of these infinite measures admits a generalized law of large numbers, and identify such laws.

1. Introduction

Motivation. The horocycle flow on the unit tangent bundle of a compact hyperbolic surface is uniquely ergodic: it admits exactly one invariant probability measure (Furstenberg [**F**]). In this paper we consider the horocycle flow on certain non-compact hyperbolic surfaces of infinite volume. The surfaces we consider (free Abelian covers of compact hyperbolic surfaces) are such that their associated horocycle flow does not preserve any invariant probability measure whatsoever; but it does preserve a family of infinite invariant Radon measures[1] [**BL2**], [**S**].

Our aim here is to prove that only one of these measures (up to normalization) is ergodic theoretically 'relevant' in a sense that is explained below. This can be viewed as a version of Furstenberg's unique ergodicity theorem in this context.

Generalized laws of large numbers. Consider a measure preserving flow φ^t on a standard measure space $(X, \mathcal{F}, m)$. If $m(X) < \infty$, then the ergodic hypothesis $\lim_{T\to\infty} \frac{1}{T}\int_0^T 1_A[\varphi^t(\omega)]dt = \frac{m(A)}{m(X)}$ holds for a.e. $\omega \in X$. But if $m(X) = \infty$ then it fails in an essential way: For every $A \in \mathcal{F}$ there exists no normalization $a(T)$ s.t. $\lim_{T\to\infty} \frac{1}{a(T)}\int_0^T 1_A[\varphi^t(\omega)]dt$ exists a.e., other than those normalizations which make the limit zero or infinity [**A2**].

The failure of the ergodic hypothesis in the infinite measure setting suggests studying the following weaker – but possible – property, due to J. Aaronson [**A1**]:

Definition 1. *A measure preserving flow φ^t on a standard measure space satisfies a* generalized law of large numbers (GLLN)*, if there is a function $L : \{0,1\}^{\mathbb{R}_+} \to \mathbb{R}_+$, $L = L[x(t)]$ s.t. for every $A \in \mathcal{B}$, $L[1_A(\varphi^t\omega)] = m(A)$ almost everywhere.*

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[1] A Radon measure is a measure which is well-defined and finite on compact sets

Example 1 (Finite measures)**.** *Any ergodic invariant probability measure admits the following generalized law of large numbers:*

$$L[x(t)] = \begin{cases} \lim\limits_{T\to\infty} \frac{1}{T}\int_0^T x(t)dt, & \text{when the expression makes sense,} \\ 0, & \text{when the expression does not make sense.} \end{cases}$$

Example 2 (Squashability [**A1**, **A4**])**.** *A measure μ is called* squashable *w.r.t. a flow φ^t, if $\exists Q : X \to X$ measurable such that $\varphi^t \circ Q = Q \circ \varphi^t$ and $\mu \circ Q = c\mu$ for $c \neq 1$. Squashable measures cannot have generalized laws of large numbers.*

Proof. The example and its proof are due to Aaronson [**A1**]. Suppose there were a generalized law of large numbers L. Fix A of positive finite measure and pick some x such that $L[1_E(h^s y)] = \mu(E)$ for $E = A, Q^{-1}A$ and $y = x, Qx$. Then

$$\mu(A) = L[1_A(h^s Qx)] = L[1_A(Qh^s x)] = L[1_{Q^{-1}A}(h^s x)] = \mu(Q^{-1}A) = \frac{1}{c}\mu(A),$$

in contradiction to $c \neq 1$.

Example 3 (Rational ergodicity [**A3**])**.** *A flow φ^t is called* rationally ergodic *if it is ergodic and there exists a measurable set A of positive finite measure such that $\int_A (\int_0^t 1_A \circ h^s ds)^2 dm = O([\int_A \int_0^t 1_A \circ h^s ds dm]^2)$. This condition implies the existence of $a(T)$ and $T_k \to \infty$ s.t.*

$$L[x(t)] = \begin{cases} \textit{Cesaro-}\lim\limits_{k\to\infty} \frac{1}{a(T_k)}\int_0^{T_k} x(t)dt, & \text{when the expression makes sense,} \\ 0, & \text{when the expression does not make sense,} \end{cases}$$

is a generalized law of large numbers.

See Aaronson [**A1**], [**A3**] for a proof.

The geodesic flow on the unit tangent bundle of a recurrent hyperbolic surface is rationally ergodic w.r.t. the volume measure (Aaronson & Sullivan [**AS**], see Roblin [**Ro**] for the variable curvature case). We prove below the rational ergodicity of the horocycle flow for a class of surfaces of negative curvature.

Horocycle flows on $\mathbb{Z}^d$–covers. Let M_0 be a compact connected orientable C^∞ Riemannian surface with negative curvature. Let $T^1(M_0)$ be the unit tangent bundle of M_0, and $g^s : T^1(M_0) \to T^1(M_0)$ the geodesic flow. Margulis [**Mrg**] and Marcus [**Mrc**] constructed a continuous flow $h^t : M_0 \to M_0$ for which the h–orbit of x is equal to $W^{ss}(x) := \{y \in T^1(M_0) : d(g^s x, g^s y) \xrightarrow[s\to\infty]{} 0\}$, and for which $\exists\mu$ such that $\forall s,t,\ g^{-s} \circ h^t \circ g^s = h^{\mu^s t}$. This is the *(stable) horocycle flow* of M_0.[2]

Now let $p : M \to M_0$ be a regular cover whose group of deck transformations

$$G := \{D : M \to M | D \text{ is an isometry, and } p \circ D = p\}$$

is isomorphic to $\mathbb{Z}^d$. Such covers are called $\mathbb{Z}^d$*–covers.* The geodesic flow and the horocycle flow of M_0 lift to continuous flows g^s, h^t on $T^1(M)$ for which:

(a) The h–orbit of x is equal to $W^{ss}(x) := \{y \in T^1(M) : d(g^s x, g^s y) \xrightarrow[s\to\infty]{} 0\}$;
(b) There exists μ such that $\forall s,t,\ g^{-s} \circ h^t \circ g^s = h^{\mu^s t}$;
(c) g^s, h^t commute with the deck transformations.

[2]The reader should note that h^t does *not*, in general, parametrize W^{ss} by length (although in the constant negative curvature case, it does).

We call h^t the *horocycle flow* of M. In the constant negative curvature case, $\mu = e$, and we get the classical (stable) horocycle flow.

The ergodic invariant Radon measures on $\mathbb{Z}^d$–covers of compact Riemannian surfaces of negative sectional curvature were identified in [**BL2**] and [**S**]: they form a family $\{cm_\varphi : c > 0, \varphi : G \to \mathbb{R}^d \text{ is a homomorphism}\}$, where

$$m_\varphi \circ dD = e^{\varphi(D)} m \quad (D \in G).$$

The parameter $\varphi \equiv 0$ corresponds to the Margulis measure, see [**BL2**], [**BM**]. (In the constant negative curvature this is the volume measure.)

The results. The measures m_φ with $\varphi \not\equiv 0$ do not admit GLLN's, because – by (c) – they are all squashable. We show that the Margulis measure $m = m_0$ does admit such laws. We thus obtain:

Theorem 1. *Let M be a $\mathbb{Z}^d$–cover of a compact connected orientable C^∞ Riemannian surface of negative curvature. The horocycle flow on $T^1(M)$ has, up to a factor, exactly one invariant Radon measure with a generalized law of large numbers: m_0.*

We proceed to describe this law of large numbers (in fact, we shall present two such laws). In what follows, $\widetilde{M_0} \subset T^1(M)$ is some fixed precompact connected fundamental domain for the action of the group of deck transformations on $T^1(M)$.

Theorem 2. *The measure m_0 is rationally ergodic. Consequently, there exists a function $a(T)$ and a sequence $T_k \uparrow \infty$ s.t. for all $f \in L^1(m_0)$,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \left(\frac{1}{a(T_k)} \int_0^{T_k} f \circ h^s ds \right) = \frac{1}{m_0(\widetilde{M_0})} \int f dm_0 \quad m_0\text{-a.e.}.$$

The corresponding GLLN is $L[x(t)] = m_0(\widetilde{M_0}) \cdot \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \left(\frac{1}{a(T_k)} \int_0^{T_k} x(s) ds \right)$ when the expression makes sense, and zero otherwise. The following theorem is a more explicit GLLN, in the spirit of the 'second order ergodic theorems' of [**ADF**]:

Theorem 3. *With the same $a(T)$ as in the previous theorem, for every $f \in L^1(m_0)$*

$$\lim_{N\to\infty} \frac{1}{\ln \ln N} \int_3^N \frac{1}{T \ln T} \left(\frac{1}{a(T)} \int_0^T f \circ h^s ds \right) dT = \frac{1}{m_0(\widetilde{M_0})} \int f dm_0 \quad m_0\text{-a.e.}$$

Next, we describe $a(T)$. We first need to make some comments concerning the geodesic flow on a $\mathbb{Z}^d$–cover.

Let $G := \{D_{\underline{a}} : \underline{a} \in \mathbb{Z}^d\}$ an enumeration of the group of deck transformations of the cover $p : M \to M_0$ s.t. $D_{\underline{a}} \circ D_{\underline{b}} = D_{\underline{a}+\underline{b}}$. Define the $\mathbb{Z}^d$*–coordinate* of $\omega \in T^1(M)$ (relative to $\widetilde{M_0}$) to be the unique $\underline{a} \in \mathbb{Z}^d$ such that $\omega \in dD_{\underline{a}}[\widetilde{M_0}]$. Now define

$$\xi_\omega(T) := \mathbb{Z}^d\text{–coordinate of } g^T(\omega).$$

It follows from the work of Ratner [**R**] and Katsuda & Sunada [**KS2**] that the (normalized) Margulis distribution of $\xi_\omega(T)/\sqrt{T}$ as ω ranges over $\widetilde{M_0}$ converges to the distribution of a multivariate Gaussian random variable N on $\mathbb{R}^d$, with a positive definite covariance matrix $\mathrm{Cov}(N)$. Set

$$\sigma := \sqrt[d]{|\det \mathrm{Cov}(N)|}.$$

Theorem 4. $a(T) \sim \frac{1}{(4\pi\sigma)^{d/2}} \frac{T}{(\log_\mu T)^{d/2}}$.

Remark 1. The value of σ is known when M is the homology cover of a compact hyperbolic surface M_0 of genus g: $\sigma = \frac{1}{2\pi}(g-1)$, see [**KS1**],[**KS2**] and [**PS**].

Our method of proof also yields the following result, which together with the central limit theorem for $\xi_\omega(T)/\sqrt{T}$, explains the fluctuations of $\frac{1}{a(T)}\int_0^T f\circ h^s ds$. Define a norm on $\mathbb{R}^d$ by $\|v\|_H := \sqrt{v^t \mathrm{Cov}(N)^{-1} v}$.

Theorem 5. *Suppose $f \in L^1(m_0)$ and $\frac{1}{m_0(\widetilde{M}_0)}\int f dm_0 = 1$. For every ε and almost every ω, if T is large enough, then*

$$2^{\frac{d}{2}-\varepsilon} e^{-\frac{1}{2}(1+\varepsilon)\left\|\frac{\xi_\omega(\log_\mu T)}{\sqrt{\log_\mu T}}\right\|_H^2} \leq \frac{1}{a(T)}\int_0^T f\circ h^s ds \leq 2^{\frac{d}{2}+\varepsilon} e^{-\frac{1}{2}(1-\varepsilon)\left\|\frac{\xi_\omega(\log_\mu T)}{\sqrt{\log_\mu T}}\right\|_H^2}$$

Thus $\frac{1}{a(T)}\int_0^T f\circ h^s ds$ is sandwiched between two fluctuating quantities, which converge in distribution, but not pointwise.

Another consequence is the following equidistribution result for the *geodesic* flow, in the constant curvature case. Define for $q \in M$, $T^1_q(M) := \{v \in T_q(M) : \|q\| = 1\}$, and $S_q(T) := g^T[T^1_q(M)]$. This is a circle with perimeter $|S_q(T)| = 2\pi \sinh T$, and the following result describes its distribution as $T \to \infty$:

Theorem 6. *Suppose M_0 has constant curvature -1, and let f be a continuous function with compact support on $T^1(M)$. Then, for all $q \in M$,*

$$\lim_{T\to\infty} (4\pi\sigma T)^{d/2}\left(\frac{1}{|S_q(T)|}\int_{S_q(T)} f\right) = \frac{1}{m_0(\widetilde{M}_0)}\int f dm_0.$$

Remark 2. In the variable curvature case, we can only prove the following: There is a function $c(q)$ s.t. for all $q \in M$,

$$\lim_{T\to\infty}\left(\frac{4\pi\sigma T}{\ln \mu}\right)^{d/2}\left(\frac{1}{|S_q(T)|}\int_{S_q(T)} f\right) = c(q)\frac{\int f dm}{m(\widetilde{M}_0)},$$

where m is the measure on T^1M which is obtained by integrating the Lebesgue measure on the unstable manifolds with respect to the transverse invariant Margulis measure.

