SPECTRAL GAP AND TRANSIENCE FOR RUELLE OPERATORS ON COUNTABLE MARKOV SHIFTS

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Abstract. We find a necessary and sufficient condition for the Ruelle operator of a weakly Hölder continuous potential on a topologically mixing countable Markov shift to act with spectral gap on some rich Banach space. We show that the set of potentials satisfying this condition is open and dense for a variety of topologies. We then analyze the complement of this set (in a finer topology) and show that among the three known obstructions to spectral gap (weak positive recurrence, null recurrence, transience), transience is open and dense, and null recurrence and weak positive recurrence have empty interior.

1. Introduction

1.1. **Overview.** Thermodynamic formalism is a branch of ergodic theory which studies, for a given dynamical system $T : X \to X$ and a given function $\phi : X \to \mathbb{R}$, the existence and properties of invariant probability measures μ_ϕ which maximize the quantity $h_\mu(T) + \int \phi d\mu$ ("*equilibrium measures*"). The key tool is the *Ruelle operator*,

$$(L_\phi f)(x) = \sum_{Ty=x} e^{\phi(y)} f(y). \tag{1.1}$$

Under fairly mild conditions, if L_ϕ acts with spectral gap on some rich enough Banach space $\mathcal{L}$, then μ_ϕ exists, and quite a lot can be said about its properties (see the books [B], [HH], [PP],[R], or theorem 1.1).

Here we ask how large is the set of functions $\phi : X \to \mathbb{R}$ for which such a space $\mathcal{L}$ can be found. We study this question within the cadre of countable Markov shifts, and weakly Hölder continuous functions $\phi : X \to \mathbb{R}$ (see below). We

(a) identify a necessary and sufficient condition on ϕ for the existence of a Banach space on which L_ϕ acts with spectral gap;
(b) analyze the topological structure of the set of functions ϕ which satisfy this condition;
(c) compare the topological properties of the various obstructions to this condition, and figure out which obstruction is the most important.

1.2. **Setting.** Let $\mathcal{S}$ be a countable set, and $A = (t_{ij})_{\mathcal{S}\times\mathcal{S}}$ be a matrix of zeroes and ones. The *countable Markov shift (CMS)* with set of *states* $\mathcal{S}$ and *transition matrix* A is the dynamical system $T : X \to X$, where

$$X := \{(x_0, x_1, \ldots,) \in \mathcal{S}^{\mathbb{N}\cup\{0\}} : t_{x_i x_{i+1}} = 1 \text{ for all } i\}, \text{ and } T(x)_i := x_{i+1}.$$

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We think of X as of the collection of one sided infinite admissible paths on a directed graph with vertices $v \in \mathcal{S}$, and edges $v_1 \to v_2$ ($v_1, v_2 \in \mathcal{S}$, $t_{v_1 v_2} = 1$).

We equip X with the metric $d(x,y) = 2^{-t(x,y)}$, $t(x,y) := \inf\{k : x_k \neq y_k\}$ (where $\inf \varnothing := \infty$). The resulting topology is generated by the *cylinder sets*

$$[a_0, \ldots, a_{n-1}] := \{x \in X : x_i = a_i \ , \ i = 0, \ldots, n-1\} \ \ (a_0, \ldots, a_{n-1} \in \mathcal{S}, n \geq 1).$$

A word $\underline{a} \in \mathcal{S}^n$ is called *admissible* if the cylinder it defines is non-empty. The *length* of an admissible word $\underline{a} = (a_0, \ldots, a_{n-1})$ is $|\underline{a}| := n$.

We assume throughout that $T : X \to X$ is *topologically mixing*. This is the case when for any two states a, b there is an $N(a,b)$ such that for all $n \geq N(a,b)$ there is an admissible word of length n which starts at a and ends at b.

Next we consider real valued functions $\phi : X \to \mathbb{R}$. We define the *variations* of a function $\phi : X \to \mathbb{R}$ to be the numbers

$$\mathrm{var}_n(\phi) := \sup\{|\phi(x) - \phi(y)| : x_0^{n-1} = y_0^{n-1}\},$$

where here and throughout $z_m^n := (z_m, \ldots, z_n)$. We say that ϕ has *summable variations*, if $\sum_{n \geq 2} \mathrm{var}_n \phi < \infty$. We say that ϕ is *θ–weakly Hölder continuous* for $0 < \theta < 1$, if there exists $A_\phi > 0$ such that $\mathrm{var}_n(\phi) \leq A_\phi \theta^n$ for all $n \geq 2$. A weakly Hölder continuous function is Hölder (with respect to the metric defined above) iff it is bounded. A bounded θ–weakly Hölder function will be called *θ–Hölder*.

The Birkhoff sums of a function ϕ are denoted by $\phi_n := \sum_{k=0}^{n-1} \phi \circ T^k$.

Suppose ϕ has summable variations and X is topologically mixing. The *Gurevich pressure* of ϕ is the limit

$$P_G(\phi) = \lim_{n \to \infty} \frac{1}{n} \log Z_n(\phi, a), \text{ where } Z_n(\phi, a) = \sum_{T^n x = x} e^{\phi_n(x)} 1_{[a]}(x), \text{ and } a \in \mathcal{S}.$$

This limit is independent of a, and if $\sup \phi < \infty$, then it is equal to $\sup\{h_\mu(T) + \int \phi d\mu\}$, where the supremum ranges over all invariant probability measures such that the sum is not of the form $\infty - \infty$ [S1].

1.3. **The Spectral Gap Property.** Recall that the *Ruelle operator* associated with ϕ is the operator $(L_\phi f)(x) := \sum_{Ty=x} e^{\phi(y)} f(y)$. This is well defined for functions f such that the sum converges for all $x \in X$. Let $\mathrm{dom}(L_\phi)$ denote the collection of such functions.

Definition 1.1. *Suppose ϕ is θ–weakly Hölder continuous, and $P_G(\phi) < \infty$. We say that ϕ has the* spectral gap property (SGP) *if there is a Banach space of continuous functions $\mathcal{L}$ s.t.*

(a) *$\mathcal{L} \subset \mathrm{dom}(L_\phi)$ and $\mathcal{L} \supseteq \{1_{[\underline{a}]} : \underline{a} \in \mathcal{S}^n \ , \ n \in \mathbb{N}\}$;*
(b) *$f \in \mathcal{L} \Rightarrow |f| \in \mathcal{L}$, $\big\||f|\big\|_{\mathcal{L}} \leq \|f\|_{\mathcal{L}}$;*
(c) *$\mathcal{L}$–convergence implies uniform convergence on cylinders;*
(d) *$L_\phi(\mathcal{L}) \subseteq \mathcal{L}$, and $L_\phi : \mathcal{L} \to \mathcal{L}$ is bounded;*
(e) *$L_\phi = \lambda P + N$ where $\lambda = \exp P_G(\phi)$, and $PN = NP = 0$, $P^2 = P$, $\dim \mathrm{Im}\, P = 1$, and the spectral radius of N is less than λ;*
(f) *If g is θ–Hölder, then $L_{\phi + zg} : \mathcal{L} \to \mathcal{L}$ is bounded, and $z \mapsto L_{\phi+zg}$ is analytic on some complex neighborhood of zero.*

The motivation is the following (compare with [R],[HH],[Li], [PP],[BS],[GH],[AD]). Suppose X is a topologically mixing CMS, and $\phi : X \to \mathbb{R}$ is a weakly Hölder continuous potential with finite Gurevich pressure, finite supremum, and the SGP. Write $L_\phi = \lambda P + N$ as above, then

Theorem 1.1. *P takes the form $Pf = h\int f d\nu$, where $h \in \mathcal{L}$ is positive, and ν is a measure which is finite and positive on all cylinders. The measure $dm_\phi = hd\nu$ is a T–invariant probability measure with the following properties:*

(a) *If m_ϕ has finite entropy, then m_ϕ is the unique equilibrium measure of ϕ.*
(b) *There is a $0<\kappa<1$ s.t. for all $g \in L^\infty(m_\phi)$ and f bounded Hölder continuous, $\exists C(f,g)>0$ s.t. $|\operatorname{Cov}_{m_\phi}(f, g\circ T^n)| \le C(f,g)\kappa^n$. (Cov =covariance.)*
(c) *Suppose ψ is a bounded Hölder continuous function such that $\mathbb{E}_{m_\phi}[\psi] = 0$. If $\psi \neq \varphi - \varphi\circ T$ with φ continuous, then $\exists\sigma > 0$ s.t. $\psi_n/\sqrt{n}$ converges in distribution (w.r.t. m_ϕ) to the normal distribution with mean zero and standard deviation σ.*
(d) *Suppose ψ is a bounded Hölder continuous function, then $t \mapsto P_G(\phi + t\psi)$ is real analytic on a neighborhood of zero.*

We remark that the assumption that m_ϕ has finite entropy is trivially satisfied for all CMS with finite Gurevich entropy $P_G(0) < \infty$.

Versions of theorem 1.1 were shown in a variety of contexts by many people [R], [GH], [HH], [AD],[Li], [BS], [G1] (this is a partial list). The proof in our context is given in appendix A.

1.4. **The problem:** *When does a potential ϕ satisfy the SGP? How common is this phenomenon? What are the most important obstructions?*

If $|\mathcal{S}|<\infty$ then every (weakly) Hölder continuous function has SGP (Ruelle [R]), but this is not the case when $|\mathcal{S}| = \infty$ because of the phenomena of null recurrence, transience [S2], and positive recurrence with sub-exponential decay of correlations [S4] or non–analytic pressure function [S5], [Lo], [PrS].

Doeblin and Fortet have given sufficient conditions for spectral gap for potentials ϕ associated to a class of countable Markov chains [DF]. Aaronson & Denker had constructed Banach spaces with spectral gap for potentials associated with Gibbs–Markov measures [AD]. The underlying CMS must satisfy a certain combinatorial condition (the "big images" property). Young had constructed Banach spaces with spectral gap for certain functions ϕ on CMS satisfying a different combinatorial condition ("tower structure"), see [Y].

1.5. **Notational convention:** $a = c \pm \varepsilon$ means $c - \varepsilon < a < c + \varepsilon$, $a = B^{\pm 1}c$ means $B^{-1} \le a/c \le B$, and $a_n \asymp c_n$ means that $\exists B$ s.t. $a_n = B^{\pm 1}c_n$ for all n large.

2. Summary of Results

2.1. **A necessary and sufficient condition for SGP.** The condition is in terms of the *discriminant*, a notion which was introduced in [S3]. We recall the definition, and refer the reader to appendix A for further information.

If one induces a CMS on one of its states $a \in \mathcal{S}$, then the result is a full shift. It is useful to fix the following notation:

(a) $\overline{\mathcal{S}} := \{[\underline{a}] = [a, \xi_1, \ldots, \xi_{n-1}] : n \ge 1, \xi_i \neq a, [a, \underline{\xi}, a] \neq \varnothing\}$;
(b) $\overline{X} := \overline{\mathcal{S}}^{\mathbb{N}\cup\{0\}}$, viewed as a countable Markov shift with set of states $\overline{\mathcal{S}}$;

(c) $\pi : \overline{X} \to [a]$; $\pi([\underline{a}_0], [\underline{a}_1], \ldots) = (\underline{a}_0, \underline{a}_1, \ldots)$. This is a conjugacy between the left shift on $\overline{X}$, and the induced (=first return) map on $[a]$.

Every function $\phi : X \to \mathbb{R}$ has an "induced version" $\overline{\phi} : \overline{X} \to \mathbb{R}$ given by

$$\overline{\phi} := \left(\sum_{k=0}^{\varphi_a - 1} \phi \circ T^k \right) \circ \pi, \text{ where } \varphi_a(x) := 1_{[a]}(x) \inf\{n \geq 1 : T^n(x) \in [a]\}.$$

It is easy to see that if ϕ is weakly Hölder continuous on X, then $\overline{\phi}$ is weakly Hölder continuous on $\overline{X}$ (moreover, $\mathrm{var}_1 \overline{\psi} < \infty$ even when $\mathrm{var}_1 \psi = \infty$).

The *a–discriminant* of ϕ is the (possibly infinite) quantity

$$\Delta_a[\phi] := \sup\{P_G(\overline{\phi + p}) : p \in \mathbb{R} \text{ s.t. } P_G(\overline{\phi + p}) < \infty\}.$$

The sign of this number has meaning [S3], see appendix A.

A weakly Hölder continuous function ϕ on a topologically mixing countable Markov shift is called *strongly positive recurrent*, if it has finite Gurevich pressure and if there is a state a s.t. $\Delta_a[\phi] > 0$. Strong positive recurrence is a generalization of the notion of *stable positive recurrence* for positive infinite matrices due to Gurevich and Savchenko [GS]. It has its roots in the classical work of Vere-Jones on the problem of geometric ergodicity for Markov chains [VJ].

Theorem 2.1. *Suppose X is a topologically mixing CMS, and $\phi : X \to \mathbb{R}$ is weakly Hölder continuous with finite Gurevich pressure, then ϕ has the spectral gap property iff ϕ is strongly positive recurrent.*

That SGP implies SPR is fairly routine, given the results of [S3]. The main part of the theorem is the other direction.

