ON THE EXISTENCE OF SRB MEASURES FOR C^∞ SURFACE DIFFEOMORPHISMS

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Abstract. We give a proof of Viana's conjecture on physical measures [V], in the special case of C^∞ surface diffeomorphisms, and using the analysis of entropy and Lyapunov exponents we developed in [BCS2]. Burguet has recently proved a stronger result in [B], using a different method.

Given a smooth diffeomorphism f of a closed Riemannian manifold M, one would like to build *physical measures*: Borel probability measures μ whose *ergodic basins*

$$\mathrm{Basin}(\mu) := \left\{ x \in M : \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \delta_{f^k(x)} \xrightarrow{weak-*} \mu \right\} \quad (\delta_y := \text{point mass at } y)$$

have positive Riemannian volume. Some diffeomorphisms do not have physical measures [Ta], and a fundamental problem in smooth dynamics is to determine when they exist. Following positive results for uniformly hyperbolic systems [R],[S],[BR], for interval maps [K], for some partially or non-uniformly hyperbolic systems [You1, ABV, BV], or assuming regularity in the spirit of Oseledets [Ts], Viana [V] conjectured the following:[1]

Viana's Conjecture: *Suppose a smooth map f has only non-zero Lyapunov exponents at Lebesgue almost every point. Then f has a physical measure.*

This has led to many works, with some notable recent advances assuming some recurrence properties (in particular, [O, CLP] and the references therein).

David Burguet's striking work on this conjecture (see below) made us realize that the ideas developed in [BCS2] allow to address the special case of C^∞ surface diffeomorphisms.

We need some notations and facts to state our result. Suppose $\dim M = 2$ and $f \in C^\infty$. The *upper Lyapunov exponent at $x \in M$* is $\lambda^+(x) := \limsup \frac{1}{n}\log\|Df^n(x)\|$. The upper Lyapunov exponent of an f-invariant but possibly non-ergodic probability measure μ is $\lambda^+(\mu) = \int \lambda^+(x)d\mu$. An ergodic measure μ is a *Sinaï-Ruelle-Bowen (SRB) measure* if its entropy $h(f,\mu)$ is positive and equal to $\lambda^+(\mu)$.[2] Ledrappier-Young Theory says that this is equivalent to the existence of a system of absolutely continuous conditional measures on local unstable manifolds [LY]. In dimension two, every SRB measure is physical [L]. The result we will prove in this paper is:

Theorem. *Let f be a C^∞ diffeomorphism on a closed surface such that, on a set of points x with positive Riemannian area, $\limsup_{n\to\infty} \frac{1}{n}\log\|Df^n(x)\| > 0$. Then f admits an SRB measure.*

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[1] In [V], the name *SRB measures* is used for what we call *physical measures*.

[2] The definition of the SRB property varies in the literature. In higher dimension, one may require as in [You2] that $h(f,\mu)$ coincides with the sum of all the positive Lyapunov exponents of μ and that it is nonzero. But in dimension bigger than two, this allows zero exponents, and it does not imply physicality.

This implies Viana's conjecture, in the special case of smooth surface diffeomorphisms.

We note that the uniqueness and finite multiplicity of these SRB measures have already been addressed elsewhere: By [RRTU], each transitive C^r surface diffeomorphism, $r > 1$, admits at most one SRB measure. By [BCS1] given any a C^∞ surface diffeomorphism, the number of ergodic SRB measures μ of satisfying $\lambda^+(\mu) > \varepsilon$ is finite for each $\varepsilon > 0$.

Burguet had obtained a stronger result in [B]:[3] For C^∞ surface diffeomorphisms, the basin of the SRB measures contains Lebesgue almost-every point satisfying $\lambda^+(x) > 0$. Moreover, this also holds for C^r surface diffeomorphisms with $1 < r < \infty$, provided the function $x \mapsto \lambda^+(x)$ is large enough on a set with positive area. Burguet's method is different from ours. It is more geometric: it uses geometric times (a strengthening of the classical notion of hyperbolic times).

Our strategy relies on the analysis of the continuity properties of entropy and Lyapunov exponents from BCS. Working inside a given curve, we select sets of good points and consider limiting measures. Possible drop in the exponent is given by "neutral blocks" in the orbits of these good points. These blocks prevent entropy creation, so too many of them would contradict the expansion at the good points. This enforces the equality $\lambda^+(\mu) = h(f, \mu)$ for the limit measure.

More specifically, we apply the reparametrization propositions as stated in [BCS2] but need to slightly modify the global reparametrizations of [BCS2, Sec. 7]. In Section 1.6 we detail these modifications by referring the reader to the specific arguments in [BCS2].

Lastly, we point out that our method is restricted to surface diffeomorphisms since it only deals with unstable curves.

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1. Proof of the theorem

Let $f : M \to M$ be a C^∞ diffeomorphism on a closed smooth surface M. For most of the proof we fix $\gamma, \eta > 0$ small, $r \geq 2$ a large integer, and work in C^r-regularity. (We will also need a condition $\gamma < \gamma_0(r, f, \eta)$.) At the end we let $\gamma, \eta \to 0$ and $r \to \infty$.

1.1. Projective dynamics.

The *projective tangent bundle of* M is the manifold $\widehat{M}$ of pairs $\xi = (x, E)$ where $x \in M$ and E is a one-dimensional linear subspace of T_xM, with the Riemannian structure inherited from TM. Let $\pi\colon \widehat{M} \to M$ be the natural projection.

The diffeomorphism f induces the diffeomorphism $\widehat{f}(x, E) = (f(x), Df(E))$ on $\widehat{M}$. Let $\varphi\colon \widehat{M} \to \mathbb{R}$ be the continuous function

$$\varphi(x, E) = \log \|Df|_E\|.$$

Lemma 1. *If $\widehat{\mu}$ is an $\widehat{f}$-ergodic invariant measure, $\int \varphi d\widehat{\mu} > 0$, and $\mu := \pi_*\widehat{\mu}$ is not supported on a source, then $\lambda^+(\mu) = \int \varphi d\widehat{\mu}$.*

The (standard) proof can be found e.g. in [BCS2, Lemma 3.3].