Other examples with unique non-squashable invariant Radon measure. The phenomenon of having a unique non-squashable invariant Radon measure is not restricted to horocycle flows. We are aware of two other examples:

Cylinder transformations. These are $R_\alpha : \mathbb{T}\times\mathbb{Z} \to \mathbb{T}\times\mathbb{Z}$, given for every $\alpha \notin \mathbb{Q}$ by $R_\alpha(x,y) = \big(x + \alpha \bmod 1, y + \varphi(x)\big)$, $\varphi = 1_{[0,\frac{1}{2})} - 1_{[\frac{1}{2},1)}$. We have:

(1) $m =$ Haar × Counting Measure is a non-squashable ergodic invariant Radon measure [**AK**];
(2) All other ergodic invariant Radon measures are squashable, and there are infinitely many such measures [**N**], [**ANSS**].

In the particular case when α is a quadratic surd, the unique non-squashable measure is rationally ergodic, and therefore admits a GLLN [**AK**].

Adic random walks. Let $\Sigma := \{-1,1\}^{\mathbb{N}}$ and $\tau : \Sigma \to \Sigma$ the adding machine, defined by $\tau(1^n(-1)*) = ((-1)^n 1*)$, and set $T : \Sigma \times \mathbb{Z} \to \Sigma \times \mathbb{Z}$ to be

$$T(\underline{x}, t) := \big(\tau(\underline{x}), t + \chi(\underline{x})\big), \quad \chi(x_1, x_2, \ldots) = x_1.$$

The terminology is because $T^n(\underline{x}, t) = (\tau^n(\underline{x}), t + x_1 + \ldots + x_n)$. Aaronson and Weiss [**AW**] (see also [**ANSS**]) proved:

1. T is rationally ergodic w.r.t. $B(\frac{1}{2},\frac{1}{2}) \times \sum_{n\in\mathbb{Z}} \delta_n$, where $B(\frac{1}{2},\frac{1}{2})$ is the Bernoulli $(\frac{1}{2},\frac{1}{2})$ measure on Σ, and δ_n is Dirac's measure at n.
2. All other measures are squashable [**ANSS**]. There are infinitely many such measures [**AW**].

Multidimensional generalizations of these random walks with non-i.i.d. jumps are considered in [**ANSS**]. They exhibit the same phenomenon.

2. The main estimate and its implications

The main estimate. Recall the definitions of $\xi_\omega(T)$, N, σ, and $\|\cdot\|_H$ from introduction. We use the following shorthand notation: $a = C^{\pm 1} b \Leftrightarrow \frac{1}{C} b \le a \le Cb$. Our results are based on the following two lemmas

Lemma 1. *For every $\varepsilon > 0$, there exists $E \subseteq \widetilde{M}_0$ Borel of finite positive m_0–measure, s.t. for some $T_1 > 0$, $\delta_0 > 0$, if $T > T_1$ and $\|\frac{\xi_\omega(\log_\mu T)}{\log_\mu T}\|_H < \delta_0$, then*

$$\int_0^T 1_E(h^s\omega)ds = \mu^{\pm 2\varepsilon} \frac{m_0(E|\widetilde{M}_0)}{(2\pi\sigma)^{d/2}} \frac{T}{(\log_\mu T)^{d/2}} e^{-\frac{1}{2}(1\pm\varepsilon)\left\|\frac{\xi_\omega(\log_\mu T)}{\sqrt{\log_\mu T}}\right\|_H^2}.$$

Lemma 2. *The following limits hold for any $f : \mathbb{R}^d \to \mathbb{R}$ s.t. $f, \widehat{f} \in L^1(\mathbb{R}^d)$:*

1. $\lim\limits_{t\to\infty} \int_E f(\frac{\xi_\omega(t)}{\sqrt{t}})dm_0 = \mathbb{E}[f(N)]m_0(E|\widetilde{M}_0)$ *for all $E \subseteq \widetilde{M}_0$ Borel;*
2. $\lim\limits_{s\to\infty} \frac{1}{\ln s}\int_3^s f(\frac{\xi_\omega(t)}{\sqrt{t}})\frac{dt}{t} = \mathbb{E}[f(N)]$ *a.e. in $\widetilde{M}_0$.*

We defer the proof of these lemmata to the next section, and explain first why they imply theorems 1,2,3,4 and 5. Henceforth we assume for simplicity that $m_0(\widetilde{M}_0) = 1$ (this can always be arranged by normalization).

Proof of Theorems 1 and 2. All measures m_φ with $\varphi \not\equiv 0$ are squashable, and therefore have no GLLN's (Example 2). Rationally ergodic measures admit GLLN's (Example 3), so it is enough to show that m_0 is rationally ergodic.

Choose E to be the set given by lemma 1 for some $\varepsilon > 0$ (it doesn't matter which), and set $I_T(\omega) := \int_0^T 1_E(h^s\omega)ds$. It suffices to show that $\|1_E I_T\|_2 = O(\|1_E I_T\|_1)$ as T tends to ∞.

Set $\Omega_t := [\|\frac{\xi_\omega(t)}{t}\|_H \ge \delta_0]$. Kifer has shown that the m_0–distributions of $\frac{\xi_\omega(T)}{T}$ satisfy the large deviations property [**K**]. Using the fact that all norms on $\mathbb{R}^d$ are equivalent, we deduce the existence of $\alpha > 0$ such that $m_0[\Omega_t] = O(\mu^{-\alpha t})$, whence

$$m_0[\Omega_{\log_\mu T}] = O(T^{-\alpha}), \text{ as } T \to \infty.$$

Since $I_T \le T$, we have that $\|1_{E\cap\Omega_{\log_\mu T}} I_T\|_1 = O\big(T^{1-\alpha}\big)$.

On the other hand, lemmas 1 and 2 give the following asymptotic lower bound for $\|1_{E\setminus\Omega_{\log_\mu T}} I_T\|_1$:

$$\frac{1}{\mu^{2\varepsilon}}\frac{m_0(E)}{(2\pi\sigma)^{d/2}}\frac{T}{(\log_\mu T)^{d/2}}\int_E e^{-\frac{1}{2}(1+\varepsilon)\left\|\frac{\xi_\omega(\log_\mu T)}{\sqrt{\log_\mu T}}\right\|_H^2} dm_0 - O(T^{1-\alpha})$$
$$\sim \frac{1}{\mu^{2\varepsilon}}\frac{m_0(E)^2}{(2\pi\sigma)^{d/2}}\frac{T}{(\log_\mu T)^{d/2}}\mathbb{E}(e^{-\frac{1}{2}(1+\varepsilon)\|N\|_H^2}).$$

Comparing this to the upper bound of $\|1_{E\cap\Omega_{\log_\mu T}} I_T\|_1$, we see that $\|1_E I_T\|_1 \sim \|1_{E\setminus\Omega_{\log_\mu T}} I_T\|_1$, whence

$$\|1_E I_T\|_1 \geq [1-o(1)]\frac{1}{\mu^{2\varepsilon}}\frac{m_0(E)^2}{(2\pi\sigma)^{d/2}}\frac{T}{(\log_\mu T)^{d/2}}\mathbb{E}\left(e^{-\frac{1}{2}(1+\varepsilon)\|N\|_H^2}\right).$$

In the same way one shows that

$$\|1_E I_T\|_2 \leq [1+o(1)]\mu^{2\varepsilon}\frac{m_0(E)^{3/2}}{(2\pi\sigma)^{d/2}}\frac{T}{(\log_\mu T)^{d/2}}\sqrt{\mathbb{E}\left(e^{-(1-\varepsilon)\|N\|_H^2}\right)}.$$

Consequently, $\|1_E I_T\|_2 = O(\|1_E I_T\|_2)$, proving rational ergodicity. □

Proof of theorem 4. The return sequence of m_0 is defined by

$$a(T) \sim \frac{1}{m_0(E)^2}\int_0^T m_0(E\cap h^{-s}E)ds \equiv \frac{1}{m_0(E)^2}\|1_E I_T\|_1.$$

whenever E satisfies $\|1_E I_T\|_2 = O(\|1_E I_T\|_1)$, $I_T = \int_0^T 1_E\circ h^s ds$ (see [**A1**]). The proof of theorems 1 and 2 shows that the sets E given by lemma 1 are such sets. That proof also shows that for T large enough

$$\frac{1}{m_0(E)^2}\|1_E I_T\|_1 = \mu^{\pm 3\varepsilon}\frac{1}{(2\pi\sigma)^{d/2}}\frac{T}{(\log_\mu T)^{d/2}}\mathbb{E}\left(e^{-\frac{1}{2}(1\pm\varepsilon)\|N\|_H^2}\right).$$

Now $\mathbb{E}\left(e^{-\frac{1}{2}\|N\|_H^2}\right) = 2^{-d/2}$, because N is Gaussian, and $\|v\|_H = \sqrt{v^T\mathrm{Cov}(N)^{-1}v}$.[3] The normal distribution is such that

$$A(\varepsilon) := \frac{\mathbb{E}\left(e^{-\frac{1}{2}(1-\varepsilon)\|N\|_H^2}\right)}{\mathbb{E}\left(e^{-\frac{1}{2}(1+\varepsilon)\|N\|_H^2}\right)} \xrightarrow[\varepsilon\to 0]{} 1.$$

It follows that for T large enough, $a(T) \sim \frac{1}{m_0(E)^2}\|1_E I_T\|_1 = \mu^{\pm 4\varepsilon}\frac{1}{(4\pi\sigma)^{d/2}}\frac{T}{(\log_\mu T)^{d/2}}$, whence $\forall\varepsilon>0\ \exists T_\varepsilon$ s.t. if $T>T_\varepsilon$ then $a(T) = \mu^{\pm 4\varepsilon}\frac{1}{(4\pi\sigma)^{d/2}}\frac{T}{(\log_\mu T)^{d/2}}$. □

Proof of theorem 3. Write $a(T) = \frac{1}{(4\pi\sigma)^{d/2}}\frac{T}{(\log_\mu)^{d/2}}$, and fix $f\in L^1(m_0)$. For every $\varepsilon>0$, construct a set E as in lemma 1. The ergodic theorem for the geodesic flow on $T^1(M_0)$ implies that $\frac{\xi_\omega(T)}{T} \xrightarrow[T\to\infty]{} 0$ m_0–almost everywhere. It follows that for almost all ω, the conditions of lemma 1 are satisfied for T large enough, whence

$$\frac{1}{a(T)}\int_0^T 1_E(h^s\omega)ds = \mu^{\pm 2\varepsilon}2^{\frac{d}{2}}\mathbb{E}\left(e^{-\frac{1}{2}(1\pm\varepsilon)\left\|\frac{\xi_\omega(\log_\mu T)}{\sqrt{\log_\mu T}}\right\|_H^2}\right)m_0(E).$$

[3]The case $d=1$ is trivial; the general case can be obtained from it by orthogonal diagonalization of $\mathrm{Cov}(N)$, and a suitable change of coordinates.

The ratio ergodic theorem says that

$$\frac{\int_0^T f\circ h^s ds}{\int_0^T 1_E\circ h^s ds}\xrightarrow[T\to\infty]{}\frac{1}{m_0(E)}\int f dm_0 \text{ almost everywhere.}$$

Consequently, for almost every ω and T large

$$\frac{1}{a(T)}\int_0^T f(h^s\omega)ds=\mu^{\pm 3\varepsilon}2^{\frac{d}{2}}\mathbb{E}\left(e^{-\frac{1}{2}(1\pm\varepsilon)\left\|\frac{\xi_\omega(\log_\mu T)}{\sqrt{\log_\mu T}}\right\|_H^2}\right)\int f dm_0.$$

Therefore, since $\varepsilon>0$ is arbitrary, it is enough to check that for a.e. ω,

$$\lim_{T\to\infty}\frac{1}{\log\log T}\int_3^T\frac{1}{t\log t}e^{-\frac{1}{2}(1\pm\varepsilon)\left\|\frac{\xi_\omega(\log_\mu t)}{\sqrt{\log_\mu t}}\right\|_H^2}dt \;=\; \frac{1}{2^{d/2}}[1+O(\varepsilon)].$$

This is part (2) of lemma 2 with $f(\underline{x})=e^{-\frac{1}{2}(1\pm\varepsilon)\|\underline{x}\|_H^2}$. □

Proof of theorem 5. Fix $\varepsilon>0$ and let E be as in lemma 1. The case $f=\frac{1}{m_0(E)}1_E$ is that lemma. The general case follows from the ratio ergodic theorem. □

Proof of theorem 6. Given a continuous function with compact support f and $\varepsilon>0$, we first choose simple[4] functions $f_i, i=1,2$ and $\delta>0$ s.t. $\int f_2/\int f_1\le e^{\varepsilon}$ and $f_1(y)\le f(x)\le f_2(y)$ whenever $d(x,y)<\delta$.