It is perhaps useful at this point to explain how to check strong positive recurrence. Define

$$Z_n^*(\phi, a) := \sum_{T^n x = x} e^{\phi_n(x)} 1_{[\varphi_a = n]}(x),$$

and let R denote the radius of convergence of $r_\phi(x) := \sum_{n \geq 1} x^n Z_n^*(\phi, a)$, then [S3] proves that either $|\Delta_a[\phi] - \log r_\phi(R)| \leq \sum_{n=2}^\infty \mathrm{var}_n \phi$ or $\Delta_a[\phi] = \log r_\phi(R) = \infty$. In particular, if $r_\phi(\cdot)$ diverges at its radius of converges, then ϕ is SPR.

It is easy to construct examples of ϕ with SGP on any topologically mixing CMS: Start with any weakly Hölder continuous $\phi : X \to \mathbb{R}$ with finite pressure, and fix some state a. One checks that $r_{\phi + t1_{[a]}}(x) = e^t r_\phi(x)$, thus $\phi + t1_{[a]}$ is strongly positive recurrent for any t large enough.

2.2. **SGP is open and dense.** Let Φ denote the collection of weakly Hölder continuous functions $\phi : X \to \mathbb{R}$ with finite Gurevich pressure. There are many different useful topologies on Φ. To list them concisely, fix an infinite sequence $\omega = (\omega_n)_{n \geq 1}$, $0 \leq \omega_n \leq \infty$ and define for a function $f : X \to \mathbb{R}$,

$$\begin{aligned} \|f\|_\omega &:= \sup|f| + \sum_{n=1}^\infty \omega_n \mathrm{var}_n(f), \text{ where } 0 \cdot \infty := 0, \\ V(\phi, \varepsilon) &:= \{\phi' \in \Phi : \|\phi - \phi'\|_\omega < \varepsilon\}. \end{aligned}$$

The *ω–topology* is the topology generated by $V(\phi, \varepsilon)$, $(\varepsilon > 0, \phi \in \Phi)$.

The choice $\omega = (0, 0, \ldots)$ is useful for the study of perturbations in the sup norm. Other important choices are $\omega = (0, \ldots, 0, \infty, \infty, \ldots)$ (finite memory), $\omega = (0, 1, 1, \ldots)$ (summable variations), and $\omega = (0, \theta^{-1}, \theta^{-2}, \ldots)$ (Hölder).

Theorem 2.2. *The set of $\phi \in \Phi$ with the spectral gap property is open and dense in Φ with respect to the ω–topology, for any $\omega = (\omega_n)_{n\geq 1}$.*

In particular, the spectral gap property is stable under perturbations in Φ with sufficiently small sup norm ($\omega = (0, 0, \ldots)$); and any $\phi \in \Phi$ can be perturbed to be strongly positive recurrent using a perturbation of arbitrarily small Hölder norm, or even finite memory of length one ($\omega = (0, \infty, \infty, \ldots)$).

This means that there is an open and dense set of $\phi \in \Phi$ which satisfy the conclusion of theorem 1.1. Loosely speaking these are potentials whose thermodynamic formalism is similar to the behavior of thermodynamic systems at equilibrium *without* a phase transition. The following works contain related results:

(1) Gurevich and Savchenko showed in [GS] that if $\phi \in \Phi$ is "stably positive recurrent" and ϕ is Markovian (i.e. $\mathrm{var}_2\, \phi = 0$), then there is an $\varepsilon > 0$ s.t. any Markovian $\phi' \in \Phi$ s.t. $\|\phi - \phi'\|_\infty < \varepsilon$ is positive recurrent (c.f. appendix A). For Markovian potentials, "stable positive recurrence" can be easily seen to be equivalent to strong positive recurrence.
(2) Gallavotti & Miracle–Sole considered in [GM] multi-dimensional lattice gas models, and showed that in a certain topology there is a dense G_δ–set of interaction potentials where pressure functional is differentiable.

Next we consider the larger set Φ_{SV} of all $\phi : X \to \mathbb{R}$ with summable variations and finite Gurevich pressure. Again, we can define the ω–topology on Φ_{SV} as the topology generated by $\{\phi' \in \Phi_{SV} : \|\phi' - \phi\|_\omega < \varepsilon\}$ for all $\varepsilon > 0, \phi \in \Phi_{SV}$.

Theorem 2.2′. *Let Φ_{SV} denote the collection of all $\phi : X \to \mathbb{R}$ with summable variations and finite Gurevich pressure, then $\{\phi \in \Phi_{SV} : \phi$ is strongly positive recurrent$\}$ is open and dense in Φ_{SV} for every ω–topology.*

Obstructions to the SGP. If a potential $\phi \in \Phi$ does not have the spectral gap property, then by theorem 2.1 it is not strongly positive recurrent, and $\Delta_a[\phi] \leq 0$.

Potentials with *strictly* negative discriminant are called *transient*. Potentials with zero discriminant are divided into two groups: null recurrent, and weakly positive recurrent (see appendix A for a summary of the definitions and properties of the various modes of recurrence – in particular see Theorem 7.3 to equate the above definition of transience with that in Definition 7.1). We ask whether one of these obstructions is more common, in some sense, than the others.

The ω–topologies are too weak to detect the difference between transience, null recurrence, and weak positive recurrence (they are all nowhere dense), so we need to use a stronger topology.

The topologies of perturbations of finite support are sufficient for this purpose. To define these topologies, fix a (nonempty) finite collection of states $B = \bigcup_{i=1}^{N} [a_i]$. The *uniform topology localized at B* (or just the 'B–uniform topology') is the topology generated by the basis

$$U(\phi; \varepsilon, B) := \{\phi' \in \Phi : \|\phi' - \phi\|_\infty < \varepsilon, \phi'|_{X\setminus B} = \phi|_{X\setminus B}\} \quad (\varepsilon > 0, \phi \in \Phi).$$

Denote the resulting topology by $\mathcal{LU}(B)$.

Theorem 2.3. *Let $\Phi(\mathrm{Tr}) := \{\phi \in \Phi : \phi$ is transient$\}$. With respect to $\mathcal{LU}(B)$, $\Phi(\mathrm{Tr})$ is open in Φ, and open and dense in $\{\phi \in \Phi : \phi$ does not have SGP$\}$.*

As a corollary of this theorem and its proof we have the following topological description of the various modes of recurrence in each of the $\mathcal{LU}(B)$–topologies:

(a) strong positive recurrence: open;
(b) transience: open
(c) weak positive recurrence and null recurrence: empty interior, contained in the boundaries of the first two sets

In other words, transience is the most common obstruction to spectral gap.

3. Proof of Theorem 2.1

3.1. **Strong Positive Recurrence implies Spectral Gap.** Assume w.l.o.g. that $P_G(\phi) = 0$ (otherwise pass to $\phi - P_G(\phi)$, c.f. §7.1). Fix some state $a \in \mathcal{S}$ s.t. $\Delta_a[\phi] > 0$. By the discriminant theorem (appendix A, Theorem 7.3), $P_G(\overline{\phi}) = 0$, where the over bar indicates induction on $[a]$. Therefore, by strong positive recurrence, there exists ε_a such that $0 < P_G(\overline{\phi + 2\varepsilon_a}) < \infty$. This ε_a must be positive, because $p(t) := P_G(\overline{\phi + t})$ is an increasing function.

The function ϕ is by assumption weakly Hölder so there exists $0 < \theta < 1$ and $A_\phi > 0$ such that $\mathrm{var}_n\, \phi \le A_\phi \theta^n$ for all $n \ge 2$. Make ε_a so small that

$$0 < \theta e^p < 1, \text{ where } p := P_G(\overline{\phi + \varepsilon_a}). \tag{3.1}$$

This is possible to do, because $p(t) := P_G(\overline{\phi + t})$ is continuous (being convex and finite) on $(-\infty, 2\varepsilon_a)$ (see (7.4) in appendix A).

Define $\psi := \phi + \varepsilon_a - p1_{[a]}$, then using the properties of $P_G(\cdot)$ listed in appendix A §7.1 it readily follows that

(1) $P_G(\overline{\psi}) = 0$, because $P_G(\overline{\psi}) = P_G(\overline{\phi + \varepsilon_a} - p) = P_G(\overline{\phi + \varepsilon_a}) - p = 0$;
(2) ψ is strongly positive recurrent, because

$$\begin{aligned} &P_G(\overline{\psi + \varepsilon_a}) \le P_G(\overline{\phi + 2\varepsilon_a}) < \infty, \text{ and} \\ &P_G(\overline{\psi + \varepsilon_a}) = P_G(\overline{\psi} + \varepsilon_a \varphi_a) \ge P_G(\overline{\psi}) + \varepsilon_a = \varepsilon_a > 0, \end{aligned}$$

so $\Delta_a[\psi] > 0$;
(3) $P_G(\psi) = 0$, because $P_G(\overline{\psi}) = 0$ and ψ is (strongly positive) recurrent, see appendix A, theorem 7.3 part (1).

Since ψ is SPR, it is positive recurrent (appendix A, theorem 7.3). By the generalized Ruelle Perron Frobenius theorem (appendix A, theorem 7.2) and the assumption that $P_G(\psi) = 0$, there exists a Borel measure ν_0, finite and positive on cylinders, and a positive continuous function $h_0 : X \to \mathbb{R}$ such that

$$L_\psi^* \nu_0 = \nu_0 \ , \ L_\psi h_0 = h_0 \ , \text{ and } \int h_0 d\nu_0 = 1.$$

Moreover, $\mathrm{var}_1[\log h_0] \le \sum_{\ell \ge 2} \mathrm{var}_\ell\, \phi$. Setting $C_0 := \exp \sum_{\ell \ge 2} \mathrm{var}_\ell\, \phi$, we see that for every x, $h_0(x) = C_0^{\pm 1} h_0[x_0]$, where $h_0[x_0] := \sup_{[x_0]} h_0$.

Define for $x, y \in X$,

$$\begin{aligned} t(x,y) &:= \min\{n : x_n \ne y_n\}, \quad \text{where } \min \varnothing = \infty, \\ s_a(x,y) &:= \#\{0 \le i \le t(x,y) - 1 : x_i = y_i = a\} \end{aligned}$$

(compare with the notion of "separation time" due to L.-S. Young [Y]).

Let $\mathcal{L}$ denote the collection of continuous functions $f : X \to \mathbb{C}$ for which

$$\|f\|_{\mathcal{L}} := \sup_{b \in \mathcal{S}} \frac{1}{h_0[b]} \left[\sup_{x \in [b]} |f(x)| + \sup\left\{ |f(x) - f(y)| / \theta^{s_a(x,y)} : x, y \in [b], x \ne y \right\} \right] < \infty.$$

It is clear that $(\mathcal{L}, \|\cdot\|_{\mathcal{L}})$ is a Banach space. We show that $L_\phi(\mathcal{L}) \subseteq \mathcal{L}$, and that $L_\phi : \mathcal{L} \to \mathcal{L}$ is a bounded operator with spectral gap.

The proof uses the strengthening of the Ionsecu-Tulcea & Marinecu theorem due to Hennion ([HH], theorem II.5). Suppose there exists a continuous semi-norm $\|\cdot\|_{\mathcal{C}}$ on $\mathcal{L}$ with the following properties:

(A) There is a constant $M > 0$ s.t. $\|L_\phi f\|_{\mathcal{C}} \le M\|f\|_{\mathcal{C}}$ for all $f \in \mathcal{L}$;
(B) Let $\rho(L_\phi)$ denote the spectral radius of $L_\phi : \mathcal{L} \to \mathcal{L}$. There are constants $n_0 \in \mathbb{N}$, $0 < r < \rho(L_\phi)$, and $R > 0$ such that

$$\|L_\phi^{n_0} f\|_{\mathcal{L}} \le r^{n_0}\|f\|_{\mathcal{L}} + R\|f\|_{\mathcal{C}}; \tag{3.2}$$

(C) Every sequence $\{f_n\}_{n\ge1} \in \mathcal{L}$ s.t. $\sup\|f_n\|_{\mathcal{L}} \le 1$ has a subsequence $\{f_{n_k}\}_{k\ge1}$ s.t. $\|L_\phi f_{n_k} - g\|_{\mathcal{C}} \xrightarrow[k\to\infty]{} 0$ for some $g \in \mathcal{L}$.

Hennion's theorem then says that $\mathcal{L} = \mathcal{F} \oplus \mathcal{N}$ where $\mathcal{F}, \mathcal{N}$ are L_ϕ–invariant subspaces such that $\dim(\mathcal{F}) < \infty$, $\rho(L_\phi|_{\mathcal{N}}) < \rho(L_\phi)$, and such that every eigenvalue of $L_\phi|_{\mathcal{F}}$ is of modulus $\rho(L_\phi)$.

As we shall see below, the theory of equilibrium measures on topologically mixing CMS implies that $\rho(L_\phi) = 1$, that the only eigenvalue on the unit circle is one, and that this eigenvalue is simple. This gives the spectral gap property with $\lambda = 1$, P the eigenprojection of one, and $N := L_\phi(I - P)$.

We will apply Hennion's theorem to $L_\phi : \mathcal{L} \to \mathcal{L}$. The semi–norm we use is $\|\cdot\|_{\mathcal{C}} := \|\cdot\|_{L^1(\nu_0)}$.

Step 1. (A) holds: $\|\cdot\|_{\mathcal{C}}$ is a continuous semi-norm on $\mathcal{L}$, and there is a constant M such that $\|L_\phi f\|_{\mathcal{C}} \le M\|f\|_{\mathcal{C}}$ for all $f \in \mathcal{L}$.