[3] The first version of [B], which preceded and motivated this work, had a correctable mistake. The mistake was noticed and corrected after our paper was submitted, in the second version of [B].

1.2. **Reduction to a curve.** A curve $\sigma\colon[0,1]\to M$ is *regular* if it is C^r, injective, and if the derivative σ' never vanishes. It lifts canonically as a curve $\widehat{\sigma}\colon[0,1]\to\widehat{M}$ defined by

$$\widehat{\sigma}(s):=(\sigma(s),\ \mathbb{R}.\sigma'(s)).$$

Fix once and for all a finite atlas for $\widehat{M}$ so we can define the C^r size as follows. Suppose $\varepsilon,\widehat{\varepsilon}\in(0,1)$. We say that $\widehat{\sigma}$ has C^r *size less than* $(\varepsilon,\widehat{\varepsilon})$ if, using the charts from the atlas, the norms of the k^{th}-order derivatives of σ with $1\le k\le r$ are bounded by ε, and k^{th}-order derivatives of $\widehat{\sigma}$ with $1\le k\le r-1$ are bounded by $\widehat{\varepsilon}$. This notion of double C^r size $(\epsilon,\widehat{\epsilon})$ is introduced in [BCS2] and is used for proving [BCS2, Proposition 5.3] (Proposition 11 below).

Lemma 2. *There exists a regular C^∞ curve σ such that, for a set of parameters $s\in[0,1]$ with positive Lebesgue measure, $\limsup\frac{1}{n}\log\|Df^n|_\sigma(\sigma(s))\|>0$ and $\sigma(s)$ is not f-periodic.*

Proof. Let U be the domain of a C^∞ chart such that $\{x\in U:\limsup\frac{1}{n}\log\|Df^n(x)\|>0\}$ has positive area. It is endowed by smooth horizontal and vertical foliations $\mathcal{H},\mathcal{V}$ with C^∞ leaves that are uniformly transverse. Observe that there is $C>0$ such that for every $x\in U$ and $n\ge 0$,

$$\|Df^n(x)\|\le C\max(\|Df^n(x)|_{T_x\mathcal{H}}\|,\|Df^n(x)|_{T_x\mathcal{V}}\|).$$

For one of the foliations, say $\mathcal{V}$, the area of $\{x\in D:\limsup\frac{1}{n}\|Df^n(x)|_{T_x\mathcal{V}}\|>0\}$ is positive.

By Fubini's theorem, there is a leaf of $\mathcal{V}$ with a parametrization $\sigma:[0,1]\to M$ such that $T:=\{s\in[0,1]:\limsup\frac{1}{n}\log\|Df^n|_\sigma(\sigma(s))\|>0\}$ has positive Lebesgue measure. The expansion implies that the periodic points in $\sigma(T)$ are countably many. □

Let $\overline{\lambda}>0$ be the essential supremum of the function $s\mapsto\limsup\frac{1}{n}\log\|Df^n|_\sigma(\sigma(s))\|$. (Here and throughout, $Df^n|_\sigma(p):=Df^n(p)|_{T_p\sigma}$ for p on σ.)

Lemma 3. *Given $0<\lambda^{min}<\overline{\lambda}<\lambda^{max}$ and $0<\rho<1$, there are n arbitrarily large such that*

$$T_n:=\{s\in[0,1]:\lambda^{min}\le\tfrac{1}{n}\log\|Df^n|_\sigma(\sigma(s))\|\le\lambda^{max},$$
$$\sigma(s)\ \textit{is not } f\textit{-periodic and}\ \forall 0\le i\le n,\ \ \|Df^i|_\sigma(f^{n-i}\sigma(s))\|\ge 1\}$$

have Lebesgue measure larger than ρ^n.

Proof. By Lemma 2, if $\varepsilon>0$ is small, then the definition of the essential supremum gives that for s in a set with positive Lebesgue measure, $\sigma(s)$ is not periodic and we have $\limsup\frac{1}{n}\log\|Df^n|_\sigma(\sigma(s))\|\in[\lambda^{min}+\varepsilon,\lambda^{max}-\varepsilon]$. By Pliss's lemma (see [M, Lemma 11.8]) there exists an arbitrarily large integer n such that $\|Df^k|_\sigma(f^{n-k}\sigma(s))\|\ge\exp(\lambda^{min}\cdot k)$ for all $0\le k\le n$. Note also that the inequality $\|Df^n|_\sigma(\sigma(s))\|\le\exp(\lambda^{max}\cdot n)$ is satisfied for all n large. Hence s belongs to infinitely many sets T_n. By the Borel-Cantelli lemma, the Lebesgue measure of T_n cannot be smaller than ρ^n for all large n. □

We continue the proof by choosing some sequences $\lambda_k^{min},\lambda_k^{max}\to\overline{\lambda}$, $\rho_k\to 1$. By Lemma 3, there exist $n_k\to\infty$ such that the sets T_{n_k} have Lebesgue measure larger than $\rho_k^{n_k}$.