The first claim is that there is δ_0 such that, if T is large enough, ω_T is a point of $S_q(T)$ and $C_{\omega,T}$ the arc of $S_q(T)$ of length $2\delta_0\sinh T$ centered at ω_T, then

$$e^{-\varepsilon}\frac{1}{2}\int_{-e^T\tan\delta_0}^{e^T\tan\delta_0}f_1(h^s_{un}\omega_T)ds\;\le\;\int_{C_{\omega,T}}f\;\le\;e^{\varepsilon}\frac{1}{2}\int_{-e^T\tan\delta_0}^{e^T\tan\delta_0}f_2(h^s_{un}\omega_T)ds,\qquad(1)$$

where h^s_{un} denotes the *unstable* horocycle flow (in the previous theorems, we used stable horocycle flows). To check (1), let r_θ denote the rotation by θ radians around q, and let $\omega\in T^1_q(M)$ be the point such that $\omega_T=g^T(\omega)$. We have

$$\int_{C_{\omega,T}}f=\sinh T\int_{-\delta_0}^{\delta_0}f[(g^T\circ r_\theta)(\omega)]d\theta,$$

$$\int_{-e^T\tan\delta_0}^{e^T\tan\delta_0}f_i(h^s_{un}\omega_T)ds=e^T\int_{-\delta_0}^{\delta_0}f_i[(h_{un}^{-e^T\tan\theta}\circ g^T)(\omega)]\frac{d\theta}{\cos^2\theta.}$$

Representing h^s_{un}, r_θ, g^T as matrix actions on $\mathrm{PSL}(2,\mathbb{R})$,[5] one calculates and sees that $d\big((h_{un}^{-e^T\tan\theta}\circ g^T)(\omega),(g^T\circ r_\theta)(\omega)\big)=O(\delta_0+\delta_0e^{-T/2})$ as $T\to\infty$ and $\delta_0\to 0$. In particular, any $\omega\in T^1_q(M)$ has some $\delta_0=\delta_0(\omega)$ such that this upper bound is less than δ for all $|\theta|<\delta_0$. Since ω ranges over the compact set $T^1_q(M)$, we can make this δ_0 uniform in ω. (1) now follows from the definition of f_1,f_2 and the fact that $\sinh T\sim\frac{1}{2}e^T$ as $T\to\infty$ and $\cos^2\theta\sim 1$ as $\theta\to 0$.

Theorem 5 applied to the unstable horocycle flow and the simple functions f_1,f_2, says that if T is large enough, then $\frac{1}{2}\int_{-e^T\tan\delta_0}^{e^T\tan\delta_0}f_i(h^s_{un}\omega_T)ds$ is within $e^{\pm\varepsilon}$ of

$$\frac{1}{2}\cdot 2a(e^T\tan\delta_0)2^{\frac{d}{2}\pm\varepsilon}e^{-\frac{1}{2}(1\pm\varepsilon)\left\|\frac{\xi_{-\omega_T}(T)}{\sqrt{T}}\right\|_H^2}\frac{1}{m_0(\widetilde{M_0})}\int f_i dm_0.$$

[4]A simple function is a finite linear combination of indicators of measurable sets.

[5]r_θ, g^T, h^s_{un} act on the right, resp., by $\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$, $\begin{pmatrix}e^{T/2} & 0\\ 0 & e^{-T/2}\end{pmatrix}$, and $\begin{pmatrix}1 & s\\ 0 & 1\end{pmatrix}$.

(The minus in front of ω_T is because we are working with the unstable, not the stable horocycle flow.) It follows that the average over $C_{\omega,T}$ of f is within $e^{\pm 2\varepsilon}$ of

$$\frac{a(e^T \tan\delta_0)}{2\delta_0 \sinh T} 2^{\frac{d}{2}\pm\varepsilon} e^{-\frac{1}{2}(1\pm\varepsilon)\left\|\frac{\xi-\omega_T(T)}{\sqrt{T}}\right\|_H^2} \frac{1}{m_0(\widetilde{M}_0)} \int f dm_0 \sim$$

$$\frac{\tan\delta_0}{\delta_0} \cdot \frac{2^{\frac{d}{2}\pm\varepsilon} T^{-d/2}}{(4\pi\sigma)^{d/2}} e^{-\frac{1}{2}(1\pm\varepsilon)\left\|\frac{\xi-\omega_T(T)}{\sqrt{T}}\right\|_H^2} \frac{1}{m_0(\widetilde{M}_0)} \int f dm_0,$$

as $T \to \infty$. If we choose δ_0 so small that $\frac{\tan\delta_0}{\delta_0} = e^{\pm\varepsilon}$, then for T large,

$$\frac{1}{|C_{\omega,T}|} \int_{C_{\omega,T}} f = e^{\pm 4\varepsilon} \frac{\frac{1}{m_0(\widetilde{M}_0)} \int f dm_0}{(2\pi\sigma T)^{d/2}} e^{-\frac{1}{2}(1\pm\varepsilon)\left\|\frac{\xi-\omega_T(T)}{\sqrt{T}}\right\|_H^2}.$$

We now average this estimate over $\omega \in S_q(T)$. The average of the LHS is just $\frac{1}{|S_q(T)|}\int_{S_q(T)} f dm_0$. The average of the RHS can be determined from the following lemma

Lemma 3. *The distribution of $\frac{\xi_\omega(T)}{\sqrt{T}}$, when ω runs over $T_q^1(M)$, convreges as $T \to \infty$ to the distribution of N.*

Proof. Let $W_\delta^s(\omega)$ be the neighbourhood of size δ of the point ω on the weak stable leaf of ω. For $\omega' \in W_\delta^s(\omega)$, $\|\frac{\xi_{\omega'}(T)}{\sqrt{T}} - \frac{\xi_\omega(T)}{\sqrt{T}}\| = O(\frac{1}{\sqrt{T}})$. Consider the set $E = \cup_{\omega \in T_q^1(M)} W_\delta^s(\omega)$. The set E has positive measure (it contains an open neighborhood of (x_0, y_0)), and the measure on E projects on $T_q^1(M)$ to the Lebesgue measure. It follows that the asymptotic distribution of $\frac{\xi_\omega(T)}{\sqrt{T}}$, when ω runs over $T_q^1(M)$ and T goes to infinity, is the same as the asymptotic distribution of $\frac{\xi_{\omega'}(T)}{\sqrt{T}}$, when ω' runs over E. This latter distribution converges to the distribution of N by lemma 2, part 1. □

3. Proof of lemmas 1 and 2

Symbolic dynamics. The proof of lemma 1 will make use of the symbolic coding of the geodesic flow. Before describing this coding, we recall some of the basic notions of symbolic dynamics that will be needed below.

A *subshift of finite type* with set of *states* S and *transition matrix* $A = (t_{ij})_{S\times S}$ ($t_{ij} = 0, 1$) is $\Sigma := \{x = (x_i) \in S^{\mathbb{Z}} : \forall i,\ t_{x_i x_{i+1}} = 1\}$ endowed with the metric $d(x,y) = \sum \frac{1}{2^{|k|}}(1 - \delta_{x_k y_k})$ and the action $\sigma : \Sigma \to \Sigma$, $\sigma(x)_k = x_{k+1}$. There is a *one–sided* version $\sigma : \Sigma^+ \to \Sigma^+$ obtained by replacing $\mathbb{Z}$ by $\mathbb{N} \cup \{0\}$. A subshift of finite type is *topologically mixing* iff $\exists m$ s.t. all the entries of A^m are positive.

Let G be a group, assumed for simplicity to be Abelian. The *skew-product* over $T : X \to X$ with the *cocycle* $f : X \to G$ is the map $T_f : X \times G \to X \times G$, $T(x,\xi) := \big(T(x), \xi + f(x)\big)$.

The *suspension semi-flow* over $T : X \to X$ and *height function* $r^* : X \to \mathbb{R}^+$ is the semi-flow $\varphi^s : X_{r^*} \to X_{r^*}$, where

$$X_{r^*} := \{(x,t) : x \in X, 0 \le t < r^*(x)\}$$

and $\varphi^s(x,t) := \big(T^n x, t+s-r_n^*(x)\big)$ where $r_n^*(x) = r^*(x)+r^*(Tx)+\cdots+r^*(T^{n-1}x)$ and n is chosen so that $0 \le t+s-r_n^*(x) < r^*(T^n x)$. If T is invertible, then this semi-flow has a unique extension to a flow. The suspension (semi)-flow can be identified with the (semi)-flow $(x,t) \mapsto (x, t+s)$ on $(X \times \mathbb{R})/\sim$ (respectively $X \times \mathbb{R}^+/\sim$) where $\sim$ is the orbit relation of the skew-product T_{-r^*}. Both descriptions shall be used below.

Now let $p : M \to M_0$ be a $\mathbb{Z}^d$–cover of a compact connected orientable C^∞ Riemannian surface M_0, and let $G \cong \mathbb{Z}^d$ be the group of deck transformations of the cover. Enumerate $G = \{D_\xi : \xi \in \mathbb{Z}^d\}$ in such a way that $D_{\xi_1} \circ D_{\xi_2} = D_{\xi_1+\xi_2}$ $(\xi_1, \xi_2 \in \mathbb{Z}^d)$.

We describe the geodesic flow on $g^s : T^1(M) \to T^1(M)$ as a suspension flow, whose base is a skew–product, whose base is a subshift of finite type. This description is well–known [**BL2**],[**Po**] and can be obtained by a straightforward lifting argument from Bowen's symbolic dynamics of $g^s : T^1(M_0) \to T^1(M_0)$ [**Bo2**].

Lemma 4. *Fix $i : T^1(M_0) \to T^1(M)$ 1-1 with connected image s.t. $dp \circ i = id$. There exist a topologically mixing two–sided subshift of finite type (Σ, T), Hölder continuous $r : \Sigma \to \mathbb{R}$ which depends only on the non-negative coordinates, $f : \Sigma \to \mathbb{Z}^d$ s.t. $f(x) = f(x_0, x_1)$, $h : \Sigma \to \mathbb{R}$ Hölder continuous, and a Hölder continuous surjection $\pi : \Sigma \times \mathbb{Z}^d \times \mathbb{R} \to T^1(M)$ with the following properties:*

1. *$r^* := r + h - h \circ \sigma$ is positive*
2. *$\pi : (\Sigma \times \{0\})_{r^*} \to i(N)$ is a bounded-to-one surjection, where*
$$(\Sigma \times \{0\})_{r^*} = \{(x, 0, t) : 0 \le t < r^*(x)\}.$$
3. *If $Q_{\xi_0,t_0}(x, \xi, t) = (x, \xi + \xi_0, t + t_0)$, then $\pi \circ Q_{(\xi_0,t_0)} = [g^{t_0} \circ D_{\xi_0}] \circ \pi$ for all $(\xi_0, t_0) \in \mathbb{Z}^d \times \mathbb{R}$;*
4. *$\pi \circ T_{(f,-r^*)} = \pi$ where $T_{(f,-r^*)}(x, \xi, t) = \big(Tx, \xi + f(x), t - r^*(x)\big)$;*
5. *Suppose $\omega = \pi(x, t, \xi), \omega' = \pi(x', t', \xi')$. Then*

$$\exists p, q \ge 0 \ s.t \begin{cases} x_p^\infty = (x')_q^\infty \\ t - t' = h(x) - h(x') + r_p(x) - r_q(x') \\ \xi - \xi' = f_q(x') - f_p(x) \end{cases} \implies \exists t \ s.t. \ \omega' = h^t(\omega).$$

Remark 3. *This construction can be made so that $\varepsilon^* := \sup r^*$ is arbitrarily small.*

Remark 4. [**Sh**, **C**] *In the context of geodesic flows of $\mathbb{Z}^d$–covers of compact connected surfaces of negative curvature, $\overline{\langle(-r_n(x), f_n(x)) : T^n x = x, \ n \in \mathbb{N}\rangle} = \mathbb{R} \times \mathbb{Z}^d$.*

Next we describe the Margulis measure in these coordinates. There exists a unique P such that $P_{top}(-Pr) = 0$ (as it turns out $P = \ln \mu$). Let $L_{-Pr} : C(\Sigma^+) \to C(\Sigma^+)$ be the operator $(L_{-Pr}F)(x) = \sum_{\sigma y = x} e^{-Pr(y)} F(y)$ (Ruelle's operator). By Ruelle's Perron-Frobenius theorem, there exists $\psi : \Sigma^+ \to \mathbb{R}^+$ Hölder continous, and a Borel probability measure ν' such that

$$L_{-Pr}\psi = \psi, \quad L^*_{-Pr}\nu' = \nu', \quad \text{and} \int \psi d\nu' = 1.$$

The measure $\psi d\nu'$ is a shift invariant probability measure which can be extended to the two-sided shift Σ. Denote this extension by ν.