Proof. To see that $\|\cdot\|_{\mathcal{C}}$ is continuous, suppose that $\|f_n - f\|_{\mathcal{L}} \to 0$. Then $f_n \to f$ pointwise, and $|f_n(x) - f(x)| \le \|f_n - f\|_{\mathcal{L}} h_0[x_0] \le C_0\|f_n - f\|_{\mathcal{L}} h_0(x)$ at every point. Since $h_0 \in L^1(\nu_0)$, $\|f_n - f\|_{\mathcal{C}} = \int |f_n - f| d\nu_0 \to 0$.

Next fix $f \in \mathcal{L}$. Then $|f| \le C_0\|f\|_{\mathcal{L}} h_0$. The identity $\phi = \psi - \varepsilon_a + p1_{[a]} \le \psi + p - \varepsilon_a$ shows that $|L_\phi f| \le e^{p-\varepsilon_a} L_\psi(C_0\|f\|_{\mathcal{L}} h_0) = C_0 e^{p-\varepsilon_a}\|f\|_{\mathcal{L}} h_0$. Integrating w.r.t ν_0, we get $\|L_\phi f\|_{\mathcal{C}} \le C_0 e^{p-\varepsilon_a}\|f\|_{\mathcal{L}}$ and the step follows with $M := C_0 \exp(p - \varepsilon_a)$.

Step 2. Proof of (3.2).

Proof. We need some notation. For every $b \in \mathcal{S}$, set

$$P^n(b) := \{\underline{p} = (p_0, \ldots, p_{n-1}) : (\underline{p}, b) \text{ is admissible}\}.$$

For every $\underline{p} = (p_0, \ldots, p_{n-1})$ admissible, let $n(\underline{p}) := \#\{0 \le i \le n-1 : p_i = a\}$, and set $P_k^n(b) := \{\underline{p} \in P^n(b) : n(\underline{p}) \ge k + 1\}$.

In what follows we fix k (to be determined later), and estimate $\|L_\phi^n f\|_{\mathcal{L}}$ for arbitrary $f \in \mathcal{L}$ and $n \ge 1$.

Part 1: Analysis of $\sup_{x\in[b]} |(L_\phi^n f)(x)|$ $(b \in \mathcal{S})$.

Suppose $x \in [b]$. Since $\phi = \psi - \varepsilon_a + p1_{[a]}$ and $|f| \le C_0\|f\|_{\mathcal{L}} h_0$,

$$|(L_\phi^n f)(x)| \le \sum_{\underline{p}\in P^n(b)} e^{\phi_n(\underline{p}x)}|f(\underline{p}x)| = \sum_{\underline{p}\in P^n(b)} e^{\psi_n(\underline{p}x) - n\varepsilon_a + pn(\underline{p})}|f(\underline{p}x)|$$

$$\begin{aligned}
&\leq \sum_{\underline{p}\in P_k^n(b)} e^{\psi_n(\underline{p}x)-n\varepsilon_a+pn(\underline{p})}|f(\underline{p}x)| + \\
&\qquad\qquad +C_0 e^{kp-n\varepsilon_a}\|f\|_{\mathcal{L}} \sum_{\underline{p}\in P^n(b)\setminus P_k^n(b)} e^{\psi_n(\underline{p}x)} h_0(\underline{p}x) \\
&\leq \sum_{\underline{p}\in P_k^n(b)} e^{\psi_n(\underline{p}x)-n\varepsilon_a+pn(\underline{p})}|f(\underline{p}x)| + C_0 e^{kp-n\varepsilon_a}\|f\|_{\mathcal{L}} h_0[b],
\end{aligned}$$

because the last sum is bounded by $(L_\psi^n h_0)(x) = h_0(x) \leq h_0[b]$.

Every $\underline{p} \in P_k^n(b)$ admits a unique decomposition $\underline{p} = (\underline{\alpha}, \underline{\beta}, \underline{\gamma})$ with $\underline{\alpha} \in A_k, \underline{\beta} \in B$ and $\underline{\gamma} \in C$, where:

$$\begin{aligned}
A_k &:= \{\underline{\alpha} : \ n(\underline{\alpha}) = k, \text{ and } (\underline{\alpha}, a) \text{ is admissible}\}, \\
B &:= \{\underline{\beta} : \ \underline{\beta} \text{ starts at } a, \text{ and } (\underline{\beta}, a) \text{ is admissible}\} \cup \{\text{empty word}\}, \\
C &:= \{\underline{\gamma} : \ \underline{\gamma} \text{ contains exactly one } a, \text{ at its beginning, and } (\underline{\gamma}, b) \text{ is admissible}\}.
\end{aligned}$$

Conversely, every triplet $(\underline{\alpha}, \underline{\beta}, \underline{\gamma}) \in A_k \times B \times C$ such that $|\underline{\alpha}| + |\underline{\beta}| + |\underline{\gamma}| = n$ (where $|\underline{w}|$:=length of $\underline{w}$) gives rise to an element of $P_k^n(b)$. Thus

$$\begin{aligned}
|(L_\phi^n f)(x)| &\leq C_0 e^{kp-n\varepsilon_a}\|f\|_{\mathcal{L}} h_0[b] + \\
&\quad + \sum_{\alpha+\beta+\gamma=n} \Bigg\{ \sum_{\underline{\gamma}\in C, |\underline{\gamma}|=\gamma} e^{\psi_\gamma(\underline{\gamma}x)-\gamma\varepsilon_a} \times \qquad (3.3) \\
&\quad \times \sum_{\underline{\beta}\in B, |\underline{\beta}|=\beta} e^{\psi_\beta(\underline{\beta}\,\underline{\gamma}x)-\beta\varepsilon_a+p[n(\underline{\beta}\,\underline{\gamma})+k]} \sum_{\underline{\alpha}\in A_k, |\underline{\alpha}|=\alpha} e^{\psi_\alpha(\underline{\alpha}\,\underline{\beta}\,\underline{\gamma}x)-\alpha\varepsilon_a}|f(\underline{\alpha}\,\underline{\beta}\,\underline{\gamma}x)| \Bigg\},
\end{aligned}$$

with the convention that $\psi_0 \equiv 0$.

We estimate the inner most sum. Since $n(\underline{\alpha}) = k$,

$$|f(\underline{\alpha}\,\underline{\beta}\,\underline{\gamma}x)| \leq \inf_{[\underline{\alpha},a]} |f| + \|f\|_{\mathcal{L}}\theta^k h_0[\alpha_0] \leq \inf_{[\underline{\alpha},a]} |f| + C_0\|f\|_{\mathcal{L}}\theta^k h_0(\underline{\alpha}\,\underline{\beta}\,\underline{\gamma}x).$$

Since $\mathrm{var}_i\,\phi = \mathrm{var}_i\,\psi$ for all $i \geq 1$,

$$e^{\psi_\alpha(\underline{\alpha}\,\underline{\beta}\,\underline{\gamma}\,x)} \leq C_0 \inf_{[\underline{\alpha},a]} e^{\psi_\alpha}.$$

We can thus estimate the inner sum by

$$\begin{aligned}
&\sum_{\underline{\alpha}\in A_k, |\underline{\alpha}|=\alpha} C_0 \inf_{[\underline{\alpha},a]} e^{\psi_\alpha-\alpha\varepsilon_a} \inf_{[\underline{\alpha},a]} |f| + C_0\|f\|_{\mathcal{L}}\theta^k e^{-\alpha\varepsilon_a} \sum_{\underline{\alpha}\in A_k, |\underline{\alpha}|=\alpha} e^{\psi_\alpha(\underline{\alpha}\,\underline{\beta}\,\underline{\gamma}x)} h_0(\underline{\alpha}\,\underline{\beta}\,\underline{\gamma}x) \\
&\leq C_0 \sum_{\underline{\alpha}\in A_k, |\underline{\alpha}|=\alpha} \inf_{[\underline{\alpha},a]} \big(e^{\psi_\alpha-\alpha\varepsilon_a}|f|\big) + C_0\|f\|_{\mathcal{L}}\theta^k e^{-\alpha\varepsilon_a} h_0[a] \quad (\because (\underline{\beta}\,\underline{\gamma})_0 = a) \\
&\leq C_0 e^{-\alpha\varepsilon_a} \sum_{\underline{\alpha}\in A_k, |\underline{\alpha}|=\alpha} \frac{1}{\nu_0[a]} \int_{[a]} e^{\psi_\alpha(\underline{\alpha}\,y)}|f(\underline{\alpha}y)| d\nu_0(y) + C_0\|f\|_{\mathcal{L}}\theta^k e^{-\alpha\varepsilon_a} h_0[a] \\
&\leq C_0 e^{-\alpha\varepsilon_a} \sum_{\underline{\alpha}\in A_k, |\underline{\alpha}|=\alpha} \frac{1}{\nu_0[a]} \int L_\psi^\alpha(1_{[\underline{\alpha},a]}|f|) d\nu_0 + C_0\|f\|_{\mathcal{L}}\theta^k e^{-\alpha\varepsilon_a} h_0[a]
\end{aligned}$$

$$= \frac{C_0 e^{-\alpha\varepsilon_a}}{\nu_0[a]} \int \sum_{\underline{\alpha}\in A_k, |\underline{\alpha}|=\alpha} 1_{[\underline{\alpha},a]}|f|d\nu_0 + C_0\|f\|_{\mathcal{L}}\theta^k e^{-\alpha\varepsilon_a} h_0[a] \quad (\because L_\psi^*\nu_0 = \nu_0)$$

$$\leq \frac{C_0 e^{-\alpha\varepsilon_a}}{\nu_0[a]}\|f\|_{\mathcal{C}} + C_0 h_0[a]\theta^k e^{-\alpha\varepsilon_a}\|f\|_{\mathcal{L}}$$

$$\leq C_1 e^{-\alpha\varepsilon_a}\left[\|f\|_{\mathcal{C}} + \theta^k\|f\|_{\mathcal{L}}\right], \text{ where } C_1 := C_0\left(\frac{1}{\nu_0[a]} + h_0[a]\right).$$

Substituting this estimate in (3.3), we see that

$$|(L_\phi^n f)(x)| \leq C_0 e^{kp-n\varepsilon_a}\|f\|_{\mathcal{L}} h_0[b] + \sum_{\alpha+\beta+\gamma=n}\sum_{\underline{\gamma}\in C, |\underline{\gamma}|=\gamma} e^{\psi_\gamma(\underline{\gamma}x)-\gamma\varepsilon_a}\times$$

$$\left[\sum_{\underline{\beta}\in B, |\underline{\beta}|=\beta} e^{\psi_\beta(\underline{\beta}\,\underline{\gamma}x)-\beta\varepsilon_a+p[n(\underline{\beta}\,\underline{\gamma})+k]}\big(C_1 e^{-\alpha\varepsilon_a}\left[\|f\|_{\mathcal{C}}+\theta^k\|f\|_{\mathcal{L}}\right]\big)\right]. \quad (3.4)$$

By construction $n(\underline{\beta}\,\underline{\gamma}) = n(\underline{\beta})+1$ and $\psi_\beta(\underline{\beta}\,\underline{\gamma}\,x) = \phi_\beta(\underline{\beta}\,\underline{\gamma}\,x) + \beta\varepsilon_a - pn(\underline{\beta})$, so the sum in the square brackets is

$$\sum_{\underline{\beta}\in B, |\underline{\beta}|=\beta} e^{\phi_\beta(\underline{\beta}\,\underline{\gamma}x)+p(k+1)}\big(C_1 e^{-\alpha\varepsilon_a}\left[\|f\|_{\mathcal{C}}+\theta^k\|f\|_{\mathcal{L}}\right]\big)$$

$$= \big(C_1 e^{-\alpha\varepsilon_a}\left[\|f\|_{\mathcal{C}}+\theta^k\|f\|_{\mathcal{L}}\right]\big)\cdot e^{p(k+1)}\sum_{\underline{\beta}\in B, |\underline{\beta}|=\beta} e^{\phi_\beta(\underline{\beta}\,\underline{\gamma}x)}$$

$$\leq C_1 e^{p(k+1)-\alpha\varepsilon_a}\left[\|f\|_{\mathcal{C}}+\theta^k\|f\|_{\mathcal{L}}\right]\cdot C_0 Z_\beta(\phi,a), \text{ where } Z_\beta(\phi,a) := \sum_{T^\beta z=z} e^{\phi_\beta(z)}1_{[a]}(z),$$

because $\underline{\beta}, \underline{\gamma}$ start with a. We claim that $\sup_\beta Z_\beta(\phi,a) \leq 2C_0$: Had there been a β with $Z_\beta(\phi,a) > 2C_0$, then we would have had $Z_{n\beta}(\phi,a) \geq [\frac{1}{C_0}Z_\beta(\phi,a)]^n \geq 2^n$, in contradiction to the assumption that $\frac{1}{n}\log Z_n(\phi,a) \xrightarrow[n\to\infty]{} P_G(\phi) = 0$. Setting $C_2 := 2C_0$, we obtain that the sum in the square brackets in (3.4) is bounded by

$$C_0 C_1 C_2 e^{(k+1)p-\alpha\varepsilon_a}\left[\|f\|_{\mathcal{C}}+\theta^k\|f\|_{\mathcal{L}}\right].$$