1.3. **Empirical measures and neutral decompositions.** Let $\xi\in\widehat{M}$ be a point. The k^{th} *empirical measure* and *Birkhoff sum* of φ at ξ are

$$p_k(\xi):=\frac{1}{n_k}\sum_{0\le i\le n_k-1}\delta_{\widehat{f}^i(\xi)},\qquad S_{n_k}\varphi(\xi):=\varphi(\xi)+\varphi(\widehat{f}(\xi))+\dots\varphi(\widehat{f}^{n_k-1}(\xi)).$$

So $\frac{1}{n_k}S_{n_k}\varphi(\xi)=\int\varphi dp_k(\xi)$. Note that by definition of φ and T_{n_k},

$$\forall\xi\in\widehat{\sigma}(T_{n_k}),\forall 0\le i\le n_k,\ \ S_i\varphi(\widehat{f}^{n_k-i}(\xi))\ge 0.$$

Neutral orbit segments. Suppose $\alpha > 0$ and $L \geq 1$. An orbit segment $(\zeta, \widehat{f}(\zeta), \dots, \widehat{f}^{\ell-1}(\zeta))$ is called *α-neutral*, if $S_i\varphi(\zeta) \leq \alpha i$ for all $0 < i \leq \ell$. An α-neutral orbit segment of length $\ell \geq L$ is called *(α, L)-neutral*.

The union of two intersecting α-neutral segments is still a neutral segment. Consequently, the union of all α-neutral sub-segments of a given finite orbit segment $(\xi, \widehat{f}(\xi), \dots, \widehat{f}^{n_k-1}(\xi))$ is a pairwise disjoint union of maximal α-neutral sub-segments.

Neutral part of an empirical measure. This is the measure

$$p_k^{\alpha,L}(\xi) := \frac{1}{n_k} \sum_{0 \leq i \leq n_k-1} 1_{N_{n_k}^{\alpha,L}(\xi)}(\widehat{f}^i(\xi)) \cdot \delta_{\widehat{f}^i(\xi)},$$

where $N_{n_k}^{\alpha,L}(\xi)$ is the union of (α, L)-neutral sub-segments of $(\xi, \dots, \widehat{f}^{n_k-1}(\xi))$.

The following proposition is a version of [BCS2, Proposition 6.2]. There we were given a sequence of ergodic measures with nonnegative averages, and we started by constructing empirical measures along generic points. Here we skip the first step and treat the empirical measures as given. To avoid cumbersome notation, the following Proposition assumes that the orbits are non-periodic so that $n \mapsto \widehat{f}^n(\xi_k)$ are bijections.

Proposition 4. *Let (ξ_k) be non-periodic points in $\widehat{M}$ satisfying $S_i\varphi(\widehat{f}^{n_k-i}(\xi_k)) \geq 0$ for each $0 \leq i \leq n_k$. Up to extracting a subsequence, there exist $\beta \in [0,1]$ and two $\widehat{f}$-invariant probability measures $\widehat{\mu}_0, \widehat{\mu}_1$ as follows:*

(a) The sequence of measures $p_k(\xi_k)$ converges to $\widehat{\mu} := (1-\beta)\widehat{\mu}_0 + \beta\widehat{\mu}_1$ weak-$*$ on $\widehat{M}$.
(b) For any pair of neighborhoods $\widehat{V}_0, \widehat{V}_1$ of $(1-\beta)\widehat{\mu}_0$ and $\beta\widehat{\mu}_1$, if $\alpha > 0$ is small and L is large, then for all k large enough, one has $p_k^{\alpha,L}(\xi_k) \in \widehat{V}_0$ and $[p_k(\xi_k) - p_k^{\alpha,L}(\xi_k)] \in \widehat{V}_1$.
(c) $(1-\beta)\int \varphi d\widehat{\mu}_0 = 0$.
(d) If $\beta > 0$, for $\widehat{\mu}_1$-almost every point ξ, the limit of $\frac{1}{n}S_n\varphi(\xi)$ is positive.

Remark 5. The proposition holds for any homeomorphism $\widehat{f}$ of a compact space $\widehat{M}$, for any continuous observable φ, and for any sequence $n_k \to +\infty$.

Proof. Up to taking a subsequence, $(p_k(\xi_k))_{k\geq 1}$ converges to an invariant measure $\widehat{\mu}$. For each pair $(\alpha, L) \in (0,\infty) \times \mathbb{N}^*$, we can find a further subsequence $(p_{k_i}^{\alpha,L}(\xi_{k_i}))_{i\geq 1}$ which converges as $k_i \to \infty$ to some measure $m_{\alpha,L}$. By a classical diagonal argument, we can find a subsequence $(k_i)_{i\geq 1}$ that works for a countable and dense set of (α, L) . Since $m_{\alpha,L} \leq m_{\alpha',L}$ if $\alpha \leq \alpha'$, this subsequence actually works for all (α, L). Since $m_{\alpha,L} \leq m_{\alpha',L'}$ if $\alpha \leq \alpha'$ and $L \geq L'$, the limit $m_0 := \inf_{\alpha,L} m_{\alpha,L}$ exists (see [BCS2, Claim 6.4] for details). Note that $\widehat{f}_* m_{\alpha,L} - m_{\alpha,L} \to 0$ as $L \to +\infty$. Hence m_0 and $m_1 := \widehat{\mu} - m_0$ are $\widehat{f}$-invariant. We set $\beta := m_1(\widehat{M})$ and $m_0 = (1-\beta)\widehat{\mu}_0$, $m_1 = \beta\widehat{\mu}_1$. The items (a) and (b) follow.

Let $(\zeta, \widehat{f}(\zeta), \dots, \widehat{f}^{\ell-1}(\zeta))$ be a maximal (α, L)-neutral segment in $(\xi, \widehat{f}(\xi), \dots, \widehat{f}^{n_k-1}(\xi))$. We have $S_\ell\varphi(\zeta) \leq \alpha \cdot \ell$. If $\widehat{f}^{\ell-1}(\zeta) \neq \widehat{f}^{n_k-1}(\xi)$, then $\alpha \cdot \ell - \|\varphi\|_\infty < S_\ell\varphi(\zeta)$ by maximality. Otherwise $S_\ell\varphi(\zeta) = S_\ell\varphi(\widehat{f}^{n_k-\ell}(\xi_k)) \geq 0$ by our assumption on the points ξ_k. Thus this gives $-\|\varphi\|_\infty/L \leq p_k^{\alpha,L}(\xi_k)(\varphi) \leq \alpha$ and taking the limit in α and L, one gets item (c).