Lemma 5. [**BM**] *The Margulis measure on $T^1(M)$, subject to the normalization $m_0(\widetilde{M_0}) = 1$, is $\frac{1}{\int r^* d\nu}(\nu \times dm_{\mathbb{Z}} \times dt)|_{(\Sigma \times \mathbb{Z}^d)_{r^*}} \circ \pi^{-1}$.*

Define for $\omega \in T^1(M)$, $W^{ss}(\omega) := \{h^t(\omega) : t \in \mathbb{R}\}$, and let ℓ_ω and $d_\omega(\cdot, \cdot)$ be the measure and metric on $W^{ss}(\omega)$ given by

$$\ell_\omega\{h^s(\omega) : a < s < b\} = b - a \text{ and } d_\omega\big(h^s(\omega), h^t(\omega)\big) = |s - t|.$$

The *symbolic local stable manifold* of $\omega = \pi(x, \xi, s + h(x))$ is defined by

$$W^{ss}_{loc}(\omega) := \pi\{(y, \xi, s + h(y)) : y_0^\infty = x_0^\infty\}.$$

Lemma 4 part (5) implies that this is a subset of $W^{ss}(\omega)$.

Lemma 6. [**BL2**] *There exists a constant c_0 such that if $\omega = \pi(x, \xi, s + h(x))$, then $\ell_\omega[W^{ss}_{loc}(\omega)] = \frac{c_0}{\mu^s}\psi(x_0, x_1, \ldots)$. We normalize the speed of h^s so that $c_0 = 1$.*[6]

Remark 5. $\inf \psi > 0$, *and the subshift Σ^+ can be recoded in such a way that $\log_\mu \sup_{x_0 = y_0} \frac{\psi(x)}{\psi(y)}$ is as small as we wish.*

(Replace the alphabet S by S^n for n large enough.)

Proof of lemma 1. We normalize m_0, for simplicity, to satisfy $m_0(\widetilde{M_0}) = 1$. Fix some $0 < \varepsilon_0 < \min r^*$. Using remark 5, we make sure that $\log_\mu \sup_{x_0 = y_0} \frac{\psi(x)}{\psi(y)} < \varepsilon_0$. Our set is going to be

$$E := \pi\big(\{(x, 0, t + h(x)) : x \in \Sigma, 0 \le t \le \varepsilon_0\}\big). \tag{2}$$

With this set in mind, define $I_T(\omega) := \int_0^T 1_E(h^s\omega)ds$.

Step 1. Fix T_0 large, and $T^* := \log_\mu T - \log_\mu T_0$. For all ω and $T > T_0$, there are $N^+, N^- \in \mathbb{N}$ and $\omega_i^* \in g^{T^*}[\{h^t\omega\}_{t=0}^T]$ $(i = 0, \ldots, N^+)$ s.t.

$$\sum_{i=0}^{N^-} J_{T^*}(\omega_i^*) \le I_T(\omega) \le \sum_{i=0}^{N^+} J_{T^*}(\omega_i^*), \text{ where } J_{T^*}(\omega_i^*) := \ell_\omega[E \cap g^{-T^*} W^{ss}_{loc}(\omega_i^*)].$$

T_0 can be chosen s.t. $\mu^{-\varepsilon^*} \le \frac{1}{T_0}\sum_{i=1}^{N^\pm} \ell_{\omega_i^*}[W^{ss}_{loc}(\omega_i^*)] \le \mu^{\varepsilon^*}$ on a set of full measure.

Proof. Fix $\omega \in E$. If $A_T := \{h^t(\omega) : 0 < t < T\}$, then $\int_0^T 1_E \circ h^t dt = \ell_\omega[E \cap A_T]$, because of the definition of ℓ_ω. By the definition of T^* and the commutation relation between the horocycle flow and the geodesic flow, $g^{T^*}[A_T]$ is also a horocyclic arc, and its length is T_0.

Let N^- be the number of different symbolic local stable manifolds which are contained in $g^{T^*}[A_T]$, and let N^+ be the number of symbolic local stable manifolds which intersect it with positive measure. These numbers are finite, because the measures of symbolic local stable manifolds is bounded below (Remark 5), and because for almost every ω, any two symbolic local stable manifolds are either equal or disjoint modulo ℓ_ω. (The disjointness is because ℓ_ω are the conditional measures of the m on the leaves of the strong stable foliation, and $|\pi^{-1}(\omega)| = 1$ for m–a.e. ω.)

Choose $\omega_i^* \in g^{T^*}(A_T)$ $(i = 1, \ldots, N^+)$ s.t. up to sets of $\ell_{g^T(\omega)}$–measure zero,

$$\bigcup_{i=1}^{N^-} W^{ss}_{loc}(\omega_i^*) \subseteq g^{T^*}(A_T) \subseteq \bigcup_{i=1}^{N^+} W^{ss}_{loc}(\omega_i^*).$$

We have: $\sum_{i=1}^{N^-} \ell_\omega[E \cap g^{-T^*} W^{ss}_{loc}(\omega_i^*)] \le \ell_\omega[E \cap A_T] \le \sum_{i=0}^{N^+} \ell_\omega[E \cap g^{-T^*} W^{ss}_{loc}(\omega_i^*)]$.

We now estimate $\frac{1}{T_0}\sum_{i=1}^{N^\pm} \ell_{\omega_i^*}[W^{ss}_{loc}(\omega_i^*)]$. We begin by showing that

$$\delta := \sup\{diam_{d_\omega}[W^{ss}_{loc}(\omega)] : \omega \in T^1(M)\} < \infty.$$

[6]Our results are invariant under time scaling, because $a(cT) \sim ca(T)$ (Theorem 4).

Assume by way of contradiction that δ is infinite. Since $d_\omega \circ dD = d_{dD(\omega)}$ for all $D \in G$, it is is possible to work on (the compact) $T^1(M_0)$ and find $\omega_n, \omega'_n \in T^1(M_0)$ on the same symbolic local stable manifold such that

$$d_{\omega_n}(\omega_n, \omega'_n) \xrightarrow[n\to\infty]{} \infty.$$

Writing ω_n, ω'_n in symbolic coordinates, we see that there must exist a subsequence n_k such that $\omega_{n_k} \to \omega$, $\omega'_{n_k} \to \omega'$ and ω, ω' are on the same symbolic local stable manifold. In particular, $\omega = h^{t_0}(\omega')$ for some t_0 (which we assume w.l.o.g. to be non-negative). For every $t > 0$, $h^t(\omega'_{n_k}) \to h^t(\omega')$. Since $d_{\omega_{n_k}}(\omega_{n_k}, \omega'_{n_k}) > t$ for all k large, $h^t(\omega'_{n_k})$ must be between[7] ω_{n_k} and ω'_{n_k} for all k large enough. It follows that $h^t(\omega')$ is between ω and $\omega' = h^{t_0}(\omega)$, whence $t_0 > t$. But this cannot be the case for all $t > 0$, a contradiction.

We can now choose T_0. We have already explained that for almost every ω, any two symbolic local stable manifolds are either equal or disjoint modulo ℓ_ω. Now, $T_0 = \ell_\omega[g^{T^*} A_T]$, and so $|\sum_{i=1}^{N^\pm} \ell_{\omega_i^*}[W^{ss}_{loc}(\omega_i^*)] - T_0| < 2\delta$. It follows that $\frac{1}{T_0}\sum_{i=1}^{N^\pm} \ell_{\omega_i^*}[W^{ss}_{loc}(\omega_i^*)] \xrightarrow[T_0\to\infty]{} 1$ uniformly on a set of full measure.

Step 2. Let $\omega = \pi(x, 0, t + h(x))$, $\omega_i^* = \pi(x_i^*, \xi_i^*, t_i^* + h(x_i^*))$, with $0 \le t < r^*(x)$, $0 \le t_i^* < r^*(x_i^*)$, and set $T_i^\# := T^* - t_i^*$ (note that $|T_i^\# - T^*| \le \max r^* = \varepsilon^*$). There is a concave real analytic $H : \mathbb{R}^d \to \mathbb{R}$, a compact neighbourhood K_0 of $0 \in \mathbb{R}^d$, such that uniformly in $\{\omega_i^* : \xi_i^*/T_i^\# \in K_0\}$,

$$J_{T^*}(\omega_i^*) = \frac{\mu^{\pm 4\varepsilon_0}(1 + o(1))}{(2\pi\sigma)^{d/2}} \frac{e^{H(\xi_i^*/T_i^\#)T_i^\#}}{(T^*)^{d/2}} \frac{\varepsilon_0}{\int r d\nu} \psi(x_i^*) \tag{3}$$

and $H(0) = \ln \mu$, $\nabla H(0) = 0$, and $H''(0) = -Cov(N)^{-1}$.

Proof. Fix $\omega = \pi(x, 0, t + h(x)) \in E$. By definition, $J_{T^*}(\omega_i^*)$ is equal to

$$\ell_\omega \left\{ \pi(y, 0, s + h(y)) : \begin{array}{l} 0 \le s \le \varepsilon_0, \text{ and} \\ g^{T^*}[\pi(y, 0, s + h(y))] \in W^{ss}_{loc}[\pi(x_i^*, \xi_i^*, t_i^* + h(x_i^*))] \end{array} \right\}$$

Calculating, we see that $g^{T^*}(\pi(y, 0, s+h(y)) = \pi\big(\sigma^n(y), f_n(y), s+h(y)+T^* - r_n^*(y)\big)$ for the unique n s.t. the third coordinate is in $[0, r^*(\sigma^n y))$. This point is in

$$W^{ss}_{loc}(\omega_i^*) = \pi\{(z, \xi_i^*, t_i^* + h(z)) : z_0^\infty = (x_i^*)_0^\infty\}$$

if and only if $y_n^\infty = (x_i^*)_0^\infty$, $f_n(y) = \xi_i^*$, and $s + h(y) + T^* - r_n^*(y) = t_i^* + h(\sigma^n y)$. Using the identity $r^* = r + h - h \circ \sigma$, we see that the last condition is equivalent to $s = r_n(y) - (T^* - t_i^*) = r_n(y) - T_i^\#$. Therefore:

$$\begin{aligned} J_{T^*}(\omega_i^*) &= (\ell_\omega \circ \pi) \left\{ (y, 0, s + h(y)) : \exists n \ge 0 \text{ such that } \begin{array}{l} \sigma^n(y)_0^\infty = (x_i^*)_0^\infty \\ f_n(y) = \xi_i^* \\ s = r_n(y) - T_i^\# \in [0, \varepsilon_0] \end{array} \right\} \\ &= \sum_{\substack{n \ge 0 \text{ and } y_0^\infty \text{ s.t. } y_n^\infty = (x_i^*)_0^\infty, \\ f_n(y) = \xi_i^*, \text{ and} \\ 0 \le r_n(y) - T_i^\# \le \varepsilon_0}} \ell_\omega\big(W^{ss}_{loc}(\omega_n[y_0^\infty])\big), \end{aligned}$$

where $\omega_n[y_0^\infty] := \pi(y, 0, r_n(y_0^\infty) - T_i^\# + h(y))$ for some $y \in [y_0^\infty]$.

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[7]The notion 'between' is well–defined, because the orbits of the horocycle flow are all homeomorphic to $\mathbb{R}$: Closed horocycles on $T^1(M)$ imply closed horocycle on $T^1(M_0)$, in contradiction to the unique ergodicity of the horocycle flow on that surface [**BM**].

By lemma 6, $\ell_\omega\big(W^{ss}_{loc}(\omega_n[y_0^\infty])\big) = \frac{\psi(y)}{\mu^{r_n(y)-T_i^\#}} = \mu^{\pm\varepsilon_0}\psi(y_0, y_1, \ldots)$. Consequently,

$$J_{T^*}(\omega_i^*) = \mu^{\pm\varepsilon_0}\sum_{n=0}^{\infty}\sum_{\sigma^n(y)=(x_i^*)_0^\infty} 1_{[0,\varepsilon_0]}(r_n(y) - T_i^\#)\delta_{\xi_i^*, f_n(y)}\psi(y),$$

where the y's in this sum take values in the *one–sided* shift Σ^+, and δ_{ij} is Kronecker's delta. The asymptotic behaviour of this sum is described by theorem 4 of [**BL2**] (this estimation is reproduced in the appendix), and the step follows.