Substituting this in (3.4), gives

$$|(L_\phi^n f)(x)| \leq \sum_{\alpha+\beta+\gamma=n}\sum_{\underline{\gamma}\in C, |\underline{\gamma}|=\gamma} e^{\psi_\gamma(\underline{\gamma}x)-\gamma\varepsilon_a} C_0 C_1 C_2 e^{(k+1)p-\alpha\varepsilon_a}\left[\|f\|_{\mathcal{C}}+\theta^k\|f\|_{\mathcal{L}}\right]$$
$$+ C_0 e^{kp-n\varepsilon_a}\|f\|_{\mathcal{L}} h_0[b]$$
$$\leq C_0 C_1 C_2 e^{(k+1)p}\left[\|f\|_{\mathcal{C}}+\theta^k\|f\|_{\mathcal{L}}\right]\sum_{\alpha+\beta+\gamma=n}\frac{C_0 e^{-(\alpha+\gamma)\varepsilon_a}}{h_0[a]}\sum_{\underline{\gamma}\in C, |\underline{\gamma}|=\gamma} e^{\psi_\gamma(\underline{\gamma}x)}h_0(\underline{\gamma}x)$$
$$+ C_0 e^{kp-n\varepsilon_a}\|f\|_{\mathcal{L}} h_0[b]$$
$$\leq C_0^2 C_1 C_2 e^{(k+1)p}\left[\|f\|_{\mathcal{C}}+\theta^k\|f\|_{\mathcal{L}}\right]\sum_{\alpha+\beta+\gamma=n}\frac{e^{-(\alpha+\gamma)\varepsilon_a}}{h_0[a]}h_0[b] \quad (\because L_\psi h_0 = h_0)$$
$$+ C_0 e^{kp-n\varepsilon_a}\|f\|_{\mathcal{L}} h_0[b].$$

It is easy to check that $\sup_{n\in\mathbb{N}}\sum_{\alpha+\beta+\gamma=n} e^{-(\alpha+\gamma)\varepsilon_a} \leq \left(\sum_{\ell\geq 0} e^{-\ell\varepsilon_a}\right)^2$. Let $C_3 = 1+\left(\sum_{\ell\geq 0} e^{-\ell\varepsilon_a}\right)^2$, then for all $x\in[b]$

$$|(L_\phi^n f)(x)| \leq e^{(k+1)p}\frac{C_0^2 C_1 C_2 C_3}{h_0[a]}\left[\|f\|_{\mathcal{C}} + (\theta^k + e^{-n\varepsilon_a})\|f\|_{\mathcal{L}}\right] h_0[b]. \tag{3.5}$$

Part 2. Analysis of the Lipschitz constant of $L_\phi^n f$ on $[b]$.

Suppose $x,y\in[b]$.

$$\begin{aligned}
&|(L_\phi^n f)(x) - (L_\phi^n f)(y)| \leq \\
&\quad\leq \sum_{\underline{p}\in P^n(b)} e^{\phi_n(\underline{p}x)}\left|1-e^{\phi_n(\underline{p}y)-\phi_n(\underline{p}x)}\right||f(\underline{p}x)| + \sum_{\underline{p}\in P^n(b)} e^{\phi_n(\underline{p}y)}|f(\underline{p}x)-f(\underline{p}y)| \\
&\quad\leq \sum_{\underline{p}\in P^n(b)} e^{\phi_n(\underline{p}x)} C_4\theta^{t(x,y)}|f(\underline{p}x)| + \sum_{\underline{p}\in P^n(b)} e^{\phi_n(\underline{p}y)} C_0 h_0(\underline{p}y)\|f\|_{\mathcal{L}}\theta^{n(\underline{p})+s_a(x,y)},
\end{aligned}$$

where $C_4 := \max\left\{1, \frac{A_\phi}{1-\theta}\sup_{|\delta|\leq\frac{A_\phi}{1-\theta}}\left|\frac{1-e^\delta}{\delta}\right|\right\}$, and $A_\phi := \sup\frac{|\phi(x)-\phi(y)|}{\theta^{t(x,y)}}$. Thus

$$\begin{aligned}
|(L_\phi^n f)(x)-(L_\phi^n f)(y)| &\leq \theta^{s_a(x,y)}\left[C_4\sup_{[b]} L_\phi^n|f| + C_0\|f\|_{\mathcal{L}}\sum_{\underline{p}\in P^n(b)} e^{\phi_n(\underline{p}y)}h_0(\underline{p}y)\theta^{n(\underline{p})}\right] \\
&=: \theta^{s_a(x,y)}[\mathrm{I}+\mathrm{II}]
\end{aligned} \tag{3.6}$$

where

$$\mathrm{I} := C_4\sup_{[b]} L_\phi^n|f| \leq e^{(k+1)p}\frac{C_0^2C_1C_2C_3C_4}{h_0[a]}\left[\|f\|_{\mathcal{C}} + (\theta^k+e^{-n\varepsilon_a})\|f\|_{\mathcal{L}}\right]h_0[b]$$

by (3.5), and

$$\begin{aligned}
\mathrm{II} &:= C_0\|f\|_{\mathcal{L}}\sum_{\underline{p}\in P^n(b)} e^{\phi_n(\underline{p}y)}\theta^{n(\underline{p})}h_0(\underline{p}y) \\
&= C_0\|f\|_{\mathcal{L}}\sum_{\underline{p}\in P^n(b)} e^{\psi_n(\underline{p}y)-n\varepsilon_a+pn(\underline{p})}\theta^{n(\underline{p})}h_0(\underline{p}y) \\
&= C_0\|f\|_{\mathcal{L}}e^{-n\varepsilon_a}\sum_{\underline{p}\in P^n(b)} e^{\psi_n(\underline{p}y)}(e^p\theta)^{n(\underline{p})}h_0(\underline{p}y) \\
&\leq C_0\|f\|_{\mathcal{L}}e^{-n\varepsilon_a}\sum_{\underline{p}\in P^n(b)} e^{\psi_n(\underline{p}y)}h_0(\underline{p}y), \quad \text{because } e^p\theta<1 \text{ by (3.1)} \\
&\leq C_0^2\|f\|_{\mathcal{L}}e^{-n\varepsilon_a}h_0[b], \quad \text{because } L_\psi h_0 = h_0 \text{ and } y\in[b] \\
&\leq e^{(k+1)p}\frac{C_0^2C_1C_2C_3C_4}{h_0[a]}e^{-n\varepsilon_a}\|f\|_{\mathcal{L}}h_0[b], \quad \text{because } p>0 \text{ and } C_1C_2C_3C_4 > h_0[a].
\end{aligned}$$

Substituting the estimates for I and II in (3.6), we see that for all $x,y\in[b]$

$$\frac{|(L_\phi^n f)(x)-(L_\phi^n f)(y)|}{\theta^{s_a(x,y)}} \leq e^{(k+1)p}\frac{C_0^2C_1C_2C_3C_4}{h_0[a]}\left[\|f\|_{\mathcal{C}} + (\theta^k+2e^{-n\varepsilon_a})\|f\|_{\mathcal{L}}\right]h_0[b]. \tag{3.7}$$

Part 3. Putting everything together to obtain (3.2).

(3.5) and (3.7), together with the fact that $C_4 \geq 1$ give

$$\begin{aligned}\|L_\phi^n f\|_{\mathcal{L}} &\leq 3e^{(k+1)p}\frac{C_0^2 C_1 C_2 C_3 C_4}{h_0[a]}\left[\|f\|_{\mathcal{C}} + (\theta^k + e^{-n\varepsilon_a})\|f\|_{\mathcal{L}}\right] \\ &\leq 3e^p \frac{C_0^2 C_1 C_2 C_3 C_4}{h_0[a]}\left[e^{kp}\|f\|_{\mathcal{C}} + ((e^p\theta)^k + e^{kp-n\varepsilon_a})\|f\|_{\mathcal{L}}\right] \qquad (3.8)\end{aligned}$$

At this stage, it is probably useful to recall the definition of the constants C_i:

$$C_0{:=}\exp\sum_{\ell=2}^{\infty}\operatorname{var}_\ell \phi, \qquad C_1{:=}C_0\left(h_0[a] + 1/\nu_0[a]\right), \qquad C_2{:=}2C_0,$$

$$C_3{:=}1 + \left(\sum_{\ell=0}^{\infty} e^{-\ell\varepsilon_a}\right)^2, \quad C_4{:=}\max\left\{1, \tfrac{A_\phi}{1-\theta}\sup_{|\delta|\leq\frac{A_\phi}{1-\theta}}\left|\tfrac{1-e^\delta}{\delta}\right|\right\}.$$

These constants do not depend on k or n. Using (3.1), it is no problem to choose first k and then n_0 so that

$$\|L_\phi^n f\|_{\mathcal{L}} \leq R\|f\|_{\mathcal{C}} + \frac{1}{2}\|f\|_{\mathcal{L}} \text{ for all } n \geq n_0, \text{ where } R := \frac{3C_0 e^{(k+1)p}}{h_0[a]}\prod_{i=0}^{4} C_i. \qquad (3.9)$$

In the particular case $n = n_0$, we get (3.2) with $r := 2^{-1/n_0}$. In the next step we shall see that $r < \rho_{\mathcal{L}}(L_\phi)$.

Step 3. L_ϕ is a bounded operator on $\mathcal{L}$ and $\rho(L_\phi) = 1$, thus (B) holds.

Proof. $\|\cdot\|_{\mathcal{C}} \leq C_0\|\cdot\|_{\mathcal{L}}$ on $\mathcal{L}$, because for every $f \in \mathcal{L}$, $|f| \leq C_0\|f\|_{\mathcal{L}} h_0$, and $\int h_0 d\nu_0 = 1$. Thus (3.8) implies that $\|L_\phi\| < \infty$, and (3.9) says that $\sup\|L_\phi^n\| < \infty$. It follows that L_ϕ is bounded, and that its spectral radius is not larger than one.

We claim that the spectral radius is equal to one. Otherwise, there is some $\kappa < 1$ such that $\|L_\phi^n\| = O(\kappa^n)$, and then $|L_\phi^n 1_{[a]}| = O(\kappa^n)$ uniformly on $[a]$. Now $L_\phi^n 1_{[a]} \asymp Z_n(\phi, a)$ uniformly on $[a]$ (appendix A, remark 7.1), so this means that $0 = P_G(\phi) = \lim_{n\to\infty}\frac{1}{n}\log Z_n(\phi, a) \leq \log\kappa < 0$, a contradiction.

Step 4. Every sequence $\{f_n\}_{n\geq 1}$ in $\mathcal{C}$ such that $\sup\|f_n\|_{\mathcal{L}} < \infty$ has a subsequence which converges w.r.t. $\|\cdot\|_{\mathcal{C}}$ to some element of $\mathcal{L}$. Since $\|L_\phi\| < \infty$, (C) holds.

Proof. Let X_0 denote the subset of X consisting of all sequences which contain the symbol a infinitely many times. This is a subset of ν_0–full measure, because ν_0 is an ergodic conservative measure which charges every partition set.

The function $\delta(x, y) := \theta^{s_a(x,y)}$ is a metric on X_0, and (X_0, δ) is a complete separable metric space. The family $\{f_n\}_{n\geq 1}$ is uniformly Lipschitz on partition sets with respect to this metric. By the Arzela–Ascoli theorem, there is a subsequence $\{f_{n_k}\}_{k\geq 1}$ which converges pointwise on X_0 to some function $g_0 : X_0 \to \mathbb{C}$. Since $|f_{n_k}(x)| \leq C_0(\sup\|f_n\|_{\mathcal{L}})h_0(x)$, and $\int h_0 d\nu_0 < \infty$, $\int_{X_0}|f_{n_k} - g_0|d\nu_0 \to 0$.

We show that $\exists g \in \mathcal{L}$ such that $g|_{X_0} = g_0$. Choose points $y^b \in [b] \cap X_0$, $(b \in \mathcal{S})$, and define a map $\vartheta : X \to X_0$ by

$$\vartheta(x) := \begin{cases} y^{x_0} & \nexists i \text{ s.t. } x_i = a, \\ (x_0, \ldots, x_k, y_1^a, y_2^a, \ldots) & \exists i \text{ s.t. } x_i = a, k := \max\{i : x_i = a\} < \infty, \\ x & \exists \text{ infinitely many } i \text{ s.t. } x_i = a. \end{cases}$$

We claim that for all $x, y \in X$, $s_a(\vartheta(x), y) \geq s_a(x, y)$. If $s_a(x, y) = 0$ or $\vartheta_a(x) = x$ then there is nothing to prove. Otherwise, x has finitely many coordinates equal to

a. Let $k := \max\{i : x_i = a, x_0^i = y_0^i\}$ and $k' := \max\{i : x_i = a, \vartheta(x)_0^i = y_0^i\}$, then $s_a(x,y) = \#\{0 \le j \le k : y_j = a\}$ and $s_a(\vartheta(x), y) = \#\{0 \le j \le k' : y_j = a\}$. By construction, $\vartheta(x)_0^k = x_0^k = y_0^k$, therefore $k' \ge k$ and $s_a(\vartheta(x), y) \ge s_a(x,y)$.