Item (d) is proved by contradiction: we assume $\lim \frac{1}{n}S_n\varphi(\xi) \leq 0$ on a set of points ξ with $\beta\widehat{\mu}_1$-measure $\chi > 0$. We also fix $\alpha > 0$ small, $L \geq 1$ large so that the mass $|m_{\alpha,L} - m_0| = (m_{\alpha,L} - m_0)(\widehat{M})$ is less than $\chi/10$. By Pliss' lemma, there exists $\ell \geq L$ such that the compact set $K_{\alpha/2,\ell}$ of points ξ which belong to some $\alpha/2$-neutral segment of length ℓ has $\beta\widehat{\mu}_1$-measure larger than $\chi/2$. The set $K_{\alpha,\ell}$ of points which belong to some α-neutral segment of length ℓ is a neighborhood of $K_{\alpha/2,\ell}$. With item (b), one deduces that for some $\alpha' < \alpha$, $L' > L$ and for any k large, $(p_k(\xi_k) - p_k^{\alpha',L'}(\xi_k))$ gives a mass larger than $\chi/3$ to

$K_{\alpha,\ell}$. By our choice of α, L, the measure $(p_k^{\alpha,L}(\xi_k) - p_k^{\alpha',L'}(\xi_k))$ has mass less than $\chi/6$, for k large. Hence $(p_k(\xi_k) - p_k^{\alpha,L}(\xi_k))$ gives a mass larger than $\chi/6$ to $K_{\alpha,\ell}$. Since $\ell \geq L$ this implies that $(\xi_k, \widehat{f}(\xi_k), \dots, \widehat{f}^{n_k-1}(\xi_k))$ contains some (α, L)-neutral segment which is not included in the support of $p_k^{\alpha,L}(\xi_k)$. This contradicts the definition of $p_k^{\alpha,L}(\xi_k)$. □

1.4. **A sequence of subsets T'_{n_k}.** We continue with the proof of the theorem and the objects introduced in the first sections: the curves $\sigma, \widehat{\sigma}$, the set of parameters T_{n_k} with Lebesgue measure $\geq \rho_k^{n_k}$, etc. We endow the space of measures with a metric compatible with the weak-$*$ topology, and denote the Lebesgue measure of $T \subset [0,1]$ by $|T|$.

Lemma 6. *Up to passing to a subsequence, there exist $\alpha_k \to 0$, $L_k \to \infty$, $\delta_k \to 0$ and subsets $T'_{n_k} \subset T_{n_k}$ with $|T'_{n_k}| > \rho_k^{2n_k}$ such that for any $\xi, \xi' \in \widehat{\sigma}(T'_{n_k})$, the empirical measures satisfy $d(p_k(\xi), p_k(\xi')) < \delta_k$ and $d(p_k^{\alpha,L}(\xi), p_k^{\alpha,L}(\xi')) < \delta_k$ for all $\alpha = \alpha_j$, $L = L_j$ with $j \leq k$.*

Proof. It is enough to show that, given a finite set of pairs (α, L) and $\delta > 0$, for k large, there exists $T'_{n_k} \subset T_{n_k}$ with $|T'_{n_k}| \geq \rho_k^{2n_k}$ such that for each of these pairs (α, L) and any $\xi, \xi' \in \widehat{\sigma}(T'_{n_k})$, one has $d(p_k(\xi), p_k(\xi')) < \delta$ and $d(p_k^{\alpha,L}(\xi), p_k^{\alpha,L}(\xi')) < \delta$.

The set of measures m with mass $m(\widehat{M}) \leq 1$ can be covered by some finite number K of balls of radius $\delta/2$. Hence, for each k there exists a set $T' \subset T_{n_k}$ with $|T'| \geq |T_{n_k}|/K^2$ such that for any $\xi, \xi' \in \widehat{\sigma}(T)$, each pair of measures $\{p_k(\xi), p_k(\xi')\}$ and $\{p_k^{\alpha,L}(\xi), p_k^{\alpha,L}(\xi')\}$ lie in some ball of the covering. Finally, for k large enough, $|T_{n_k}|/K^2 \geq \rho_k^{2n_k}$. □

From now on, we consider a sequence of integers $k \to \infty$ as in the previous Lemma. We choose points $\xi_k \in \widehat{\sigma}(T'_{n_k})$ and obtain the limits $\widehat{\mu} := (1-\beta)\widehat{\mu}_0 + \beta\widehat{\mu}_1$ from Proposition 4. By Lemma 6, $p_k(\xi'_k) \to \widehat{\mu}$ and $p_k^{\alpha,L}(\xi'_k) \to (1-\beta)\widehat{\mu}_0$ for any choice of the sequence of $\xi'_k \in T'_{n_k}$.

Corollary 7. *The measure $\mu_1 := \pi_*(\widehat{\mu}_1)$ gives zero measure to all repelling hyperbolic periodic orbits, and $\beta\lambda^+(\mu_1) = \overline{\lambda} > 0$. Thus $\beta \neq 0$.*

Proof. By construction $\int \varphi dp_k(\xi_k) \in [\lambda_k^{min}, \lambda_k^{max}]$. Since $(1-\beta)\int \varphi d\widehat{\mu}_0 = 0$ by item (c) of Proposition 4, at the limit one gets $\overline{\lambda} = \int \varphi d\widehat{\mu} = \beta \int \varphi d\widehat{\mu}_1$. In particular $\beta > 0$.

The ergodic components $\widehat{\mu}'_1$ of $\widehat{\mu}_1$ satisfy $\int \varphi d\widehat{\mu}'_1 > 0$ by Proposition 4 (d). We claim that their projections $\mu'_1 = \pi_*\widehat{\mu}'_1$ are not supported on repelling hyperbolic periodic orbits.