Step 3. Completion of the proof.

Proof. Set $C_0 = \frac{1}{(2\pi\sigma)^{d/2}}\frac{\varepsilon_0}{\int r d\nu}$. The estimate $|T^* - T_i^\#| \leq \varepsilon^*$ together with the previous steps show that uniformly on $\{\omega : \xi_i^*/T_i^\# \in K_0\}$,

$$\begin{aligned} I_T(\omega) &\leq C_0 e^{\varepsilon^* H(0)}\mu^{4\varepsilon_0}[1+o(1)]\frac{e^{H(0)T^*}}{(T^*)^{d/2}}\sum_{i=1}^{N^+} e^{[H(\frac{\xi_i^*}{T_i^\#})-H(0)]T_i^\#}\psi(x_i^*), \\ I_T(\omega) &\geq C_0 e^{-\varepsilon^* H(0)}\mu^{-4\varepsilon_0}[1+o(1)]\frac{e^{H(0)T^*}}{(T^*)^{d/2}}\sum_{i=1}^{N^-} e^{[H(\frac{\xi_i^*}{T_i^\#})-H(0)]T_i^\#}\psi(x_i^*). \end{aligned}$$

Recall that $\nabla H(0) = 0$ and that $H''(0) = \mathrm{Cov}(N)^{-1}$ is negative definite. The function $\|w\|_H := \sqrt{-w^t H''(0) w}$ is a norm on $\mathbb{R}^d$, and $H(w) = H(0) - \frac{1}{2}\|w\|_H^2 + o(\|w\|_H^2)$ as $\|w\|_H \to 0$ (all norms on $\mathbb{R}^d$ are equivalent). Fixing an $\varepsilon > 0$, we see that there exists δ_0 such that $-(\frac{1}{2}+\delta_0)\|w\|_H^2 \leq H(w) - H(0) \leq -(\frac{1}{2}-\delta_0)\|w\|_H^2$ on the ellipsoid $[\|w\|_H < \delta_0]$. Recall that for every i, $\frac{\xi_i^*}{T_i^\#} \sim \frac{\xi_\omega(T^*)}{T^*}$ uniformly as $T \to \infty$. Therefore, we can make δ_0 so small that $\|\frac{\xi_\omega(T)}{T}\|_H < \delta_0 \Rightarrow \frac{\xi_i^*}{T_i^\#} \in K_0$ and

$$-\frac{1}{2}(1+\varepsilon)\|\tfrac{\xi_\omega(T^*)}{\sqrt{T^*}}\|_H^2 \leq [H(\tfrac{\xi_i^*}{T_i^\#}) - H(0)]T_i^\# \leq -\frac{1}{2}(1-\varepsilon)\|\tfrac{\xi_\omega(T^*)}{\sqrt{T^*}}\|_H^2.$$

Next recall that by the definition of ψ (Lemma 6) and the choice of T_0 (step 1),

$$\sum_{i=1}^{N^\pm}\psi(x_i^*) = \mu^{\pm\varepsilon^*}T_0\cdot\frac{1}{T_0}\sum_{i=1}^{N^\pm}\ell_{\omega_i^*}[W^{ss}_{loc}(\omega_i^*)] = \mu^{\pm 2\varepsilon^*}T_0.$$

Using the identities $e^{H(0)} = \mu$, $\mu^{T^*} = T/T_0$, $m_0(E) = \frac{\varepsilon_0}{\int r d\nu}$, we obtain

$$\begin{aligned}
I_T(\omega) &\leq C_0 \mu^{\varepsilon^*} \mu^{4\varepsilon_0}[1+o(1)] \frac{\mu^{T^*}}{(T^*)^{d/2}} e^{-\frac{1}{2}(1-\varepsilon)\|\frac{\xi_\omega(T^*)}{\sqrt{T^*}}\|_H^2} \sum_{i=1}^{N^+} \psi(x_i^*) \\
&\leq C_0 T_0 \mu^{3\varepsilon^*+4\varepsilon_0}[1+o(1)] \frac{\mu^{T^*}}{(T^*)^{d/2}} e^{-\frac{1}{2}(1-\varepsilon)\|\frac{\xi_\omega(T^*)}{\sqrt{T^*}}\|_H^2} \\
&= \frac{\mu^{3\varepsilon^*+4\varepsilon_0}[1+o(1)]}{(2\pi\sigma)^{d/2}} \frac{T}{(\log_\mu T)^{d/2}} e^{-\frac{1}{2}(1-\varepsilon)\|\frac{\xi_\omega(\log_\mu T)}{\sqrt{\log_\mu T}}\|_H^2} m_0(E) \\
I_T(\omega) &\geq C_0 \mu^{-\varepsilon^*} \mu^{-4\varepsilon_0}[1+o(1)] \frac{\mu^{T^*}}{(T^*)^{d/2}} e^{-\frac{1}{2}(1+\varepsilon)\|\frac{\xi_\omega(T^*)}{\sqrt{T^*}}\|_H^2} \sum_{i=1}^{N^-} \psi(x_i^*) \\
&= \frac{\mu^{-(3\varepsilon^*+4\varepsilon_0)}[1+o(1)]}{(2\pi\sigma)^{d/2}} \frac{T}{(\log_\mu T)^{d/2}} e^{-\frac{1}{2}(1+\varepsilon)\|\frac{\xi_\omega(\log_\mu T)}{\sqrt{\log_\mu T}}\|_H^2} m_0(E)
\end{aligned}$$

These estimate are valid and uniform for ω in $[\|\frac{\xi_\omega(T^*)}{T^*}\|_H \leq \delta_0]$. Recalling that $\varepsilon^*, \varepsilon_0$ can be made arbitrarily small, and that m_0 was assumed to be normalized so that $m_0(\widetilde{M}_0) = 1$, we see that lemma 1 is proved. □

Proof of lemma 2. Define for $\underline{\eta} \in \mathbb{R}^d$, $\varphi_{\underline{\eta}}(\cdot) := \langle \underline{\eta}, \cdot \rangle$. Denker and Philipp [**DP**] show how to construct for every $\underline{\eta}$ a Brownian motion $\{B_{\underline{\eta}}(\omega,t)\}_{t>0}$ on $(T^1(\widetilde{M}_0), \mathcal{B}(\widetilde{M}_0, m_0))$ such that

$$\left| B_{\underline{\eta}}(\omega,t) - \varphi_{\underline{\eta}}(\xi_\omega(t)) \right| = o(\sqrt{t}), \text{ almost surely as } t \to \infty. \tag{4}$$

Ratner's CLT for $\frac{\xi_\omega(T)}{\sqrt{T}}$ guarantees that $B_{\underline{\eta}}(\cdot, 1)$ is distributed like $\varphi_{\underline{\eta}}(N)$.

We denote by $\mathcal{B}_{\underline{\eta}}$ the σ–algebra generated by $\{B_{\underline{\eta}}(\cdot,t)\}_{t>0}$. The statement that $B_{\underline{\eta}}(\omega,t)$ is Brownian motion is the same as saying that $(\widetilde{M}_0, \mathcal{B}_{\underline{\eta}}, m_0/m(\widetilde{M}_0))$ is measure algebra isomorphic to $(\Omega_{\underline{\eta}}, \mathcal{F}_{\underline{\eta}}, dW_{\underline{\eta}})$: Wiener's measure space of Brownian Motion normalized to be distributed like $\varphi_{\underline{\eta}}(N)$ at time $t = 1$.

This measure space is endowed with the following renormalization flow

$$\Phi_t : \Omega_{\underline{\eta}} \to \Omega_{\underline{\eta}}, \quad \Phi_t[\{B(t)\}_{t>0}] := \{e^{-t/2} B(e^t)\}_{t>0}.$$

This flow is strongly mixing (in fact, Bernoulli [**Fi**]).

We now use the inversion formula for the Fourier transform to write

$$\begin{aligned}
\int_E f(\frac{\xi_\omega(t)}{\sqrt{t}}) dm_0 &= \frac{1}{(2\pi)^d} \int_E \int_{\mathbb{R}^d} e^{i\varphi_{\underline{\eta}}(\frac{\xi_\omega(t)}{\sqrt{t}})} \widehat{f}(\underline{\eta}) d\underline{\eta} dm_0(\omega) \\
&= \frac{1}{(2\pi)^d} \int_{\widetilde{M}_0} 1_E(\omega) \int_{\mathbb{R}^d} e^{i\frac{B_{\underline{\eta}}(\omega,t)}{\sqrt{t}}} \widehat{f}(\underline{\eta}) d\underline{\eta} dm_0(\omega) + o(1) \\
&= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{f}(\underline{\eta}) \int_{\widetilde{M}_0} \mathbb{E}(1_E | \mathcal{B}_{\underline{\eta}}) e^{i\frac{B_{\underline{\eta}}(\omega,t)}{\sqrt{t}}} dm_0(\omega) d\underline{\eta} + o(1).
\end{aligned}$$

Using the measure algebra isomorphism between $(\widetilde{M}_0, \mathcal{B}_{\underline{\eta}}, m_0/m(\widetilde{M}_0))$ and $(\Omega_{\underline{\eta}}, \mathcal{F}_{\underline{\eta}}, dW_{\underline{\eta}})$, we represent the inner integral as

$$\frac{1}{m_0(\widetilde{M}_0)} \int_{\widetilde{M}_0} \mathbb{E}(1_E | \mathcal{B}_{\underline{\eta}}) e^{i\frac{B_{\underline{\eta}}(\omega,t)}{\sqrt{t}}} dm_0(\omega) = \int_{\Omega_{\underline{\eta}}} F_1 \cdot F_2 \circ \Phi_{\ln t} dW_\eta,$$

where F_1 is the Wiener space model of $\mathbb{E}(1_E|\mathcal{B}_{\underline{\eta}})$ and $F_2(\{B(t)\}_{t>0}) = e^{iB(1)}$. The strong mixing of Φ_t now implies that

$$\int_{\widetilde{M}_0} \mathbb{E}(1_E|\mathcal{B}_{\underline{\eta}}) e^{i\frac{B_{\underline{\eta}}(\omega,t)}{\sqrt{t}}} dm_0(\omega) \sim m(\widetilde{M}_0)\mathbb{E}(F_1)\mathbb{E}(F_2) = m_0(E)\mathbb{E}(e^{i\varphi_{\underline{\eta}}(N)}).$$

Since this integral is uniformly bounded (by $m_0(E)$), and since $\widehat{f}$ is absolutely integrable, we can replace the order of the limit and the integral, and obtain

$$\begin{aligned}\lim_{t\to\infty}\int_E f(\frac{\xi_\omega(t)}{\sqrt{t}})dm_0 &= m_0(E)\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\widehat{f}(\underline{\eta})\mathbb{E}(e^{i\varphi_{\underline{\eta}}(N)})d\underline{\eta} \\ &= m_0(E)\mathbb{E}\left(\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\widehat{f}(\underline{\eta})e^{i\varphi_{\underline{\eta}}(N)}d\underline{\eta}\right) = m_0(E)\mathbb{E}[f(N)],\end{aligned}$$

proving the first part of the lemma.