Now set $g := g_0 \circ \vartheta$. Since $\vartheta|_{X_0} = id$, $g|_{X_0} = g_0$. If $x \in [b]$, then $\vartheta(x) \in [b]$, so $|g(x)| = |g_0(\vartheta(x))| \le \sup |f_n(\vartheta(x))| \le h_0[b] \sup_{n\ge1} \|f_n\|_{\mathcal{L}}$. If $x, y \in [b]$, then

$$|g(x) - g(y)| \le |g_0(\vartheta(x)) - g_0(\vartheta(y))| \le \sup_n |f_n(\vartheta(x)) - f_n(\vartheta(y))|$$
$$\le \sup_n \|f_n\|_{\mathcal{L}} \cdot h_0[b]\theta^{s_a(\vartheta(x),\vartheta(y))} \le \sup_n \|f_n\|_{\mathcal{L}} \cdot h_0[b]\theta^{s_a(x,y)}.$$

We conclude that $g \in \mathcal{L}$, and that $\int_X |f_{n_k} - g| d\nu_0 = \int_{X_0} |f_{n_k} - g_0| d\nu_0 \to 0$.

Step 5. $L_\phi : \mathcal{L} \to \mathcal{L}$ satisfies parts (a)–(e) of the spectral gap property.

Proof. It is clear that every element of $\mathcal{L}$ is continuous, and that $\mathcal{L}$ contains all indicators of cylinder sets. Parts (a) and (d) of the spectral gap property were shown in step 3. Parts (b) and (c) are obvious from the definition of $\|\cdot\|_{\mathcal{L}}$.

We prove part (e). The previous steps show that the conditions of Hennion's theorem are satisfied and that $\rho(L_\phi) = 1$. It follows that $\mathcal{L} = \mathcal{F} \oplus \mathcal{N}$ where $L_\phi(\mathcal{F}) \subseteq \mathcal{F}$, $L_\phi(\mathcal{N}) \subseteq \mathcal{N}$, $\mathcal{F}$ is a finite dimensional space, the eigenvalues of $L_\phi|_{\mathcal{F}}$ are all of modulus one, and the spectral radius of $L_\phi|_{\mathcal{N}}$ is strictly less than one.

We show that $\mathcal{F} = \text{span}\{h\}$ for some function h s.t. $L_\phi h = h$. Once this is done, we let $P : \mathcal{L} \to \mathcal{F}$ denote the eigenprojection of the eigenvalue 1, and $N := L_\phi(I - P)$. It is clear that $L_\phi = P + N$, $P^2 = P$, $PN = NP = 0$, and $\dim \operatorname{Im} P = \dim \mathcal{F} = 1$. To see that $\rho(N) < 1$, we use the facts $\rho(L_\phi|_{\mathcal{N}}) < 1$ and $L_\phi|_{\mathcal{F}} = id$ to see that $L_\phi^n = P + N^n \to P$, whence

$$\mathcal{N} = \{f \in \mathcal{L} : L_\phi^n f \xrightarrow[n\to\infty]{} 0\}.$$

It follows that $\mathcal{N} = \ker P$. Thus $N = L_\phi(I - P)$ is equal to zero on $\mathcal{F}$ and equal to L_ϕ on $\mathcal{N}$. Since $\mathcal{F}$ and $\mathcal{N}$ are L_ϕ–invariant and $\rho(L_\phi|_{\mathcal{N}}) < 1$, $\rho(N) < 1$ and (e) is proved.

Step 5.1: 1 is an eigenvalue of $L_\phi|_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}$.

We construct an eigenfunction. Recall that ϕ is (strongly) positive recurrent with pressure zero. By the generalized RPF theorem (appendix A, theorem 7.2) there is a positive continuous function h and a Borel measure ν such that $L_\phi h = h$, $L_\phi^* \nu = \nu$, $\int h d\nu = 1$. The measure $d\mu = h d\nu$ is known to be an exact invariant probability measure, and for every cylinder $[\underline{a}]$, $L_\phi^n 1_{[\underline{a}]} \xrightarrow[n\to\infty]{} h\nu[\underline{a}]$ pointwise [S2].

We claim that $h \in \mathcal{L}$. By (3.8), $\sup_{n\ge1} \|L_\phi^n 1_{[\underline{a}]}\|_{\mathcal{L}} < \infty$. By step 4, $\exists n_k \to \infty$ such that $L_\phi^{n_k} 1_{[\underline{a}]} \xrightarrow[n\to\infty]{L^1(\nu_0)} g \in \mathcal{L}$. The limit must agree with the pointwise limit of $L_\phi^{n_k} 1_{[\underline{a}]}$, whence with h. Thus $h = (1/\nu[\underline{a}])g$ ν_0–almost everywhere, whence by continuity — everywhere. Thus $h \in \mathcal{L}$.

We claim that $h \in \mathcal{F}$. Write $h = h_1 + h_2$ where $h_1 \in \mathcal{F}$ and $h_2 \in \mathcal{N}$. Since $L_\phi h = h$, $h = L_\phi^n h_1 + L_\phi^n h_2$. The first summand stays inside $\mathcal{F}$, and the second summand tends to zero in norm, because $\rho(L_\phi|_{\mathcal{N}}) < 1$. It follows that $h \in \overline{\mathcal{F}}$. But $\dim \mathcal{F} < \infty$ so $\overline{\mathcal{F}} = \mathcal{F}$. Thus $h \in \mathcal{F}$.

Since $h \in \mathcal{F}$ and $L_\phi h = h$, 1 is an eigenvalue of $L_\phi : \mathcal{F} \to \mathcal{F}$.

Step 5.2: 1 is the only eigenvalue of $L_\phi|_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}$. This eigenvalue is simple.

By the definition of $\mathcal{F}$, all the eigenvalues of $L_\phi|_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}$ have modulus one. Suppose $f \in \mathcal{F} \setminus \{0\}$ and $L_\phi f = e^{i\theta} f$, we show that $e^{i\theta} = 1$ and $f = \text{const}\, h$.

We claim that $f \in L^1(\nu)$. Since L_ϕ is positive, $L_\phi|f| \geq |L_\phi f| = |f|$, whence

$$\sum_{n=1}^{N} L_\phi^n[L_\phi|f| - |f|] \leq L_\phi^{N+1}|f| \leq C_0\big(\sup_n \|L_\phi^n\|\big)\|f\|_{\mathcal{L}} h_0 \text{ for all } N.$$

But ν is conservative and ergodic and $L_\phi^*\nu = \nu$, so for every $F \geq 0$ such that $\int F d\nu \neq 0$, $\sum L_\phi^n F = \infty$ ([A], proposition 1.3.2). Thus $L_\phi|f| = |f|$ ν–almost everywhere. It follows that $|f|$ is an absolutely continuous invariant density of ν. An ergodic conservative measure can have at most one invariant density, so $|f| = \text{const}\, h$ ν–a.e., whence $f \in L^1(\nu)$.

We claim that f is proportional to h and $e^{i\theta} = 1$. Let $d\mu := hd\nu$. Since $f \in L^1(\nu)$, $f/h \in L^1(\mu)$. The transfer operator of μ is $\widehat{T}\psi = \frac{1}{h}L_\phi(h\psi)$.[1] Since μ is exact, $\|\widehat{T}^n(f/h) - \int(f/h)d\mu\|_{L^1(\mu)} \to 0$ ([A], theorem 1.3.3). But $\widehat{T}^n(f/h) = e^{in\theta}(f/h)$, so this can only happen if $e^{i\theta} = 1$ and $f/h = \text{const}$ almost everywhere. Since f, h are continuous and ν has global support, $f/h = \text{const}$.

Step 5.3: $\dim \mathcal{F} = 1$.

Since the spectrum of $L_\phi|_{\mathcal{F}}$ consists of a single simple eigenvalue equal to one, and since (by construction) $\dim \mathcal{F} < \infty$, $\mathcal{F}$ has a basis with respect to which $L_\phi : \mathcal{F} \to \mathcal{F}$ is represented by $\dim(\mathcal{F}) \times \dim(\mathcal{F})$ Jordan block with ones on the diagonal. The iterates of such a matrix diverge when $\dim(\mathcal{F}) > 1$ (the $(1,2)$–entry escapes to infinity). This cannot be the case, because $\sup \|L_\phi^n\| < \infty$ by (3.8). The conclusion is that $\dim(\mathcal{F}) = 1$.

We conclude that $\mathcal{F} = \text{span}\{h\}$ where $L_\phi h = h$. By the discussion above, part (e) of SGP is proved.

Step 6. Proof of part (f) of SGP.

Part (f) of SGP says that if $f \in \mathcal{F}$ is θ–Hölder, then $z \mapsto L_{\phi+zf}$ is analytic at zero. Write for every θ–Hölder continuous function g,

$$\|g\|_\theta := \sup|g| + \sup\{|g(x) - g(y)|/\theta^{t(x,y)} : x, y \in X\}.$$

It is easy to verify that $\|gf\|_{\mathcal{L}} \leq \|g\|_\theta \|f\|_{\mathcal{L}}$ for all $f \in \mathcal{L}$.

It follows that the operator $M_n : f \mapsto L_\phi(g^n f)$ is bounded, and that $\|M_n\| \leq \|L_\phi\| \|g\|_\theta^n$. Thus the series $\sum_{n=0}^\infty \frac{z^n}{n!} M_n$ converges absolutely in the operator norm for all $|z| < 1/\|g\|_\theta$. As a result, $L_{\phi+zg} \equiv \sum_{n=0}^\infty \frac{z^n}{n!} M_n$ is analytic on $\{z \in \mathbb{C} : |z| < 1/\|g\|_\theta\}$. This shows (f), and completes the proof of SGP. □

3.2. **Spectral Gap implies Strong Positive Recurrence.** Suppose ϕ has the spectral gap property, and write $L_\phi = \lambda P + N$ with $\lambda = \exp P_G(\phi)$ and P, N as above.

Since $PN = NP = 0$ and $P^2 = P$, $L_\phi^n = \lambda^n P + N^n$. Since the spectral radius of N is less than λ, $\|\lambda^{-n}N^n\| = O(\kappa^n)$ where $0 < \kappa < 1$. Thus for (any) fixed $x \in [a]$, $\lambda^{-n} Z_n(\phi, a) \asymp \lambda^{-n}(L_\phi^n 1_{[a]})(x) = P1_{[a]}(x) + O(\kappa^n)$ (appendix A, remark (7.1)).

[1] The transfer operator of a measure μ s.t. $\mu \circ T^{-1} \ll \mu$ is the operator $\widehat{T} : L^1(\mu) \to L^1(\mu)$ whose value on a function $f \in L^1(\mu)$ is determined by the condition $\int \varphi \widehat{T} f d\mu = \int \varphi \circ T \cdot f d\mu$ for all test functions $\varphi \in L^\infty(\mu)$.

It is impossible for $P1_{[a]}(x)$ to vanish, because this would imply that $Z_n(\phi, a) = O((\kappa\lambda)^n)$, whereas $\frac{1}{n}\log Z_n(\phi, a) \to \log\lambda$ and $\kappa < 1$. Thus $P1_{[a]}(x) \neq 0$.

According to the theory of analytic perturbations of linear operators [K], there exists $\varepsilon > 0$ s.t. every $L : \mathcal{L} \to \mathcal{L}$ which satisfies $\|L - L_\phi\| < \varepsilon$ can be written in the form

$$L = \lambda(L)P(L) + N(L)$$

where $P(L), N(L)$ are bounded linear operators s.t. $P(L)^2 = P(L)$, $\dim \operatorname{Im} P(L) = 1$, $N(L)P(L) = P(L)N(L) = 0$, and such that the spectral radius of $N(L)$ is smaller than $|\lambda(L)|$. Moreover, if $\varepsilon > 0$ is sufficiently small, then $L \mapsto \lambda(L), P(L), N(L)$ are analytic on $\{L : \|L - L_\phi\| < \varepsilon\}$.

Since $g := 1_{[a]}$ is Hölder continuous, $t \mapsto L_{\phi+tg}$ is real analytic, whence continuous, at zero. So $\exists\delta > 0$ such that if $|t| < \delta$, then $\|L_{\phi+tg} - L_\phi\| < \varepsilon$ and

$$L_{\phi+tg} = \lambda_t P_t + N_t,$$

where $\lambda_t := \lambda(L_{\phi+tg})$, $P_t := P(L_{\phi+tg})$, $N_t := N(L_{\phi+tg})$.

Since $\mathcal{L}$–convergence implies pointwise convergence, $P_t 1_{[a]}(x) \xrightarrow[t\to 0]{} P1_{[a]}(x)$. We saw above that for any $x \in [a]$, $P1_{[a]}(x) \neq 0$. Choosing our δ sufficiently small, we can ensure that $(P_t 1_{[a]})(x) \neq 0$ for all $|t| < \delta$ for some $x \in [a]$.

We now repeat the argument above for $\phi + tg$ and see that for all t real such that $|t| < \delta$, $|\lambda_t|^{-n} Z_n(\phi + tg, a) \asymp |\lambda_t|^{-n}(L^n_{\phi+tg}1_{[a]})(x) = |(P_t 1_{[a]})(x) + o(1)|$, whence $|\lambda_t|^{-n} Z_n(\phi + tg, a) \asymp 1$.

This implies that for all $|t| < \delta$, $|\lambda_t| = \exp P_G(\phi + tg)$ and $\phi + tg$ is recurrent. By the discriminant theorem, $\Delta_a[\phi + tg] \geq 0$ for all $|t| < \delta$.

But $\Delta_a[\phi + tg] = \Delta_a[\phi + t1_{[a]}] = \Delta_a[\phi] + t$ (appendix A, lemma 7.1). If this is non-negative for all $|t| < \delta$, then it must be the case that $\Delta_a[\phi] > 0$. □

4. Strong Positive Recurrence is Open and Dense

The material in this section relies on the theory of modes of recurrence, which we summarized for the convenience of the reader in appendix A.