Otherwise there are $\chi > 0$ and an arbitrarily small open neighborhood U of a repelling hyperbolic periodic orbit such that $\widehat{\mu}(\partial U) = 0$, and $p_k(\xi_k)(U) > \chi$. By Lemma 6, $S_k := \{s \in [0,1],\ p_k(\sigma(s))(U) > \chi/2\}$ contains T'_{n_k} for every k large. But $|T'_{n_k}| \geq \rho_k^{2k}$ with $\rho_k \to 1$ whereas, since U is a small repelling neighborhood, $|S_k| \to 0$ exponentially, a contradiction.

So $\widehat{\mu}_1$ does not charge sources. By Lemma 1, $\lambda^+(\mu'_1) = \int \varphi d\widehat{\mu}'_1$. So $\lambda^+(\mu_1) = \int \varphi d\widehat{\mu}_1$. □

1.5. **Reparametrizations: tools.** Let $N \geq 1$ and $\varepsilon, \widehat{\varepsilon} \in (0,1)$ (to be specified later).

A *reparametrization of* σ is a non-constant affine map $\psi\colon [0,1] \to [0,1]$.

A *family of reparametrizations over a subset* $T \subset [0,1]$ is a collection $\mathcal{R}$ of reparametrizations such that $T \subset \bigcup_{\psi \in \mathcal{R}} \psi([0,1])$.

A reparametrization ψ is $(N, \varepsilon, \widehat{\varepsilon})$-*admissible up to time* n, if there exist $n_0, \dots, n_\ell$ s.t.

- $n_0 = 0$, $n_\ell = n$, and $1 \leq n_j - n_{j-1} \leq N$ for each $1 \leq j \leq \ell$,
- for each $0 \leq j \leq \ell$ the curve $f^{n_j} \circ \sigma \circ \psi$ has C^r size less than $(\varepsilon, \widehat{\varepsilon})$.

Lemma 8. *The following holds for all $\varepsilon, \widehat{\varepsilon}$ small enough. Let $\mathcal{R}_k$ be a family of reparametrizations of σ over T'_{n_k} which are $(N, \varepsilon, \widehat{\varepsilon})$-admissible up to time n_k. Then*

$$\mathrm{Card}(\mathcal{R}_k) \geq \rho_k^{2n_k} \exp(\lambda_k^{min}\, n_k) \min_{0 \leq t \leq 1} \|\sigma'(t)\|. \tag{1}$$

Proof. For any reparametrization ψ, the curve $f^{n_k} \circ \sigma \circ \psi$ has C^r size less than $(\varepsilon, \widehat{\varepsilon})$, hence length smaller than 1. Consequently the set $\sigma(T'_{n_k} \cap \psi[0,1])$ has a length smaller than $\exp(-\lambda_k^{min}\ n_k)$ (by definition of T_{n_k}). Since $\mathcal{R}_k$ is a family of parameterizations over T'_{n_k}, one gets $\mathrm{Card}(\mathcal{R}_k)\exp(-\lambda_k^{min}\ n_k) \geq \sum_{\psi\in\mathcal{R}_k} |\sigma(T'_{n_k} \cap \psi[0,1])| \geq |\sigma(T'_{n_k})| \geq \min_t \|\sigma'(t)\| \cdot |T'_{n_k}|$. □

Next we present some tools for constructing reparametrizations. To do this we need some notation and a couple of more definitions. Balls in $\widehat{M}$ with center ξ and radius r will be denoted by $B(\xi, r)$. We let $\|D\widehat{f}^n\|_{\sup} := \sup_{\xi\in\widehat{M}} \|D\widehat{f}^n(\xi)\|$, and define the *asymptotic dilation*

$$\lambda(\widehat{f}) := \lim_{n\to\infty} \tfrac{1}{n} \log \|D\widehat{f}^n\|_{\sup}.$$

The *supremum entropy* $\overline{h}(f, \mu)$ of an invariant measure μ is the essential supremum of the entropies of its ergodic components, with respect to the natural measure on the space of ergodic components.

The following result is a version of Yomdin's theorem [Yom] adapted to our setting (see [BCS2, Theorem 4.13]). We recall that we have fixed $r \geq 2$.

Theorem 9 (Yomdin)**.** *There exist $\Upsilon > 0$ (depending only on r) and $\varepsilon_0 > 0$ (depending only on r and f) such that for every $\varepsilon, \widehat{\varepsilon} \in (0, \varepsilon_0)$, every regular curve σ_0 with C^r size less than $(\varepsilon, \widehat{\varepsilon})$ and every $s \in [0,1]$, there is a family $\mathcal{R}$ of reparametrizations of σ_0 over the set*

$$T := \{t \in [0,1] : f(\sigma_0(t)) \in B(f(\sigma_0(s)), \varepsilon) \text{ and } \widehat{f} \circ \widehat{\sigma}_0(t) \in B(f(\widehat{\sigma}_0(s)), \widehat{\varepsilon})\}$$

so that the curves $f \circ \sigma_0 \circ \psi$ have C^r size less than $(\varepsilon, \widehat{\varepsilon})$ for each $\psi \in \mathcal{R}$, and

$$\mathrm{Card}(\mathcal{R}) \leq \Upsilon \|D\widehat{f}\|_{\sup}^{1/(r-1)}.$$

The next two propositions, taken from [BCS2, Section 5], provide small families of admissible parametrizations over certain dynamically defined subsets of σ_0. (We state them for $f \in C^\infty$, but they only require $f \in C^r$, $r \geq 2$.)