The second step is done in a similar way, but using the Birkhoff ergodic theorem. We begin by considering the case $f(\underline{x}) = e^{i\varphi_{\underline{\eta}}(\underline{x})}$ for some fixed $\underline{\eta} \in \mathbb{R}^d$. For every ω which satisfies (4) w.r.t. this $\underline{\eta}$:

$$\begin{aligned}\frac{1}{\ln s}\int_3^s \frac{1}{t}e^{i\varphi_{\underline{\eta}}\left(\frac{\xi_\omega(t)}{\sqrt{t}}\right)}dt &= \frac{1}{\ln s}\int_3^s \frac{1}{t}e^{i[\frac{B_{\underline{\eta}}(\omega,t)}{\sqrt{t}}+o(1)]}dt \\ &= \frac{1}{\ln s}\int_3^s \frac{1}{t}(F_2\circ\Phi_{\ln t})(\{B_{\underline{\eta}}(\omega,t)\}_{t>0})dt + o(1) \\ &= \frac{1}{\ln s}\int_{\ln 3}^{\ln s}(F_2\circ\Phi_T)(\{B_{\underline{\eta}}(\omega,t)\}_{t>0})dT + o(1).\end{aligned}$$

Applying the Birkhoff ergodic theorem for $dW_{\underline{\eta}}$, we see that:

$$\text{For every } \underline{\eta}\in\mathbb{R}^d \ \left\{\lim_{s\to\infty}\frac{1}{\ln s}\int_3^{\ln s} e^{i\varphi_{\underline{\eta}}(\frac{\xi_\omega(t)}{\sqrt{t}})}\frac{dt}{t} = \mathbb{E}[e^{i\varphi_{\underline{\eta}}(N)}] \text{ for a.e. } \omega\right\}.$$

By Fubini's theorem, the following set has full measure in $\widetilde{M}_0\times\mathbb{R}^d$ with respect to $m\times$Lebesgue: $\Omega := \left\{(\omega,\underline{\eta})\in\widetilde{M}_0\times\mathbb{R}^d : \lim_{s\to\infty}\frac{1}{\ln s}\int_3^s e^{i\varphi_{\underline{\eta}}(\frac{\xi_\omega(t)}{\sqrt{t}})}\frac{dt}{t} = \mathbb{E}[e^{i\varphi_{\underline{\eta}}(N)}]\right\}$. It follows that

$$\text{For almost every } \omega \ \left\{\lim_{s\to\infty}\frac{1}{\ln s}\int_3^s e^{i\varphi_{\underline{\eta}}(\frac{\xi_\omega(t)}{\sqrt{t}})}\frac{dt}{t} = \mathbb{E}[e^{i\varphi_{\underline{\eta}}(N)}] \text{ for a.e. } \underline{\eta}\right\}.$$

Integrating this limit against $\widehat{f}(\underline{\eta})$ (assumed to be in $L^1(\mathbb{R}^d)$), we have by the dominated convergence theorem and the inversion formula for the Fourier transform that

$$\lim_{s\to\infty}\frac{1}{\ln s}\int_3^s f(\frac{\xi_\omega(t)}{\sqrt{t}})\frac{dt}{t} = \mathbb{E}[f(N)]$$

for almost every $\omega\in\widetilde{M}_0$. □

4. Appendix

Let Σ^+ be a one–sided subshift of finite type, and let $r,\psi:\Sigma^+\to\mathbb{R}$, $f:\Sigma^+\to\mathbb{Z}^d$ two Hölder continuous functions, with r cohomologous (as a function on Σ) to a positive function $r^*:\Sigma\to\mathbb{R}$. We assume using remark 4 that

$$\overline{\langle(-r_n(x),f_n(x)) : T^n x = x,\ n\in\mathbb{N}\rangle} = \mathbb{R}\times\mathbb{Z}^d, \tag{5}$$

and explain how to obtain the asymptotic expansion of

$$A(x,\xi,T)=\sum_{n=0}^{\infty}\sum_{\sigma^n y=x}1_{[0,\varepsilon_0]}(r_n(y)-T)\delta_\xi\big(f_n(y)\big)\psi(y).$$

that was crucial to the proof of lemma 1. What follows is an expanded form of the the proof of theorem 4 in [**BL2**], and is not new. We decided to include it for completeness, and because we needed to emphasize that the following estimates are uniform.

Step 1. Rewriting $A(x,\xi,T)$ in terms of complex Ruelle operators.

Fix $\delta_0>0$, and introduce two parameters $P\in\mathbb{R}$ and $u\in\mathbb{R}^d$ (which shall later be calibrated to obtain optimal asymptotics). Construct two even functions $\gamma_1,\gamma_2\in L^1(\mathbb{R})$ such that $e^{-sP}\gamma_1(s)\le\frac{1}{(2\pi)^d}1_{[0,\varepsilon_0]}(s)\le e^{-sP}\gamma_2(s)$ in such a way that their Fourier transforms $\widehat{\gamma}_1,\widehat{\gamma}_2$ have compact support, belong to $C^N(\mathbb{R})$ for $N>2d+10$, and satisfy $e^{-\delta_0}\le\widehat{\gamma}_1(0)/\widehat{\gamma}_2(0)<e^{\delta_0}$. We have:[8]

$$\widehat{\gamma}_i(0)=\frac{\varepsilon_0}{(2\pi)^d}e^{\pm(\delta_0+\varepsilon_0P)}. \tag{6}$$

Set $\mathbb{T}^d:=[0,2\pi]^d$. The identity $\int_{\mathbb{T}^d}e^{i\langle w,\xi-f_n(y)\rangle}\frac{dw}{(2\pi)^d}=\delta_\xi\big(f_n(y)\big)$ shows that $A_1(x,\xi,T)\le A(x,\xi,T)\le A_2(x,\xi,T)$ where

$$A_i(x,\xi,T):=\sum_{n=0}^{\infty}\sum_{\sigma^n y=x}e^{P[T-r_n(y)]}\gamma_i(r_n(y)-T)e^{\langle u,f_n(y)-\xi\rangle}\psi(y)\int_{\mathbb{T}^d}e^{i\langle w,\xi-f_n(y)\rangle}dw.$$

Fourier's inversion formula gives

$$A_i(x,\xi,T)=\frac{1}{2\pi}e^{T(P-\langle u,\frac{\xi}{T}\rangle)}\sum_{n=0}^{\infty}\int_{\mathbb{R}\times\mathbb{T}^d}\Big[e^{-iT(\alpha-\langle w,\frac{\xi}{T}\rangle)}\widehat{\gamma}_i(\alpha)\times$$
$$\times\sum_{\sigma^n y=x}e^{(-P+i\alpha)r_n(y)+\langle u-iw,f_n(y)\rangle}\psi(y)\Big]\,d\alpha dw$$

We simplify this expression using *Ruelle operators* [**Bo2**]. Define for $s:=P-i\alpha$, $z:=u-iw$ the following operator on $C(\Sigma^+)$:

$$(L_{s,z}\varphi)(x)=\sum_{\sigma y=x}e^{-sr(y)+\langle z,f(y)\rangle}\varphi(y).$$

One easily checks that $(L^n_{s,z}\varphi)(x)=\sum_{\sigma^n y=x}e^{-sr_n(y)+\langle z,f_n(y)\rangle}\varphi(y)$. Consequently,

$$A_i(x,\xi,T)=\frac{1}{2\pi}e^{T(P-\langle u,\frac{\xi}{T}\rangle)}\sum_{n=0}^{\infty}\int_{\mathbb{R}\times\mathbb{T}^d}e^{-iT(\alpha-\langle w,\frac{\xi}{T}\rangle)}\widehat{\gamma}_i(\alpha)(L^n_{s,z}\psi)(x)\,d\alpha dw, \tag{7}$$

with $s:=P-i\alpha$, $z:=u-iw$. It is much easier to study the quantities

$$\widetilde{A_i}(x,\xi,T)=\frac{1}{2\pi}e^{T(P-\langle u,\frac{\xi}{T}\rangle)}\int_{\mathbb{R}\times\mathbb{T}^d}e^{-iT(\alpha-\langle w,\frac{\xi}{T}\rangle)}\widehat{\gamma}_i(\alpha)\sum_{n=0}^{\infty}(L^n_{s,z}\psi)(x)\,d\alpha dw, \tag{8}$$

with $s:=P-i\alpha$, $z:=u-iw$, because of the possibility of bringing in operator theoretic methods to study $\sum_{n=0}^{\infty}(L^n_{s,z}\psi)(x)$.

[8]Here and throughout the Fourier transform is $\widehat{f}(s)=\int_{\mathbb{R}}e^{-isx}f(x)dx$.

We shall do so below, and analyze $\widetilde{A_i}(x,\xi,T)$; The information that we will gather in the process will eventually allow us justify the summation under in the integral and show that $\widetilde{A_i}(x,\xi,T) = A_i(x,\xi,T)$ (step 7).

Step 2. Summation of $\sum_{n=0}^{\infty}(L_{s,z}^n\psi)(x)$, leading to the first constraint on P and u.

Let κ be a (common) exponent of Hölder continuity of r, ψ and f. It is well–known that $L_{s,z}$ acts continuously on

$$\mathcal{H}_\kappa := \{F : \|F\|_\kappa := \|F\|_\infty + \sup |F(x) - F(y)|/d(x,y)^\kappa\}.$$

Set $L = L_{-P,u} : \mathcal{H}_\kappa \to \mathcal{H}_\kappa$. The following is known [**PP**]:

(a) L has a positive eigenvalue λ. This eigenvalue is equal to $\exp P_{top}(-Pr + \langle u, f\rangle)$, where $P_{top} : C(\Sigma^+) \to \mathbb{R}$ is the topological pressure functional.
(b) $L = \lambda Q + N$, where $\lambda > 0$, $\dim[\mathrm{Im}(Q)] = 1$, $LQ = Q$, $QN = NQ = 0$, and the spectral radius of N is strictly less than λ;
(c) The spectral radius of $L_{-P+i\alpha,u-iw}$ is smaller than λ when $(\alpha, w) \neq (0,0)$.

The first and second statements are a re-formulation of Ruelle's Perron-Frobenius theorem. The third statement follows from (5) and theorem 4.5 in [**PP**].

The map $(s,z) \mapsto L_{s,z}$ is a holomorphic map from $\mathbb{C} \times \mathbb{C}^d$ to $\mathrm{Hom}(\mathcal{H}_\kappa, \mathcal{H}_\kappa)$. Standard analytic perturbation techniques provide $\varepsilon_1 > 0$ and $\lambda(s,z), Q_{s,z}, N_{s,z}$ holomorphic in $\{(s,z) \in \mathbb{C} \times \mathbb{C}^d : |s - P|, \|z - u\| < \varepsilon_1\}$ such that

$$L_{s,z} = \lambda(s,z)Q_{s,z} + N_{s,z}$$

where $Q_{s,z}^2 = Q_{s,z}$, $\dim[\mathrm{Im}(Q_{s,z})] = 1$, $Q_{s,z}N_{s,z} = N_{s,z}Q_{s,z} = 0$, and $N_{s,z}$ has spectral radius strictly less than $|\lambda_{s,z}|$. Calculating, we see that

$$L_{s,z}^n = \lambda(s,z)^n Q_{s,z} + N_{s,z}^n.$$

Now, the spectral radius of $N_{s,z}$ is strictly less than $\lambda = \lambda(P,u)$. If this number were equal to one, then $\sum N_{s,z}^n$ would converge in norm. This is our first constraint on P and u:

$$P \text{ and } u \text{ should satisfy } P_{top}(-Pr + \langle u, f\rangle) = 0.$$

We will discuss the possibility of choosing such P and u later. For the moment, we note that if this condition is satisfied, then we can sum over n and obtain

$$\sum_{n=0}^{\infty}(L_{s,z}^n\psi)(x) = \frac{A_1(s,z)(x)}{1 - \lambda(s,z)} + B_1(s,z)(x) \tag{9}$$

where $A_1(s,z) := P_{s,z}\psi$ and $B_1(s,z) := (I - N_{s,z})^{-1}\psi$. Note that these functions are analytic in an ε_1–neighbourhood of $s = P, z = u$. We can also arrange for $A_1(s,z)$ not vanish in this neighbourhood.[9] We have:

$$A_1(P(0),0) = Q_{P(0),0}\psi = \psi(x). \tag{10}$$

We now discuss the constraint $P_{top}(-Pr+\langle u, f\rangle) = 0$. The following are standard properties of the topological pressure functional (see e.g. [**Bo2**]):

(a) $P_{top}(t\phi_1 + (1-t)\phi_2) \le tP_{top}(\phi_1) + (1-t)P_{top}(\phi_2)$;
(b) $P_{top}(\phi_1) + \min \phi_2 \le P_{top}(\phi_1 + \phi_2) \le P_{top}(\phi_1) + \max \phi_2$.
(c) $P_{top}(f + h - h \circ \sigma) = P_{top}(f)$ for all f, h continuous.

[9] Here is how: $A_1(P,u)$ is the eigenfunction of L. It is easy to check directly from the definition of L that this eigenfunction is strictly positive, and therefore bounded away from zero. If ε_1 is small enough, its perturbations must also be strictly positive.

Fix u and consider $p_u(t) := P_{top}(-tr + \langle u, f\rangle)$. The first property says that $p_u(t)$ is convex, whence continuous. The third says that we can replace r by the positive r^*. The second now implies that $p_u(t) \xrightarrow[t\to\pm\infty]{} \mp\infty$. It follows that $\exists t$ such that $p_u(t) = 0$. Another look at the second property shows that $p_u(t)$ is strictly decreasing, and therefore this t is unique. Consequently, for every $u \in \mathbb{R}^d$, there exists a unique $P = P(u)$ for which $P_{top}(-Pr + \langle u, f\rangle) = 0$. This function can be easily seen to be continuous in u.

One conclusion is that the set $\Lambda := \{(P, u) \in \mathbb{R} \times \mathbb{R}^d : P_{top}(-Pr + \langle u, f\rangle) = 0\}$ is pathwise connected. In particular, Λ is connected. It follows that the analytic functions $A_1(s, z), B_1(s, z), \lambda(s, z)$ which were defined locally in the neighbourhoods of $(P, u) \in \Lambda$ patch up to well-defined $\mathcal{H}_\kappa$–valued analytic functions in an open complex neighbourhood of Λ, and (9) holds in this neighbourhood.