Main Lemma. As we shall see below, it is fairly easy to approximate a recurrent potential by a strongly positive recurrent potential. Here we show that every potential can be approximated by a recurrent potential.

Lemma 4.1. *If $\phi \in \Phi$ is a transient potential, $a \in \mathcal{S}$, and ψ is a non-positive bounded weakly Hölder function s.t.* $\operatorname{supp}\psi \subset [a]$, *then $\phi + \psi \in \Phi$, $\phi + \psi$ is transient, and $P_G(\phi + \psi) = P_G(\phi)$.*

Proof. Since $\phi + \psi \leq \phi$ we have $P_G(\phi + \psi) \leq P_G(\phi)$. To see the other inequality, we note that since ϕ is transient,

$$\begin{aligned} P_G(\phi) &= \limsup_{n\to\infty} \frac{1}{n}\log Z_n^*(\phi, a) && \text{(appendix A, (7.6))} \\ &= \limsup_{n\to\infty} \frac{1}{n}\log Z_n^*(\phi + \psi, a) && (\because \operatorname{supp}\phi \subset [a] \text{ and } \sup|\psi| < \infty) \\ &\leq P_G(\phi + \psi). && (\because (7.5)) \end{aligned}$$

This shows that $P_G(\phi) = P_G(\phi + \psi)$.

Using the transience of ϕ and the non-positivity of ψ, we see that

$$\sum_{n=0}^{\infty} e^{-nP_G(\phi+\psi)} Z_n(\phi+\psi,a) = \sum_{n=0}^{\infty} e^{-nP_G(\phi)} Z_n(\phi+\psi,a) \leq \sum_{n=0}^{\infty} e^{-nP_G(\phi)} Z_n(\phi,a) < \infty,$$

so $\phi+\psi$ is transient. □

Lemma 4.2 (Main Lemma). *Suppose $\phi \in \Phi$ is transient, then for any $\epsilon > 0$ there exists a recurrent $\varphi \in \Phi$ so that $\|\varphi - \phi\|_\infty \leq \epsilon$ and* $\mathrm{var}_1[\varphi - \phi] = 0$.

Proof. Recall that $\mathcal{S}$ denotes the set of states. We write $a \xrightarrow{k} b$ for $a, b \in \mathcal{S}$ if there is an admissible word with $k+1$ symbols which starts with a and ends with b.

Fix $\varepsilon > 0$ and $b \in \mathcal{S}$. We construct finite sets of states $\{c_1^k, \ldots, c_{r_k}^k\}$ $(k \geq 0)$ by induction as follows. When $k = 0$, let $r_0 := 1$, and $c_1^0 := b$. Now suppose we have carried the construction for each $\ell < k$. Let $b_1^k, b_2^k, b_3^k, \ldots$ be the list of all different states c for which $b \xrightarrow{\ell} c$ for $\ell \leq k$. If this collection is finite, let r_k be its size, and set $\{c_1^k, \ldots, c_{r_k}^k\} := \{b_1^k, \ldots, b_{r_k}^k\}$. If it is infinite, observe that

$$Z_n^*\Big(\phi + \epsilon \sum_{i=1}^{\infty} 1_{[b_i^k]}, b\Big) \geq e^{n\epsilon} Z_n^*(\phi, b) \quad (1 \leq n \leq k).$$

since for any x with $T^n x = x$ and $x_0 = b$ we have added an extra factor of ϵ to the potential at states $x_0, x_1, \ldots, x_{n-1}$. Therefore we can find $s_k \in \mathbb{N}$ such that

$$Z_n^*\Big(\phi + \epsilon \sum_{i=1}^{s_k} 1_{[b_i^k]}, b\Big) \geq e^{n \cdot \frac{\epsilon}{2}} Z_n^*(\phi, b) \quad (1 \leq n \leq k). \tag{4.1}$$

We let $\{c_1^k, \ldots, c_{r_k}^k\}$ be the set $\{c_1^{k-1}, \ldots, c_{r_{k-1}}^{k-1}\} \cup \{b_1^k, \ldots, b_{s_k}^k\}$ where, in this case, r_k is the number of different states c_i^k so defined.

Set $\phi[0] := \phi$, and define for $k \geq 1$

$$\phi[k] = \phi + \epsilon \sum_{i=1}^{r_k} 1_{[c_i^k]}.$$

We interpolate these potentials. Observe that for all $k \geq 1$,

$$\phi[k] = \phi[k-1] + \epsilon \sum_{i=1}^{m_k} 1_{[d_i^k]}, \text{ where } \{d_1^k \ldots, d_{m_k}^k\} := \{c_1^k \ldots, c_{r_k}^k\} \setminus \{c_1^{k-1}, \ldots, c_{r_{k-1}}^{k-1}\},$$

with m_k defined by the above identity. Define for $k \geq 1$ and $0 \leq i \leq m_k$

$$\phi[k,i] := \phi[k-1] + \epsilon \sum_{j=1}^{i} 1_{[d_j^k]}.$$

Then $\phi[k,0] = \phi[k-1]$, and $\phi[k,m_k] = \phi[k]$.

We claim that there must be some k, i such that $\phi[k,i]$ is recurrent. Assume by way of contradiction that this is not the case: $\phi[k,i]$ is transient for all k, i.

In this case, the sequence

$$\phi[k] = \phi[k,m_k] \geq \phi[k,m_k - 1] \geq \cdots \geq \phi[k,1] \geq \phi[k-1,m_{k-1}] \geq \cdots$$

is a decreasing sequence of *transient* potentials where each term is equal to its predecessor minus ε times the indicator of some partition set. By lemma 4.1, all terms in the sequence have the same Gurevich pressure. Since the sequence terminates after finitely many steps at $\phi[0]=\phi$,

$$P_G(\phi[k]) = P_G(\phi) \text{ for all } k. \tag{4.2}$$

Consider now the power series

$$\begin{aligned} t_k(x) &:= 1+\sum_{i=1}^{\infty} Z_i(\phi[k],b)x^i \\ r_k(x) &:= \sum_{i=1}^{\infty} Z_i^*(\phi[k],b)x^i \end{aligned}$$

Both have radius of convergence $\exp[-P_G(\phi)]$: the first by the definition of the Gurevich pressure and (4.2), and the second because of the assumption that $\phi[k]$ is transient (appendix A, (7.6)). They are related by the following inequality for all $0<x<\exp[-P_G(\phi)]$ (appendix A, (7.2)):

$$\frac{1}{B^2}[t_k(x)-1] \le t_k(x)r_k(x) \le B^2[t_k(x)-1], \text{ where } B:=\exp\sum_{n=2}^{\infty}\mathrm{var}_n\,\phi. \tag{4.3}$$

By (4.3), $r_k(x)\le B^2$ for all $0<x<\exp[-P_G(\phi)]$ and $k\ge 1$.

But this cannot be the case, because for $\exp[-P_G(\phi)-\frac{\epsilon}{2}]<x<\exp[-P_G(\phi)]$

$$\begin{aligned} r_k(x) &\ge \sum_{n=1}^{k} Z_n^*(\phi[k],b)x^n \ge \sum_{n=1}^{k} e^{n\cdot\frac{\epsilon}{2}} Z_n^*(\phi,b)x^n \qquad \text{(by (4.1))} \\ &\xrightarrow[k\to\infty]{} \sum_{n=1}^{\infty} Z_n^*(\phi,b)(e^{\epsilon/2}x)^n = \infty. \end{aligned}$$

This contradiction shows that there must be some k_0, i_0 for which $\varphi := \phi[k_0,i_0]$ is recurrent. By construction $\varphi\in\Phi$, $\mathrm{var}_1[\varphi-\phi]=0$, and $\|\varphi-\phi\|_\infty=\epsilon$. □

Proof of Theorem 2.2. The proof has two parts:

(a) If $\phi\in\Phi$, then for every $\varepsilon>0$ there is a strongly positive recurrent potential $\varphi\in\Phi$ s.t. $\|\varphi-\phi\|_\infty<\epsilon$ and $\mathrm{var}_1[\varphi-\phi]=0$.

(b) The set of strongly positive recurrent potentials is open w.r.t the sup norm on Φ.

The first part shows that the set of strongly positive recurrent potentials is dense in the strongest possible ω–topology; the second step shows that it is open in the weakest possible ω–topology.

Part 1. Approximating general potentials by strongly positive recurrent potentials.

Fix $\phi\in\Phi$ and $\varepsilon>0$. By Lemma 4.2, there exists a recurrent $\psi\in\Phi$ such that $\|\phi-\psi\|_\infty<\varepsilon/2$ and $\mathrm{var}_1[\phi-\psi]=0$.

We now appeal to the discriminant theorem (appendix A, theorem 7.3): Fix some $a\in\mathcal{S}$, then the recurrence of ψ implies that $\Delta_a[\phi]\ge 0$. If $\varphi:=\psi+\frac{\varepsilon}{2}\cdot 1_{[a]}$, then

$$\Delta_a[\varphi]=\Delta_a[\psi]+\frac{\varepsilon}{2} \quad \text{(appendix A, lemma 7.1)},$$

so φ is strongly positive recurrent. It is obvious that $\|\phi-\varphi\|_\infty<\varepsilon$ and $\mathrm{var}_1[\varphi-\phi]=\mathrm{var}_1[\psi-\phi]=0$.

Part 2. For every strongly positive recurrent $\phi \in \Phi$ there exists a $\delta > 0$ such that if $\varphi \in \Phi$ and $\|\varphi - \phi\|_\infty < \delta$, then φ is strongly positive recurrent.

We fix some $a \in \mathcal{S}$ and work with the induced system on $[a]$, $\overline{X}$, as defined in §2.1. By the definition of the discriminant, if $\phi \in \Phi$ is strongly positive recurrent then there exists $p \in \mathbb{R}$ such that $0 < P_G(\overline{\phi + p}) < \infty$. W.l.o.g. $P_G(\overline{\phi + p'}) < \infty$ for some $p' > p$. The map $x \mapsto P_G(\overline{\phi + x})$ is convex and finite on $(-\infty, p')$, whence continuous on $(-\infty, p]$ (appendix A (7.4)). It is also strictly increasing (because $\overline{\phi + x + h} \geq (\overline{\phi + x}) + h$ for all $h > 0$).

Hence, there exist numbers $p_1 < p_2$ s.t. $0 < P_G(\overline{\phi + p_1}) < P_G(\overline{\phi + p_2}) < \infty$. Take $p_0 := (p_1 + p_2)/2$ and $\delta := (p_2 - p_1)/2$. If $\varphi \in \Phi$ and $\|\varphi - \phi\|_\infty < \delta$, then $\phi + p_1 \leq \varphi + p_0 \leq \phi + p_2$ so

$$0 < P_G(\overline{\phi + p_1}) < P_G(\overline{\varphi + p_0}) < P_G(\overline{\phi + p_2}) < \infty,$$

proving that $\Delta_a[\varphi] > 0$. This shows that the set of strongly positive recurrent potentials is $\|\cdot\|_\infty$-open. □

Proof of Theorem 2.2′. The proof is identical to the proof of theorem 2.2 with the words "weakly Hölder" replaced by "summable variations". □

5. Transience is open and dense in the set of non-strongly positive recurrent potentials

The reader is referred to appendix A for the definition and properties of transient, null recurrent, and weakly positive recurrent potentials.

Proof of theorem 2.3. Lemma 7.1 in appendix A says that for every $a \in \mathcal{S}$ and $t \in \mathbb{R}$, $\Delta_a[\phi + t \cdot 1_{[a]}] = \Delta_a[\phi] + t$.

Suppose $B = \biguplus_{i=1}^r [a_i]$, and $\phi \in \Phi$ is transient. Then $\Delta_{a_1}[\phi] < 0$. Find $\varepsilon_1 > 0$ s.t. $\phi^{(1)} := \phi + \varepsilon_1 \cdot 1_{[a_1]}$ satisfies $\Delta_{a_1}[\phi^{(1)}] < 0$. Then $\phi^{(1)}$ is transient. The transience of $\phi^{(1)}$ means that $\Delta_{a_2}[\phi^{(1)}] < 0$, so we can find $\varepsilon_2 > 0$ s.t. $\phi^{(2)} := \phi^{(1)} + \varepsilon_2 \cdot 1_{[a_2]}$ satisfies $\Delta_{a_2}[\phi^{(2)}] < 0$. So $\phi^{(2)}$ is also transient. Continuing in this manner, we obtain $\varepsilon_1, \ldots, \varepsilon_r > 0$ s.t.

$$\psi := \phi^{(r)} = \phi + \sum_{i=1}^{r} \varepsilon_i \cdot 1_{[a_i]} \text{ is transient.}$$

Take $\delta := \min\{\varepsilon_1, \ldots, \varepsilon_r\}$. We claim that every $\varphi \in \Phi$ such that $\|\varphi - \phi\|_\infty < \delta$ and $\phi|_{X \setminus B} = \varphi|_{X \setminus B}$ is transient. To see this, we observe that φ can be obtained from ψ by subtracting the r non-negative functions $(\psi - \varphi)1_{[a_i]}$. By lemma 4.1 each subtraction preserves transience, so the end result φ is transient.