Proposition 10 (Reparametrization during neutral periods)**.** *The following holds for some $\gamma_0(r, f, \eta) > 0$. For all $0 < \gamma \leq \gamma_0(r, f, \eta)$, for all N large enough, and for any $\widehat{f}$-invariant probability measure $\widehat{\mu}_0$, there exist $\varepsilon, \widehat{\varepsilon} > 0$ arbitrarily small, $\bar{n}_0 \geq 1$, and an open set $\widehat{U}_0$ such that $\widehat{\mu}_0(\widehat{U}_0) > 1 - \gamma^2$, $\widehat{\mu}_0(\partial\widehat{U}_0) = 0$ and:*

For any regular curve σ_0 with C^r size less than $(\varepsilon, \widehat{\varepsilon})$, and any $n \geq \bar{n}_0$, there is a family $\mathcal{R}$ of reparametrizations of σ_0 over the set

$$\widehat{\sigma}_0^{-1}\left\{\xi : (\xi, \ldots, \widehat{f}^{n-1}(\xi)) \text{ is } \tfrac{\eta}{10}\text{-neutral and } \mathrm{Card}\{\widehat{f}^j(\xi) \in \widehat{U}_0,\ 0 \leq j < n\} > (1-\gamma)n\right\}$$

which is $(N, \varepsilon, \widehat{\varepsilon})$-admissible up to time n and has cardinality $\mathrm{Card}(\mathcal{R}) \leq \exp\left[n(\frac{\lambda(\widehat{f})}{r-1} + \eta)\right]$.

In the previous statement ε has to be chosen much smaller than $\widehat{\varepsilon}$. This is the reason why we need to work with different scales on M and $\widehat{M}$.

Proposition 11 (Reparametrization during typical orbit segments)**.** *Let us take $\varepsilon, \widehat{\varepsilon} > 0$ small and N large and $\gamma > 0$. For any $\widehat{f}$-invariant probability measure $\widehat{\mu}_1$, if n_1 is large enough, then there exists an open set $\widehat{U}_1$ satisfying $\widehat{\mu}_1(\widehat{U}_1) > 1 - \gamma^2$, $\widehat{\mu}_1(\partial\widehat{U}_1) = 0$ and the following:*

For any regular curve σ_0 with C^r size less than $(\varepsilon, \widehat{\varepsilon})$, there is a family $\mathcal{R}$ of reparametrizations over $\widehat{\sigma}_0^{-1}(\widehat{U}_1)$ which is $(N, \varepsilon, \widehat{\varepsilon})$-admissible up to time n_1 and with cardinality

$$\mathrm{Card}(\mathcal{R}) \leq \exp\left[n_1(\overline{h}(f, \pi_*\widehat{\mu}_1) + \tfrac{\lambda(\widehat{f})}{r-1} + \eta)\right].$$

1.6. **Reparametrizations: construction.** Let σ be the regular C^∞ curve from section 1.2 with selected sets of parameters $T'_{n_k} \subset [0,1]$ and let $\widehat{\mu} = \lim_k p_k(\xi_k)$ the limit measure independent of the choices of $\xi_k \in T'_{n_k}$. Let also $\widehat{\mu} = (1-\beta)\widehat{\mu}_0 + \beta\widehat{\mu}_1$ be the neutral decomposition from Proposition 4. Set $\mu_1 := \pi_*(\widehat{\mu}_1)$.

Proposition 12. *Given the regularity $r \geq 2$, there exist families of C^r reparametrizations $\mathcal{R}_k$ of σ over T'_{n_k} which are $(N, \varepsilon, \widehat{\varepsilon})$-admissible up to the time n_k for some numbers N (arbitrarily large) and $\varepsilon, \widehat{\varepsilon}$ (arbitrarily small), and which satisfy for all k large*

$$\mathrm{Card}(\mathcal{R}_k) \leq \exp(\beta h(f, \mu_1) n_k + c(r) n_k), \tag{2}$$

where $c(r) \to 0$ as $r \to \infty$.

Proof. We follow [BCS2, section 7.4] closely, using the same division into steps as there.

Steps 1–5. This is essentially the same as in [BCS2, section 7.4], so we only sketch the construction and refer the reader to [BCS2] for details:

- We fix $0 < \eta < 1/r$ small.
- We decompose $\widehat{\mu}_1 = \sum a_c \widehat{\mu}_{1,c}$ such that $\sum a_c = 1$ into $\ell \asymp 1/\eta$ mutually singular invariant measures satisfying: almost all the ergodic components of $\mu_{1,c} = \pi_* \widehat{\mu}_{1,c}$ have their entropy in an interval $[h_c, h_c + \eta)$, for each c (so $\overline{h}(f, \mu_{1,c}) \approx h(f, \mu_{1,c}) \approx h_c$ up to error η).
- We choose $0 < \gamma \ll \eta$ small with $\gamma < \gamma_0(r, f, \eta)$ and N large as in Propositions 10 and 11.
- We apply Proposition 10 to $\widehat{\mu}_0$ and get $\varepsilon, \widehat{\varepsilon}$ small (small enough so that Proposition 11 applies), $\bar{n}_0$ and an open set $\widehat{U}_0$ with $\widehat{\mu}(\partial U_0) = 0$.
- We apply Proposition 11 to each measure $\widehat{\mu}_{1,c}$ and large distinct integers $n_{1,c} > 1/\gamma$. We get open sets $\widehat{U}_{1,c}$ with $\widehat{\mu}_{1,c}(\widehat{U}_{1,c}) > 1 - \gamma^2$ and $\widehat{\mu}(\partial \widehat{U}_{1,c}) = 0$. We can reduce them so that for $c \neq c'$ and $0 \leq j \leq n_{1,c}$, we have $\widehat{\mu}_{1,c}(\widehat{U}_{1,c'}) < \gamma^2$ and the closures of $\widehat{f}^j(\widehat{U}_{1,c})$ and $\widehat{U}_{1,c'}$ are disjoint.
- We choose small neighborhoods $\widehat{V}_0, \widehat{V}_1$ of the measures $(1-\beta)\widehat{\mu}_0$ and $\beta\widehat{\mu}_1$ such that all measures in a same neighborhood $\widehat{V}_i$ give the same mass to $\widehat{M}$ up to an error smaller than γ^2, and similarly give the same mass to each set $\widehat{U}_0, \widehat{U}_{1,1}, \ldots, \widehat{U}_{1,\ell}$ up to γ^2.
- We choose the neutral parameters $0 < \alpha < \frac{\eta}{10}$ and $L > 2\max\{\bar{n}_0, n_{1,1}, \ldots, n_{1,\ell}\}/\gamma$ such that for k large enough, item (b) of Proposition 4 is satisfied for $\widehat{V}_0, \widehat{V}_1$.