Step 3. Expansion of the eigenvalue $\lambda(s, z)$, and absolute integrability of $\sum_{n\geq 0} L^n_{s,z}\psi$.

We expand $\lambda(s, z)$ in the neighbourhood of Λ. Again, the connectedness of Λ allows us to argue locally in a neighbourhood of a fixed (P, u) such that $P_{top}(-Pr + \langle u, f\rangle) = 0$. Write, using the Weierstrass preparation theorem

$$1 - \lambda(s, z) = A_2(s, z)[s - P(z)],$$

where A_2, P are analytic and $A_2 \neq 0$ in a neighbourhood of (P, u). Note that $P(u) = P$, because $\lambda(P, u) = 1$. In particular, $1 - \lambda(P - i\alpha, 0) = -i\alpha A_2(P - i\alpha, 0)$. Differentiating in α, we get (see e.g. [**PP**] proposition 4.10),

$$A_2(P(0), 0) = -\left.\frac{d\lambda(s, 0)}{ds}\right|_{s=P} = -\int r^* d\nu \neq 0. \tag{11}$$

Taylor's expansion gives for $s = P - i\alpha$, $z = u - iw$,

$$1 - \lambda(s, z) = -A_2(s, z)[i(\alpha - \langle\nabla P(u), w\rangle) - R(u - iw)],$$

where $R(u - iw) = \frac{1}{2}w^t P''(u)w + o(\|w\|^2)$, and $P''(u) := \big(\frac{\partial^2 P}{\partial u_i \partial u_j}(u)\big)_{d\times d}$. As explained in [**BL1**], page 30,

$$P''(0) = \mathrm{Cov}(N).$$

In particular, $P''(0)$ is positive definite.

Summarizing, we see that there is an open set $U \supset \Lambda$ and holomorphic functions $A(s, z), B(s, z) \in \mathcal{H}_\kappa$, $R(z)$ on U such that for every $u \in \mathbb{R}^d$ and α, w such that $(s, z) := (P(u) - i\alpha, u - iw) \in U$,

$$\sum_{n=0}^{\infty}(L^n_{s,z}\psi)(x) = \frac{A(s, z)(x)}{i(\alpha - \langle\nabla P(u), w\rangle) - R(u - iw)} + B(s, z)(x), \tag{12}$$

where $R(u - iw) = \frac{1}{2}w^t P''(u)w + o(\|w\|^2)$ as $w \to 0$, and $A(s, z) := -\frac{A_1(s,z)}{A_2(s,z)} \neq 0$. We also note that, by equations (10) and (11),

$$A(P(0), 0)(x) = \frac{1}{\int r d\nu}\psi(x). \tag{13}$$

We now explain how to modify (12) so that it holds for *all* $(\alpha, w) \in \mathbb{R} \times \mathbb{T}^d$. We have already mentioned that thanks to (5), the spectral radius of $L_{s,z}$ is strictly less than one for $(\alpha, w) \neq (0, 0)$. Consequently, the left hand side of (12) is holomorphic outside U, and is equal to the right hand side inside U. It is now a standard procedure using C^∞–bump functions to redefine A, B and R in such a way that

(12) holds on $\mathbb{R}\times\mathbb{T}^d$. The only sacrifice we need to make is that A, B, R are become C^∞, but not necessarily holomorphic.

Next, note that Re $R(u-iw)$ is positive for $\|w\|$ small because $P''(u)$ is positive definite. Using the method of C^∞–functions we can modify $A(s,z)$, $B(s,z)$ and $R(z)$ so that Re $R(u-iw)$ is positive for all $w\in\mathbb{T}^d$ and all u in any given compact neighbourhood. (The neighbourhood we need is $(\nabla P)^{-1}(K_0)$ where K_0 is described by the next step.)

Step 4. A useful change of variables, leading to the second constraint on P and u, the definition of $H(\cdot)$, and the choice of K_0.

The expansion we got for $\sum(L^n_{s,z}\psi)(x)$ gives for $(s,z)=(P(u)-i\alpha, u-iw)$:

$$\begin{aligned}\widetilde{A_i}(x,\xi,T) = \frac{1}{2\pi}e^{T(P-\langle u,\frac{\xi}{T}\rangle)}\int_{\mathbb{R}\times\mathbb{T}^d}\frac{e^{-iT(\alpha-\langle w,\frac{\xi}{T}\rangle)}\widehat{\gamma}_i(\alpha)A(s,z)(x)}{i(\alpha-\langle\nabla P(u),w\rangle)-R(u-iw)}d\alpha dw\\ +\frac{1}{2\pi}e^{T(P-\langle u,\frac{\xi}{T}\rangle)}\int_{\mathbb{R}\times\mathbb{T}^d}e^{-iT(\alpha-\langle w,\frac{\xi}{T}\rangle)}\widehat{\gamma}_i(\alpha)B(s,z)(x)d\alpha dw\end{aligned}$$

The change of variables $\alpha\to\alpha+\langle w,\frac{\xi}{T}\rangle$ would make the $\int_{\mathbb{R}}d\alpha$ integral the Fourier transform at T of two functions, which give rise to integrals whose asymptotic behavior can then be determined. This change of variables would be particularly convenient, if we require the following second constraint on P and u:

$$P \text{ and } u \text{ should satisfy: } \nabla P(u)=\xi/T,$$

because together with the condition $P=P(u)$, it would the lead to

$$\begin{aligned}\widetilde{A_i}(x,\xi,T) = \frac{1}{2\pi}e^{TH(\frac{\xi}{T})}\int_{\mathbb{R}\times\mathbb{T}^d}e^{-iT\alpha}\frac{a(\alpha,w)(x)}{i\alpha-R(u-iw)}d\alpha dw\\ +\frac{1}{2\pi}e^{TH(\frac{\xi}{T})}\int_{\mathbb{R}\times\mathbb{T}^d}e^{-iT\alpha}b(\alpha,w)(x)d\alpha dw,\ \text{where}\end{aligned}\tag{14}$$

- H :=minus the Legendre transform of P, so $H''(0)=-P''(0)^{-1}=-\mathrm{Cov}(N)^{-1}$;
- $a(\alpha,w)(x):=\widehat{\gamma}_i(\alpha+\langle w,\frac{\xi}{T}\rangle)A(P(u)-i\alpha-i\langle w,\frac{\xi}{T}\rangle,u-iw)(x)$;
- $b(\alpha,w)(x):=\widehat{\gamma}_i(\alpha+\langle w,\frac{\xi}{T}\rangle)B(P(u)-i\alpha-i\langle w,\frac{\xi}{T}\rangle,u-iw)(x)$.

We now discuss the possibility of choosing P and u such that $\nabla P(u)=\frac{\xi}{T}$ and $P=P(u)$ ($\Leftrightarrow P_{top}(-Pr+\langle u,f\rangle)=0$). As shown in [**BL1**], $u\mapsto\nabla P(u)$ is a diffeomorphism from $\mathbb{R}^d$ onto an open set $V\subset\mathbb{R}^d$, and $\nabla P(0)=0\in V$. Fix a compact neighbourhood K_0 of 0 inside V. If $\frac{\xi}{T}\in K_0$, then there exists $u\in\mathbb{R}^d$ such that $\nabla P(\frac{\xi}{T})=u$. Once we have this u, choose $P=P(u)$.

We see that as long as $\frac{\xi}{T}\in K_0$, our expansion is valid. We also note that $a(\alpha,w)(x)$, $b(\alpha,w)(x)$, $R(u-iw)$ all depend in a C^N way on $\frac{\xi}{T}$ (via their dependence on u and P).

Finally we make K_0 small enough that if $\frac{\xi}{T}\in K_0$ then u is so close to zero that $A(P(u),u)=e^{\pm\delta_0}A(P(0),0)$.

Step 5. Asymptotic analysis of the integrals in (14).

Throughout this step $\xi/T\in K_0$, P, and u are fixed so that $\nabla P(u)=\frac{\xi}{T}$ and $P_{top}(-Pr+\langle u,f\rangle)=0$.

The integral $\int_{\mathbb{R}\times\mathbb{T}^d}e^{-iT\alpha}b(\alpha,w)(x)d\alpha dw$ is, up to a constant, the Fourier transform of $\int_{\mathbb{T}^d}b(\cdot,w)(x)dw$ evaluated at T. The map $\alpha\mapsto\int_{\mathbb{T}^d}b(\cdot,w)(x)dw\in\mathcal{H}_\kappa$ is

C^{2d+10}. Consequently, it has $2d+10$ many derivatives (all of which are elements in $\mathcal{H}_\kappa$), and

$$\sup_{x\in\Sigma^+}\left\|\int_{\mathbb{T}^d} b(\cdot,w)(x)dw\right\|_{C^{2d+10}}<\infty.$$

Since $\int_{\mathbb{T}^d} b(\cdot,w)dw$ is supported inside $\{x-\langle y,z\rangle : x\in \text{supp }\widehat{\gamma}_i, y\in\mathbb{T}^d, z\in K_0\}$, we obtain the bound

$$\int_{\mathbb{R}\times\mathbb{T}^d} e^{-iT\alpha}b(\alpha,w)(x)d\alpha dw = O(T^{-N}) \text{ for } N>\frac{d}{2}+1$$

uniformly in x, ξ, and T such that $\frac{\xi}{T}\in K_0$.

We now treat the integral $\int_{\mathbb{R}\times\mathbb{T}^d} e^{-iT\alpha}\frac{a(\alpha,w)(x)}{i\alpha-R(u-iw)}d\alpha dw$. By construction, $a(\alpha,w)$ is a C^{2d+10} $\mathcal{H}_\kappa$–valued function. Fixing α, we expand $a(\alpha,\cdot)$ into a Taylor series up to $(d+2)$–order terms: $a(\alpha,w)=p(\alpha,w)+\varphi(\alpha,w)$ where

$$p(\alpha,w)=\sum_{k_1,\ldots,k_d=0}^{d+2} c_{k_1,\ldots,k_d}(\alpha)(x)w_1^{k_1}\cdots w_d^{k_d}.$$

The expansion $R(u-iw)=\frac{1}{2}w^tP''(u)w+o(\|w\|^2)$ shows that $(\alpha,w)\mapsto\frac{\varphi(\alpha,w)(x)}{i\alpha-R(u-iw)}$ is a C^{2d+2} map into $\mathcal{H}_\kappa$, whence

$$\sup_{x\in\Sigma^+,\frac{\xi}{T}\in K_0}\left\|\frac{\varphi(\alpha,w)(x)}{i\alpha-R(u-iw)}\right\|_{C^{2d+2}}<\infty.$$

It follows that

$$\int_{\mathbb{R}} e^{iT\alpha}\left(\int_{\mathbb{T}^d}\frac{a(\alpha,w)(x)}{i\alpha-R(u-iw)}dw\right)d\alpha = \\ = \int_{\mathbb{R}} e^{iT\alpha}\left(\int_{\mathbb{T}^d}\frac{p(\alpha,w)(x)}{i\alpha-R(u-iw)}dw\right)d\alpha + O(T^{-(\frac{d}{2}+1)})$$

uniformly in $x\in\Sigma^+$ and $\frac{\xi}{T}\in K_0$.

We estimate $\int_{\mathbb{R}\times\mathbb{T}^d} e^{-iT\alpha}\frac{p(\alpha,w)(x)}{i\alpha-R(u-iw)}d\alpha dw$. By construction $p(\alpha,w)(x)$ is a finite sum of functions of the form $c(\alpha)(x)b(w)$ where $b(w)$ is a multinomial of degree $\leq d+2$, and $c(\alpha)(x)$ has compact support and depends (as a vector in $\mathcal{H}_\kappa$) in a C^{d+10}–manner on $\alpha\in\mathbb{R}$ and $\frac{\xi}{T}$. We analyze each summand $\int_{\mathbb{R}\times\mathbb{T}^d} e^{-iT\alpha}\frac{c(\alpha)b(w)}{i\alpha-R(w)}d\alpha dw$ separately, and then add the results.