This proves that the set of transient potentials is $\mathcal{LU}(B)$–open. We claim that it is dense in the complement of the strongly positive recurrent potentials. To see this, it is enough to show that every $\phi \in \Phi$ s.t. $\Delta_{a_1}[\phi] = 0$ can be approximated in $\mathcal{LU}(B)$ by a transient potential. Take $\phi + t \cdot 1_{[a_1]}$ with $t \to 0^-$. □

6. More on Transience

The previous arguments suggest the following new characterization of transience:

Theorem 6.1. *$\phi \in \Phi$ is transient if and only if there exists $\psi \in \Phi$ such that $\psi \geq \phi$, $\psi \not\equiv \phi$, and $P_G(\psi) = P_G(\phi)$.*

Proof. If ϕ is transient, then for any $a \in \mathcal{S}$, $\psi := \phi + t \cdot 1_{[a]}$ is transient for all $t > 0$ sufficiently small (theorem 2.3). By lemma 4.1, $P_G(\psi - s1_{[a]}) = P_G(\psi)$ for all $s > 0$. In the particular case $s = t$ we get $P_G(\psi) = P_G(\phi)$, and ψ is as required.

We will show that if ϕ is recurrent then no such ψ can exist. Suppose by way of contradiction that $\exists \psi \in \Phi$ such that $\psi \not\equiv \phi$, $\psi \geq \phi$, and $P_G(\psi) = P_G(\phi)$. Find some word $[\underline{a}] := [a_1, \ldots, a_n]$ such that $\psi - \phi > \alpha$ on $[\underline{a}]$ for some $\alpha > 0$. Since $\phi \leq \phi + \alpha \cdot 1_{[\underline{a}]} \leq \psi$ and $P_G(\cdot)$ is increasing, $P_G(\phi + \alpha \cdot 1_{[\underline{a}]}) = P_G(\phi)$.

The potential $\varphi := \phi + \alpha \cdot 1_{[\underline{a}]}$ must be recurrent, because

$$\sum_{n=1}^{\infty} Z_n(\varphi, a) e^{-nP_G(\varphi)} = \sum_{n=1}^{\infty} Z_n(\varphi, a) e^{-nP_G(\phi)} \geq \sum_{n=1}^{\infty} Z_n(\phi, a) e^{-nP_G(\phi)} = \infty,$$

by the recurrence of ϕ. Therefore there exists a positive continuous function h such that $L_\varphi h = e^{P_G(\varphi)} h = e^{P_G(\phi)} h$ (appendix A, theorem 7.2). This and $\phi \leq \varphi$ implies that $L_\phi h \leq e^{P_G(\phi)} h$, and it is easy to see that this inequality is strict on $T[\underline{a}]$. Now consider the non-negative function $f := h - e^{-P_G(\phi)} L_\phi h$. This is a non-negative continuous function, not everywhere equal to zero, such that

$$\sum_{k=0}^{\infty} e^{-kP_G(\phi)} L_\phi^k f \leq h < \infty \text{ everywhere.}$$

In particular the series on the left (all of whose summands are non-negative) converges almost surely.

But this is impossible: ϕ is recurrent, so L_ϕ has a conservative ergodic eigenmeasure ν, $L_\phi^* \nu = e^{P_G(\phi)} \nu$. Since $L_\phi^* \nu = e^{P_G(\phi)} \nu$, $e^{-P_G(\phi)} L_\phi$ is the transfer operator of ν, whence $\sum L_\phi^k f = \infty$ ν–almost everywhere (c.f. [A], proposition 1.3.2) , whence at least at one point. This contradiction shows that ψ cannot exist. □

The result should be compared with the results of S. Ruette [Rt] on the transience of $\phi \equiv 0$.

7. Appendix A: The discriminant and the three modes of recurrence

The purpose of this section is to summarize the results of [S1], [S2], and [S3] concerning the thermodynamic formalism of countable Markov shifts.

Throughout this section we assume that X is a *topologically mixing* CMS with set of states $\mathcal{S}$ and transition matrix A, which we think of as the set of one sided admissible paths on a directed graph $\mathcal{G}$. We use the notation introduced in §1.2.

7.1. **Gurevich Pressure.**

Suppose ϕ has summable variations, and define as always $\phi_n := \phi + \phi \circ T + \cdots + \phi \circ T^{n-1}$. The *Gurevich pressure* of ϕ is

$$P_G(\phi) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(\phi, a), \text{ where } Z_n(\phi, a) := \sum_{T^n x = x} e^{\phi_n(x)} 1_{[a]}(x).$$

The limit exists, is independent of the choice of a, and satisfies [S1]:

(a) For every constant c, $P_G(\phi + c) = P_G(\phi) + c$;
(b) $\phi \leq \psi \Rightarrow P_G(\phi) \leq P_G(\psi)$;
(c) if ϕ, ψ have summable variations, then $P_G(t\phi + (1-t)\psi) \leq tP_G(\phi) + (1-t)P_G(\psi)$ for all $t \in [0, 1]$.

Theorem 7.1 (Variational Principle [S1])**.** *If $\sup\phi < \infty$ and ϕ has summable variations, then $P_G(\phi) = \sup\{h_\mu(T) + \int \phi d\mu\}$ where the supremum ranges over all T–invariant Borel probability measures such that $(h_\mu(T), \int \phi d\mu) \neq (\infty, -\infty)$.*

Remark 7.1. *If X is topologically mixing and ϕ has summable variations, then $L_\phi^n 1_{[a]} \asymp Z_n(\phi, a)$ uniformly on $[a]$.*

7.2. **Modes of Recurrence.** Recall that $\varphi_a(x) := 1_{[a]}(x) \inf\{n \geq 1 : T^n(x) \in [a]\}$, and set

$$Z_n^*(\phi, a) = \sum_{T^n x = x} e^{\phi_n(x)} 1_{[\varphi_a = n]}(x).$$

$Z_n(\phi, a)$ and $Z_n^*(\phi, a)$ are related by the following "approximate renewal equation": set $B := \exp\left(2\sum_{n=2}^\infty \mathrm{var}_n(\phi)\right)$, then

$$Z_n(\phi, a) = B^{\pm 1}\big(Z_{n-1}(\phi, a)Z_1^*(\phi, a) + \cdots + Z_1(\phi, a)Z_{n-1}^*(\phi, a) + Z_n^*(\phi, a)\big) \quad (7.1)$$

Passing to the generating functions

$$t_\phi(x) = 1 + \sum_{n=1}^\infty Z_n(\phi, a)x^n \text{ and } r_\phi(x) = \sum_{n=1}^\infty Z_n^*(\phi, a)x^n,$$

we obtain

$$\frac{1}{B^2} t_\phi(x) r_\phi(x) \leq t_\phi(x) - 1 \leq B^2 t_\phi(x) r_\phi(x) \quad (7.2)$$

for every $x \in [0, R)$, where $R = e^{-P_G(\phi)}$ is the radius of convergence of $t_\phi(\cdot)$.

Definition 7.1. *Set $\lambda = e^{P_G(\phi)}$. We call ϕ*

- transient, *if $t_\phi(\lambda^{-1}) < \infty$;*
- positive recurrent, *if $t_\phi(\lambda^{-1}) = \infty$ but $r_\phi'(\lambda^{-1}) < \infty$;*
- null recurrent, *if $t_\phi(\lambda^{-1}) = \infty$ and $r_\phi'(\lambda^{-1}) = \infty$.*

We have the following [S2, Theorem 1]:

Theorem 7.2 (Generalized Ruelle-Perron-Frobenius Theorem [S2])**.** *ϕ is recurrent iff there exist $\lambda > 0$, a conservative measure ν, finite and positive on cylinders, and a positive continuous function h such that $L_\phi^* \nu = \lambda \nu$ and $L_\phi h = \lambda h$. In this case $\lambda = e^{P_G(\phi)}$ and $\exists\, a_n \nearrow \infty$ such that for every cylinder $[\underline{a}]$ and $x \in X$*

$$\frac{1}{a_n} \sum_{k=1}^n \lambda^{-k} (L_\phi^k 1_{[\underline{a}]})(x) \xrightarrow[n\to\infty]{} h(x)\nu[\underline{a}],$$

where $\{a_n\}_n$ satisfies $a_n \sim (\int_{[a]} h d\nu)^{-1} \sum_{k=1}^n \lambda^{-k} Z_k(\phi, a)$ for every $a \in \mathcal{S}$. Furthermore,

1. *if ϕ is positive recurrent then $\nu(h) < \infty$, $a_n \sim n \cdot const$ and for every $[\underline{a}]$, $\lambda^{-n} L_\phi^n 1_{[\underline{a}]} \xrightarrow[n\to\infty]{} h\nu[\underline{a}]/\nu(h)$ uniformly on compacts;*
2. *if ϕ is null recurrent then $\nu(h) = \infty$, $a_n = o(n)$ and for every cylinder $[\underline{a}]$, $\lambda^{-n} L_\phi^n 1_{[\underline{a}]} \xrightarrow[n\to\infty]{} 0$ uniformly on compacts.*

It is not difficult to see, using the representation of h as the limit above, that $\mathrm{var}_1[\log h] \leq \sum_{n\geq 2} \mathrm{var}_n \phi$.

7.3. **The Discriminant.** Fix a state $a \in \mathcal{S}$, and recall the operation of passing from the pair (X, ϕ) to $(\overline{X}, \overline{\phi})$ as explained in §2.1. Define $p_a^*[\phi] := \sup\{p \mid P_G(\overline{\phi + p}) < \infty\}$ (the bar means that we induce on the state a). This number can be calculated by the formula [S3]

$$p_a^*[\phi] = -\limsup_{n\to\infty} \frac{1}{n} \log Z_n^*(\phi, a). \tag{7.3}$$

Moreover, the map

$$p(t) = P_G(\overline{\phi + t}) \tag{7.4}$$

is convex, strictly increasing and continuous on $\{t : t \le p_a^*[\phi]\}$ ([S3, Proposition 3]).

The *discriminant* of ϕ at $a \in \mathcal{S}$ is defined to be

$$\Delta_a[\phi] := \sup\{P_G(\overline{\phi + p}) \mid p < p_a^*[\phi]\}.$$

The following is frequently useful so we state it as a lemma.

Lemma 7.1. *If X is topologically mixing and ϕ has summable variations and finite pressure then $\Delta_a[\phi + t \cdot 1_{[a]}] = \Delta_a[\phi] + t$.*

Proof. $P_G(\overline{\phi + t \cdot 1_{[a]} + p}) = P_G(\overline{\phi + p} + t) = P_G(\overline{\phi + p}) + t$, so $p_a^*[\phi + t \cdot 1_{[a]}] = p_a^*[\phi]$ and $\Delta_a[\phi + t \cdot 1_{[a]}] = \Delta_a[\phi] + t$. □

The main interest in the discriminant is that it detects modes of recurrence:

Theorem 7.3 (Discriminant Theorem [S3])**.** *Let X be a topologically mixing countable Markov shift and let $\phi : X \to \mathbb{R}$ be some function with summable variations and finite Gurevich pressure. Let $a \in \mathcal{S}$ be some arbitrary fixed state.*

(1) *The equation $P_G(\overline{\phi + p}) = 0$ has a unique solution $p(\phi)$ if $\Delta_a[\phi] \ge 0$ and no solution if $\Delta_a[\phi] < 0$. The Gurevich pressure of ϕ is given by*

$$P_G(\phi) = \begin{cases} -p(\phi) & \text{if } \Delta_a[\phi] \ge 0 \\ -p_a^*[\phi] & \text{if } \Delta_a[\phi] < 0 \end{cases};$$

(2) *ϕ is positive recurrent if $\Delta_a[\phi] > 0$ and transient if $\Delta_a[\phi] < 0$. In the case $\Delta_a[\phi] = 0$, ϕ is either positive recurrent or null recurrent.*

In particular, strong positive recurrence implies positive recurrence.

Definition 7.2. *We say that ϕ is* weakly positive recurrent *if it is positive recurrent but not strongly positive recurrent.*

Corollary 7.1. *Suppose X is topologically mixing and ϕ has summable variations and finite Gurevich pressure. If ϕ is recurrent then*

$$P_G(\phi) \ge \limsup_{n\to\infty} \frac{1}{n} \log Z_n^*(\phi, a) \tag{7.5}$$

and if ϕ is transient then

$$P_G(\phi) = \limsup_{n\to\infty} \frac{1}{n} \log Z_n^*(\phi, a). \tag{7.6}$$

The first equation is by definition of the pressure and $Z_n(\phi, a) \ge Z_n^*(\phi, a)$. The second equation is the discriminant theorem and (7.3).

8. Appendix B: Proof of Theorem 1.1

Throughout this section, assume that $T : X \to X$ is a topologically mixing countable Markov shift, and that $\phi \in \Phi$. We use the thermodynamic formalism for CMS as summarized in appendix A.