Step 6. We decompose each non-periodic orbit $(\xi, \ldots, \widehat{f}^{n_k}(\xi))$ into pairwise disjoint subsegments $(\widehat{f}^t(\xi), \ldots, \widehat{f}^{t'-1}(\xi))$, falling into one of the following classes:

(a) *Blank segments*: (α, L)-neutral, with $\mathrm{Card}\{\widehat{f}^j(\xi) \in \widehat{U}_0,\ t \leq j < t'\} > (1-\gamma)(t'-t)$.
(b) *Segments with color c*: such that $\widehat{f}^t(\xi) \in \widehat{U}_{1,c}$ and $t' - t = n_{1,c}$.
(c) *Fillers*: orbit segments with length 1.

Note that the length of a segment determines its class.

The construction is the same as in [BCS2, section 7.4], so we only sketch it and refer the reader to [BCS2] for details. First, one considers all the (α, L)-neutral sub-segments of $(\xi, \ldots, \widehat{f}^{n_k}(\xi))$ which are maximal for the inclusion. Those which meet $\widehat{U}_0$ with a density larger than $1-\gamma$ are tagged "blank," the others are declared to be made of fillers. One then considers the complement of the union of all the (α, L)-neutral sub-segments: Scanning from the earliest iterate onwards, one inductively selects segments which qualify to be colored segments, and which are disjoint from the segments that have been previously identified. The iterates that have not been selected at the end of this process are declared to be fillers.

Lemma 13. *For any k large enough and $\xi \in \widehat{\sigma}(T'_{n_k})$, inside the orbit segment $(\xi, \ldots, \widehat{f}^{n_k}(\xi))$,*

(a) blank segments have total length at least $(1-\beta)n - 4\gamma n_k$*,*
(b) segments with color c have total length at most $\beta a_c n + \gamma n_k$*,*
(c) fillers have total length at most $6\gamma n_k$*.*

Proof. Item (b) of Proposition 4 and Lemma 6 imply that for k large enough, any point $\xi \in \widehat{\sigma}(T'_{n_k})$ satisfies $p_k^{\alpha,L}(\xi) \in \widehat{V}_0$ and $p_k(\xi) - p_k^{\alpha,L}(\xi) \in \widehat{V}_1$. Hence the proportion of the orbit segment $(\xi, \dots, \widehat{f}^{n_k}(\xi))$ and of its (α, L)-neutral part spent in each set $\widehat{U}_0$, $\widehat{U}_{1,c}$ is equal to the masses given to these sets by $(1-\beta)\widehat{\mu}_0$ and $\beta\widehat{\mu}_1$, up to an error smaller than γ^2.

We can now repeat the argument in [BCS2, Lemma 7.5] verbatim, using $\widehat{\mu}_0(\widehat{U}_{1,c}) > 1-\gamma^2$, $\mu_{1,c}(\widehat{U}_{1,c}) > 1-\gamma^2$, $\mu_{1,c}(\widehat{U}_{1,c'}) < \gamma^2$ $(c \neq c')$, $\gamma < \gamma_0(r, f, \eta)$ and $\gamma \ll \eta$. □

Step 7. The non-periodic orbits $(\xi, \dots, \widehat{f}^{n_k-1}(\xi))$ have been decomposed into segments which begin at iterates $\widehat{f}^{t_0}(\xi)$, ..., $\widehat{f}^{t_m}(\xi)$. The sequence $\theta = (t_0, \dots, t_m)$ is the *type of the decomposition.* (We recall that $t_{i+1} - t_i$ determines the class of the segment starting at t_i.)

Lemma 14. *Let* $H(t) := t\log\frac{1}{t} + (1-t)\log\frac{1}{1-t}$*. For k large, the number of types of decomposition* θ *of the orbits* $(\xi, \dots, \widehat{f}^{n_k-1}(\xi))$ *with* $\xi \in \widehat{\sigma}(T'_{n_k})$ *is bounded by* $\exp[H(10\gamma)n_k]$*.*

Idea of the proof. There are at most $6\gamma n_k$ fillers, and segments of other classes have lengths $\geq 1/\gamma$, hence every type θ has at most $\lfloor 8\gamma n_k \rfloor + 1$ elements. So the number of possible θ is bounded by the number of representations of n_k as an ordered sum of $m \leq \lfloor 8\gamma n_k \rfloor$ positive numbers. The lemma follows from a standard combinatorial computation, and De Moivre's estimate for $\binom{n}{pn}$. See [BCS2, Claim 7.6] for details. □

[**Step 8.** This step is not needed in the present proof (since n_k, T'_{n_k} are already built).]

Step 9. We fix a type θ and set $T^\theta_{n_k} = \{t \in T'_{n_k} : (\widehat{f}^i\sigma(t))_{i=0}^{n_k-1}$ has a decomposition of type $\theta\}$.