The estimation is done precisely as in the proof of lemma 2.3 in [**BL1**]. Using the change of coordinates $\frac{1}{z}=-\int_0^\infty e^{zT'}dT'$ valid for every $z\in\mathbb{C}$ such that $\operatorname{Re} z<0$, and the expansion $R(u-iw)=\frac{1}{2}w^tP''(u)w+o(\|w\|^2)$ we see that $\int_{\mathbb{R}\times\mathbb{T}^d} e^{-iT\alpha}\frac{c(\alpha)b(w)}{i\alpha-R(u-iw)}d\alpha dw=$

$$\begin{aligned}
&= -\int_{\mathbb{R}\times\mathbb{T}^d}\int_0^\infty e^{-iT\alpha}c(\alpha)b(w)e^{iT'\alpha-T'R(u-iw)}dT'd\alpha dw \\
&= -\int_{\mathbb{T}^d}\int_0^\infty \widehat{c}(T-T')b(w)e^{-T'R(u-iw)}dT'dw, \text{ where } \widehat{c}(\xi)=\int_{\mathbb{R}} e^{-i\xi\alpha}c(\alpha)d\alpha \\
&= -\int_0^\infty \widehat{c}(T-T')\left(\int_{\mathbb{T}^d} b(w)e^{-T'R(u-iw)}dw\right)dT'
\end{aligned}$$

$$= \quad -\int_{T/2}^{\infty} \widehat{c}(T-T') \left(\int_{\mathbb{T}^d} b(w) e^{-T'R(u-iw)} dw \right) dT' + O\left(\frac{\|c\|_{C^{\frac{d}{2}+10}}}{T^{\frac{d}{2}+10}}\right),$$

where the big Oh is uniform in x and $\frac{\xi}{T}$ (we have used Re $R(u-iw) > 0$ for $u \in (\nabla P)^{-1}(K_0)$, $w \in \mathbb{T}^d$). Since c depends continuously on $\frac{\xi}{T}$, we get that uniformly in $x \in \Sigma^+$, $\frac{\xi}{T} \in K_0$

$$\begin{aligned}
&= \quad -\int_{T/2}^{\infty} \widehat{c}(T-T') \left(\int_{\mathbb{T}^d} b(w) e^{-T'R(u-iw)} dw \right) dT' + o(T^{-\frac{d}{2}}) \\
&= \quad -\int_{T/2}^{\infty} \frac{\widehat{c}(T-T')}{(T')^{d/2}} \left(\int_{\sqrt{T'}\mathbb{T}^d} b(w/\sqrt{T'}) e^{-T'R(u-iw/\sqrt{T'})} dw \right) dT' + o(T^{-\frac{d}{2}}) \\
&= \quad \int_{-\infty}^{T/2} \frac{\widehat{c}(T'')}{(T-T'')^{d/2}} \int_{\sqrt{T-T''}\mathbb{T}^d} b(\tfrac{w}{\sqrt{T-T''}}) e^{-(T-T'')R(u-\frac{iw}{\sqrt{T-T''}})} dw dT'' + o(T^{-\frac{d}{2}}).
\end{aligned}$$

Now, $b(\frac{w}{\sqrt{T-T''}}) = b(0) + o(\frac{1}{\sqrt{T}})$, $(T-T'')R(u - \frac{iw}{\sqrt{T-T''}}) = \frac{1}{2} w^t P''(u) w + o(\frac{1}{T})$ uniformly in $\frac{\xi}{T} \in K_0$. Consequently,

$$\int_{\sqrt{T-T''}\mathbb{T}^d} b(\tfrac{w}{\sqrt{T-T''}}) e^{-(T-T'')R(u-\frac{iw}{\sqrt{T-T''}})} dw \xrightarrow[T\to\infty]{} C(u) b(0)$$

where $C(u) := \int_{\mathbb{R}^d} e^{-\frac{1}{2}w^t P''(u) w} dw$, and the convergence is uniform in $\frac{\xi}{T} \in K_0$ and $x \in \Sigma^+$.

The function $\widehat{c}(T)(x)$ is absolutely integrable with integral $\int_{\mathbb{R}} \widehat{c}(T'') dT'' = 2\pi c(0)(x)$. Since $|\widehat{c}(T)| \leq \|c(x)\|_{C^{10}} T^{-10}$, we see that the absolute integrability is uniform in $x \in \Sigma^+$, $\frac{\xi}{T} \in K_0$. Consequently,

$$\int_{\mathbb{R}\times\mathbb{T}^d} e^{-iT\alpha} \frac{c(\alpha)(x) b(w)}{i\alpha - R(u-iw)} d\alpha dw = 2\pi C(u) c(0)(x) b(0)[1+o(1)] \frac{1}{T^{d/2}}$$

uniformly in $x \in \Sigma^+$ and $\xi/T \in K_0$.

Putting this all together, we see that

$$\int_{\mathbb{R}\times\mathbb{T}^d} e^{-iT\alpha} \frac{a(\alpha,w)(x)}{i\alpha - R(u-iw)} d\alpha dw = 2\pi C(u) a(0,0)(x)[1+o(1)] \frac{1}{T^{d/2}}$$

uniformly in $x \in \Sigma^+$ and $\xi/T \in K_0$. Recalling the way a and K_0 were defined, and (6) and (13), we see that

$$a(0,0)(x) = \widehat{\gamma}_i(0) A(P(u), u)(x) = e^{\pm(2\delta_0 + P\varepsilon_0)} \frac{\varepsilon_0}{(2\pi)^d} \frac{1}{\int r d\nu} \psi(x).$$

Consequently,

$$\widetilde{A_i}(x,\xi,T) = e^{\pm(\delta_0 + P\varepsilon_0)} C(u) \frac{\varepsilon_0}{(2\pi)^d} \frac{1}{\int r d\nu} \frac{e^{TH(\frac{\xi}{T})}}{T^{d/2}} \psi(x)$$

uniformly in $x \in \Sigma^+$, $\frac{\xi}{T} \in K_0$.

Since δ_0 was arbitrary, we get

$$\widetilde{A_i}(x,\xi,T) = e^{\pm\varepsilon} e^{\pm\varepsilon_0 P} C(u) \frac{\varepsilon_0}{(2\pi)^d} \frac{1}{\int r d\nu} \frac{e^{TH(\frac{\xi}{T})}}{T^{d/2}} \psi(x)$$

uniformly in $x \in \Sigma^+$, $\frac{\xi}{T} \in K_0$, with $C(u) = \int_{\mathbb{R}^d} e^{-\frac{1}{2}w^t P''(u) w} dw = \frac{(2\pi)^{d/2}}{\sqrt{|\det P''(u)|}}$ and $u = u(\frac{\xi}{T})$ such that $\nabla P(u) = \frac{\xi}{T}$. Note that C is continuous at the origin. In particular, if K_0 is small enough, $C(u) = \mu^{\pm\varepsilon_0} \frac{(2\pi)^{d/2}}{\sqrt{|\det P''(0)|}} = \mu^{\pm\varepsilon_0} \frac{(2\pi)^{d/2}}{\sigma^{d/2}}$. In the same manner, if K_0 is small enough then P is close to to $\ln \mu$ (the solution of $P_{top}(-Pr) = 0$), so $e^{\pm\varepsilon_0 P} = \mu^{\pm 2\varepsilon_0}$. Putting this all together we get

$$\widetilde{A_i}(x, \xi, T) = \mu^{\pm 3\varepsilon_0} \frac{1 + o(1)}{(2\pi\sigma)^{d/2}} \frac{\varepsilon_0}{\int r d\nu} \frac{e^{TH(\frac{\xi}{T})}}{T^{d/2}} \psi(x)$$

uniformly in $x \in \Sigma^+$, $\frac{\xi}{T} \in K_0$.

Now that we have obtained the asymptotic behaviour of $\widetilde{A_i}(x, \xi, T)$, we show that $\widetilde{A_i}(x, \xi, T) = A_i(x, \xi, T)$. Since $A(x, \xi, T)$ is sandwiched between $A_i(x, \xi, T)$, $i = 1, 2$, this will give us our final goal: an (approximate) asymptotic expansion for $A(x, \xi, T)$.

Step 7. $\widetilde{A_i}(x, \xi, T) = A_i(x, \xi, T)$ $(i = 1, 2)$.

Starting from the definition of $\widetilde{A_i}(x, \xi, T)$ in (8), we see that it is enough to show that $\left|\widehat{\gamma}_i(\alpha) \sum_{n \geq N} (L^n_{s,z}\psi)(x)\right|$ are dominated by a (single) absolutely integrable function of (u, w).

We shall show that this holds (uniformly) for $x \in \Sigma^+$, $\xi/T \in K_0$. Recalling the definition of $\|\cdot\|_\kappa$, the proof of step 2, and the convention $P_{top}(-Pr + \langle u, f\rangle) = 0$, we see that

$$\left|\widehat{\gamma}_i(\alpha) \sum_{n \geq N} (L^n_{s,z}\psi)(x)\right| \leq |\widehat{\gamma}_i(\alpha)| \|L^N_{s,z}\| \left\|\sum_{n=0}^{\infty} (L^n_{s,z}\psi)(x)\right\|_\kappa \leq$$

$$\leq |\widehat{\gamma}_i(\alpha)| \left(\lambda^N_{s,z}\|Q_{s,z}\| + \|N^N_{s,z}\|\right) \left\|\sum_{n=0}^{\infty} (L^n_{s,z}\psi)(x)\right\|_\kappa \leq \text{const.} |\widehat{\gamma}_i(\alpha)| \left\|\sum_{n=0}^{\infty} (L^n_{s,z}\psi)(x)\right\|_\kappa .$$

The constant is uniform in N, because $|\lambda_{s,z}| \leq |\lambda_{P,u}| = 1$ due to the first constraint on P and u, and because $\|Q_{s,z}\|, \|N_{s,z}\|$ are bounded on compacts (e.g. the support of $\widehat{\gamma}_i$ times $\mathbb{T}^d$).

It follows that it is enough to show the absolute integrability of $\left\|\sum_{n=0}^{\infty} (L^n_{s,z}\psi)(x)\right\|$ on compact subsets of $\mathbb{R} \times \mathbb{T}^d$. Equation (12) shows that it is enough to prove to absolute integrability of

$$\frac{\|A(s, z)(\cdot)\|_\kappa}{i(\alpha - \langle \nabla P(u), w\rangle) - R(u - iw)}$$

on a compacts. After the change of coordinates $\alpha \to \alpha + \langle w, \frac{\xi}{T}\rangle$ suggested in step 4, this becomes the absolute integrability of

$$\frac{\|a(\alpha, w)(\cdot)\|_\kappa}{i\alpha - R(u - iw)} = O\left(\left|\frac{1}{i\alpha - R(u - iw)}\right|\right)$$

on compacts. (Note that this change of coordinates preserves the property of being absolutely integrable on compacts, because of the assumption that $\xi/T \in K_0$ where K_0 is compact.)

Recalling the choice of K_0 at the end of step 3, we see that $i\alpha - R(u - iw)$ vanishes only at the origin. Therefore, it is enough to check the convergence of $\int_{-a}^{a} d\alpha \int_{[-b,b]^d} |1/[i\alpha - R(u-iw)]|dw$ for some $a, b > 0$. We choose $b > 0$ so small that for some $c > 0$

$$\begin{array}{l}\operatorname{Re} R(u-iw) \geq c\|w\|^2 \\ \operatorname{Im} R(u-iw) \leq c\|w\|^3\end{array} \qquad (w \in [-b,b]^d).$$

We make c so small that $|x+iy| \geq c(|x|+|y|)$ for all $x, y \in \mathbb{R}$. Using the identity $\frac{1}{A} = \int_0^\infty e^{-A\tau} d\tau$ valid for all $A > 0$, we write

$$\begin{aligned}
\int_{-a}^{a}\int_{[-b,b]^d} \frac{d\alpha dw}{|i\alpha - R(u-iw)|} &= \int_{-a}^{a} d\alpha \int_{[-b,b]^d} dw \int_0^\infty e^{-\tau|i\alpha + R(u-iw)|} d\tau \\
&= \text{const.} + \int_1^\infty d\tau \int_{-a}^{a} d\alpha \int_{[-b\sqrt{\tau}, b\sqrt{\tau}]^d} e^{-|i\alpha\tau + \tau R(u - i\frac{v}{\sqrt{\tau}})|} \frac{dv}{\tau^{d/2}} \\
&\leq \text{const.} + \int_1^\infty d\tau \int_{-a}^{a} d\alpha \int_{\mathbb{R}^d} e^{-c(|\tau\alpha \pm c\frac{\|v\|^3}{\sqrt{\tau}}| + c\|v\|^2)} \frac{dv}{\tau^{d/2}} \\
&\leq \text{const.} + \int_1^\infty \frac{d\tau}{\tau^{d/2}} \int_{\mathbb{R}^d} e^{-c^2\|v\|^2} \left(\int_{-a}^{a} \left[e^{-\frac{c}{2}|\tau\alpha|} + 1_{[|\alpha| \leq \frac{c\|v\|^3}{2\tau^{3/2}}]} \right] d\alpha \right) dv \\
&\leq \text{const.} \left[1 + \int_1^\infty \frac{d\tau}{\tau^{1+d/2}} \int_{\mathbb{R}^d} e^{-c^2\|v\|^2} (1 + \|v\|^3) dv \right] < \infty.
\end{aligned}$$

Absolute integrability, and with it step 7, is proved.

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