Lemma 15. *There exists a family* $\mathcal{R}^\theta_k$ *of reparametrizations of* σ *over the set* $T^\theta_{n_k}$ *that are* $(N, \varepsilon, \widehat{\varepsilon})$*-admissible up to time* n_k *with* $\mathrm{Card}(\mathcal{R}^\theta_n) \leq \exp\left(\sum_{i=1}^m \kappa_i(\theta)(t_i - t_{i-1})\right)$*, where*

$$\kappa_i(\theta) := \begin{cases} (r-1)^{-1}\lambda(\widehat{f}) + \eta & \text{if } t_i - t_{i-1} \geq L, \quad \text{(blank segment)},\\ h_c + (r-1)^{-1}\lambda(\widehat{f}) + 2\eta & \text{if } t_i - t_{i-1} = n_{1,c}, \quad \text{(c-colored segment)},\\ \text{some } C \text{ depending only on } r \text{ and } f & \text{if } t_i - t_{i-1} = 1, \quad \text{(filler)}. \end{cases}$$

Idea of the proof. The construction is identical to the step 9 in [BCS2]. One builds, inductively on i, a family of reparametrizations which are $(N, \varepsilon, \widehat{\varepsilon})$-admissible up to the time t_i. Given such a reparametrization ψ which is admissible up to the time t_{i-1}, one considers the curve $\sigma_0 = f^{t_{i-1}} \circ \sigma \circ \psi$ and applies Proposition 10, Proposition 11, or Theorem 9, depending on the class of the orbit segment between times t_{i-1} and t_i (respectively blank, colored or filler). One gets a family of reparametrizations φ of σ_0 which are admissible up to time $t_i - t_{i-1}$ over the set $\psi^{-1}(T^\theta_{n_k})$. The set of compositions $\psi \circ \varphi$ defines a family of reparametrizations of σ that are admissible up to time t_i. □

Step 10. We estimate the cardinality of a suitable family of reparametrizations $\mathcal{R}_k$ as in [BCS2]. For each type θ, the bounds on the total lengths of the different classes of segments (Lemma 13), the estimates $\sum a_c h_c \leq h(f, \mu_1)$ and $\gamma \ll \eta$ together give:

$$\mathrm{Card}(\mathcal{R}^\theta_n) \leq \exp\left(\beta h(f, \mu_1)n + \frac{\lambda(\widehat{f})}{r-1}n + 4\eta n\right).$$

We can take $\mathcal{R}_k := \bigcup_\theta \mathcal{R}^\theta_k$. Its cardinality is at most the above bound multiplied by the number of types, estimated in Lemma 14. Remembering that we have taken $\eta < 1/r$, this concludes the proof of Proposition 12. □

1.7. **Summary up to this point and conclusion of the proof.** Fix a regular C^∞ curve $\sigma : [0,1] \to M$ so that $\limsup \frac{1}{n} \log \|Df^n|_\sigma(\sigma(s))\| > 0$ for a set of positive Lebesgue measure of parameters $s \in [0,1]$ (Lemma 2).

Let $\overline{\lambda} := \operatorname{ess\,sup}_{s\in[0,1]} \limsup \frac{1}{n} \log \|Df^n|_\sigma(\sigma(s))\| > 0$, and choose sequences $\rho_k \uparrow 1$, $\lambda_k^{min} \uparrow \overline{\lambda}$ and $\lambda_k^{max} \downarrow \overline{\lambda}$. In sections 1.2–1.4 we constructed $n_k \uparrow \infty$, and a sequence of sets

$$T'_{n_k} \subset \{s \in [0,1] : \lambda_k^{min} \leq \frac{1}{n_k} \log \|Df^{n_k}|_\sigma(\sigma(s))\| \leq \lambda_k^{max},$$
$$\sigma(s) \text{ is not } f\text{-periodic and } \forall 0 \leq i \leq n_k,\ \|Df^i|_\sigma(f^{n_k-i}\sigma(s))\| \geq 1\}$$

with Lebesgue measure $|T'_{n_k}| \geq \rho_k^{2n_k}$.

We then took points $\xi_{n_k} \in \widehat{\sigma}(T'_{n_k})$, and replaced n_k by a further subsequence (which abusing notation we also called n_k), for which the $\widehat{f}$-empirical measures $p_k(\xi_{n_k})$ satisfy

$$p_k(\xi_{n_k}) \xrightarrow[k\to\infty]{\text{weak-}*} \widehat{\mu} = (1-\beta)\widehat{\mu}_0 + \beta\widehat{\mu}_1,$$

with $\widehat{\mu}_0$, $\widehat{\mu}_1$ as in Proposition 4. We saw in Corollary 7 that $\beta > 0$ and that the projection of $\widehat{\mu}_1$ to M is an f-invariant measure such that

$$\beta\lambda^+(\mu_1) = \overline{\lambda}. \tag{3}$$

We then analyzed the Yomdin-theoretic C^r complexity of $\sigma|_{T'_{n_k}}$, by estimating the size of the family $\mathcal{R}_k$ of admissible reparametrizations needed to cover $\sigma(T'_{n_k})$. Using tools developed in [BCS2], we were able to show the existence of families $\mathcal{R}_k$ such that for $k \gg 1$,

$$\rho_k^{2n_k} \exp(\lambda_k^{min} n_k) \min_t \|\sigma'(t)\| \leq \operatorname{Card}(\mathcal{R}_k) \leq \exp(\beta h(f,\mu_1) n_k + c(\eta,\gamma,r) n_k), \tag{4}$$

with η, γ arbitrarily small, r arbitrarily large, and $c(\eta,\gamma,r) \to 0$ as $\gamma, \eta \to 0, r \to \infty$, see (1),(2). Since $\lambda_k^{min} \to \overline{\lambda}$ and $\rho_k \to 1$, (4) implies that $\overline{\lambda} \leq \beta h(f,\mu_1)$.

Combining this with (3) and with Ruelle's inequality, we obtain:

$$\beta\lambda^+(\mu_1) = \overline{\lambda} \leq \beta h(f,\mu_1) \leq \beta\lambda^+(\mu_1) \qquad \text{with } \overline{\lambda} > 0.$$

Hence $\lambda^+(\mu_1) = h(f,\mu_1) > 0$. Note that almost every ergodic component μ'_1 of μ_1 satisfies $\lambda^+(\mu'_1) = h(f,\mu'_1)$ by Ruelle's inequality and $\lambda^+(\mu'_1) > 0$ by Proposition 4 (d), hence is an SRB measure. □

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