8.1. **Some technical implications of SGP.**

Lemma 8.1. *If ϕ has SGP, then the P in definition 1.1 has the form $Pg = h \int g d\nu$ for all $g \in \mathcal{L}$, where $h \in \mathcal{L}$ is positive and bounded away from zero on cylinders, ν is finite and positive on cylinders, and $L_\phi h = \lambda h$, $L_\phi^* \nu = \lambda \nu$, $\int h d\nu = 1$.*

Proof. We show that ϕ is positive recurrent (appendix A, definition 7.1). The idea is to fix $a \in \mathcal{S}$ and show that $\lambda^{-n} Z_n(\phi, a) \asymp 1$, where $Z_n(\phi, a) = \sum_{T^n x = x} e^{\phi_n(x)} 1_{[a]}(x)$. This implies recurrence by definition, and rules out null recurrence because if ϕ were null recurrent, then $\lambda^{-n} Z_n(\phi, a) \asymp \lambda^{-n} L_\phi^n 1_{[a]} \xrightarrow[n\to\infty]{} 0$ on $[a]$ because of theorem 7.2 part (2), which contradicts $\lambda^{-n} Z_n(\phi, a) \asymp 1$.

Write $L_\phi = \lambda P + N$ with $\lambda = \exp P_G(\phi)$ and P, N as in definition 1.1. Since $PN = NP = 0$ and $P^2 = P$, $L_\phi^n = \lambda^n P + N^n$. Since the spectral radius of N is less than λ, $\|\lambda^{-n} N^n\| = O(\kappa^n)$ where $0 < \kappa < 1$. We have for (any) fixed $x \in [a]$, $\lambda^{-n} Z_n(\phi, a) \asymp \lambda^{-n}(L_\phi^n 1_{[a]})(x) = P1_{[a]}(x) + O(\kappa^n)$ (see (7.1) in appendix A). It is impossible for $P1_{[a]}(x)$ to vanish, because this would imply that $Z_n(\phi, a) = O((\kappa\lambda)^n)$, whereas $\frac{1}{n} \log Z_n(\phi, a) \to \log \lambda$ and $\kappa < 1$. Thus $P1_{[a]}(x) \neq 0$. It follows that $\lambda^{-n} Z_n(\phi, a) \asymp 1$, whence the positive recurrence of ϕ.

By the generalized RPF theorem (appendix A, theorem 7.2), $\exists h$ positive, continuous, and bounded away from zero on cylinders, and $\exists \nu$ positive and finite on cylinders such that $L_\phi h = \lambda h$, $L_\phi^* \nu = \lambda h$, $\int h d\nu = 1$. Moreover, $\lambda^{-n} L_\phi^n 1_{[\underline{a}]} \xrightarrow[n\to\infty]{} \nu[\underline{a}] h$ pointwise. But $\|\lambda^{-n} L_\phi^n 1_{[\underline{a}]} - P1_{[\underline{a}]}\|_{\mathcal{L}} \le \lambda^{-n} \|N^n 1_{[\underline{a}]}\|_{\mathcal{L}} \to 0$, so $\lambda^{-n} L_\phi^n 1_{[\underline{a}]} \to P1_{[\underline{a}]}$ pointwise. We see that $P1_{[\underline{a}]} = \nu[\underline{a}] h$. Since $P(\mathcal{L}) \subseteq \mathcal{L}$, $h \in \mathcal{L}$.

Since $\dim \operatorname{Im} P = 1$, there exists $\varphi \in \mathcal{L}^*$ s.t. $Pg = \varphi(g) h$ for all $g \in \mathcal{L}$. We show that $\varphi(g) = \int g d\nu$ for all $g \in \mathcal{L}$.

Let $m_\phi := h d\nu$. The relations $L_\phi^* \nu = \lambda \nu$ and $L_\phi h = \lambda h$ can be used to see that m_ϕ is T–invariant measure. The methods of [ADU] show that it is mixing (even exact).

Suppose $g \in \mathcal{L} \cap L^1(\nu)$, then $gh^{-1} \in L^1(m_\phi)$, and the mixing of m_ϕ implies that $\int (gh^{-1}) 1_{[a]} \circ T^n dm_\phi \xrightarrow[n\to\infty]{} m_\phi[a] \int g d\nu$. On the other hand

$$\begin{aligned}\int (gh^{-1}) 1_{[a]} \circ T^n dm_\phi &= \int g 1_{[a]} \circ T^n d\nu = \int \lambda^{-n} L_\phi^n (g 1_{[a]} \circ T^n) d\nu \\ &= \int_{[a]} \lambda^{-n} L_\phi^n g d\nu = \int_{[a]} \big[Pg + \lambda^{-n} N^n g\big] d\nu \xrightarrow[n\to\infty]{} \varphi(g) m_\phi[a],\end{aligned}$$

because $\|\lambda^{-n} N^n g\|_{\mathcal{L}} \to 0$, whence $\lambda^{-n} N^n g \to 0$ uniformly on $[a]$. Comparing the limits we see that $\varphi(g) = \int g d\nu$ for all $g \in \mathcal{L} \cap L^1(\nu)$.

It remains to see that $\mathcal{L} \subset L^1(\nu)$. Otherwise there exists $f \in \mathcal{L}$ s.t. $\int |f| d\nu = \infty$. Since $f \in \mathcal{L}$, $g := |f| \in \mathcal{L}$, and $\int g h^{-1} dm_\phi = \infty$. The mixing of m_ϕ implies that

$$\int (gh^{-1}) 1_{[a]} \circ T^n dm_\phi \xrightarrow[n\to\infty]{} \infty$$

(bound gh^{-1} from below by a bounded function with large integral). But $g \in \mathcal{L}$, so we can write as before $\int (gh^{-1})1_{[a]} \circ T^n dm_\phi = \int_{[a]} \lambda^{-n} L_\phi^n g d\nu \xrightarrow[n\to\infty]{} \varphi(g) m_\phi[a]$. This limit is finite, so we arrive at a contradiction. □

Lemma 8.2. *Let ν be as in the previous lemma, then there exists some constant C_0 s.t. $\|\cdot\|_{L^1(\nu)} \leq C_0 \|\cdot\|_{\mathcal{L}}$.*

Proof. Suppose $f \in \mathcal{L}$. By assumption, $\mathcal{L}$ has the lattice property: $f \in \mathcal{L} \Rightarrow |f| \in \mathcal{L}$. By the previous lemma, $P|f| = h \int |f| d\nu$, so $\|f\|_{L^1(\nu)} = \big\|P|f|\big\|_{\mathcal{L}} / \|h\|_{\mathcal{L}} \leq \frac{\|P\|}{\|h\|_{\mathcal{L}}} \big\||f|\big\|_{\mathcal{L}} \leq \frac{\|P\|}{\|h\|_{\mathcal{L}}} \|f\|_{\mathcal{L}}$. So take $C_0 := \|P\| / \|h\|_{\mathcal{L}}$. □

Lemma 8.3. *Suppose $\phi \in \Phi$ has the SGP. If ψ is (bounded and) Hölder continuous, then $\psi f \in dom(L_\phi)$ and $L_\phi(\psi f) \in \mathcal{L}$ for all $f \in \mathcal{L}$. The operator $f \mapsto L_\phi(\psi f)$ is a bounded linear operator on $\mathcal{L}$.*

Proof. If $f \in \mathcal{L}$, then $|f| \in \mathcal{L}$. Since $\mathcal{L} \subseteq dom(L_\phi)$, $|f| \in dom(L_\phi)$. If ψ is bounded, then $|\psi f| < C|f|$ for some C, so $|\psi f| \in dom(L_\phi)$, whence $\psi f \in dom(L_\phi)$.

Next, by assumption, $t \mapsto L_{\phi+t\psi}$ is a real analytic $\mathrm{Hom}(\mathcal{L}, \mathcal{L})$–valued map on a neighborhood of zero. This means that for every $f \in \mathcal{L}$, $t \mapsto L_{\phi+t\psi} f$ is a real analytic $\mathcal{L}$–valued map on a neighborhood of zero. Differentiating at zero, we see that there exists $g \in \mathcal{L}$ s.t.

$$\frac{1}{t}[L_{\phi+t\psi} f - L_\phi f] \xrightarrow[t\to 0]{\mathcal{L}} g \in \mathcal{L}.$$

Since $\mathcal{L}$-convergence implies pointwise convergence,

$$g(x) = \lim_{t\to 0} \frac{1}{t}[L_{\phi+t\psi} f - L_\phi f](x) = L_\phi(\psi f)(x) + \lim_{t\to 0} L_\phi \left(\left(\frac{e^{t\psi} - 1}{t} - \psi \right) f \right)(x)$$

for every $x \in X$. Since ψ is bounded and $|f| \in \mathcal{L} \subset \mathrm{dom}(L_\phi)$,

$$\left| L_\phi \left(\left(\frac{e^{t\psi} - 1}{t} - \psi \right) f \right) \right| \leq \sup_{|\tau| \leq \|\psi\|_\infty} \left| \frac{e^{\tau t} - 1}{t\tau} - 1 \right| \|\psi\|_\infty (L_\phi |f|)(x) \xrightarrow[t\to 0]{} 0.$$

Thus $g(x) = L_\phi(\psi f)(x)$ for all x, whence $L_\phi(\psi f) = g \in \mathcal{L}$.

We estimate $\|L_\phi(\psi f)\|_{\mathcal{L}}$. We just saw that $L_\phi(\psi f)$ is the derivative at zero of the $\mathcal{L}$–valued function $t \mapsto L_{\phi+t\psi} f$. By SGP, this function extends to a holomorphic function $z \mapsto L_{\phi+z\psi}$ on some complex neighborhood U of the origin. Let C be a circle with center zero and radius r so small that $C \subset U$, then for every $f \in \mathcal{L}$:

$$\|L_\phi(\psi f)\|_{\mathcal{L}} = \left\| \frac{1}{2\pi i} \oint_C \frac{1}{z^2} L_{\phi+z\psi} f dz \right\|_{\mathcal{L}} \leq \frac{1}{r} \max_{z\in C} \|L_{\phi+z\psi}\| \cdot \|f\|_{\mathcal{L}}.$$

It follows that $f \mapsto L_\phi(\psi f)$ is a bounded operator. □

8.2. **Equilibrium measures.** It was proved in [BS] that if a weakly Hölder continuous function ϕ with finite pressure and supremum has an equilibrium measure, then this measure is of the form $hd\nu$ with $h > 0$ continuous and ν s.t. $L_\phi h = \lambda h$, $L_\phi^* \nu = \lambda \nu$, $\int h d\nu = 1$. Here we show the converse: If h, ν are as above, and $dm = hd\nu$ has finite entropy, then it is an equilibrium measure (by [BS] the unique one). Let $\alpha := \{[a] : a \in \mathcal{S}\}$ denote the natural generator.

Lemma 8.4 (Rokhlin)**.** *Let μ be a shift invariant measure on a CMS X, and let α be the natural generator. Then $h_\mu(T) \geq H_\mu(\alpha|\alpha_1^\infty)$, with equality when $H_\mu(\alpha) < \infty$.*

Proof. The equality when $H_\mu(\alpha)<\infty$ is standard, so we focus on the case when $H_\mu(\alpha)=\infty$. We use the following notational conventions for partitions. Suppose γ is a measurable partition of X, then $\sigma(\gamma)$:=the sigma algebra generated by γ; $\gamma_m^n := \bigvee_{k=m}^n T^{-k}\gamma$ =the smallest partition s.t. $\sigma(\gamma_m^n)\supseteq \bigcup_{k=m}^n \sigma(T^{-k}\gamma)$; and γ_1^∞ :=the smallest sigma–algebra which contains $\bigcup_{n\geq 1}\sigma(\gamma_1^n)$.

Take an increasing sequence of finite partitions $\beta^{(n)}$ such that $\sigma(\beta^{(n)})\uparrow\sigma(\alpha)$. For every fixed n, since $H_\mu(\beta^{(n)})<\infty$,

$$\begin{aligned} h_\mu(T,\beta^{(n)}) &= \lim_{k\to\infty}\frac{1}{k}H_\mu((\beta^{(n)})_0^{k-1}) = \lim_{k\to\infty}\frac{1}{k}\sum_{\ell=1}^{k-1}[H_\mu((\beta^{(n)})_0^{\ell}) - H_\mu((\beta^{(n)})_0^{\ell-1})] \\ &= \lim_{k\to\infty}\frac{1}{k}\sum_{\ell=1}^{k-1}H_\mu(\beta^{(n)}|(\beta^{(n)})_1^{\ell}) \geq \lim_{k\to\infty}\frac{1}{k}\sum_{\ell=1}^{k-1}H_\mu(\beta^{(n)}|\alpha_1^{\ell}), \end{aligned} \tag{8.1}$$

because $\sigma(\alpha)\supset\sigma(\beta^{(n)})$. We claim that

$$H_\mu(\beta^{(n)}|\alpha_1^\ell)\xrightarrow[\ell\to\infty]{} H_\mu(\beta^{(n)}|\alpha_1^\infty). \tag{8.2}$$

This is because

(a) $I_\mu(\beta^{(n)}|\alpha_1^\ell)\xrightarrow[\ell\to\infty]{} I_\mu(\beta^{(n)}|\alpha_1^\infty)$ μ–a.e. (Martingale convergence theorem)
(b) $\int\sup_{\ell\geq 1} I_\mu(\beta^{(n)}|\alpha_1^\ell)d\mu<\infty$, by the Chung–Neveu Lemma ([P], lemma 2.1);
(c) the dominated convergence theorem.

By (8.1) and (8.2), for all n, $h_\mu(T,\beta^{(n)})\geq H_\mu(\beta^{(n)}|\alpha_1^\infty)\equiv\int I_\mu(\beta^{(n)}|\alpha_1^\infty)d\mu$. Now $I_\mu(\beta^{(n)}|\alpha_1^\infty)\uparrow I_\mu(\alpha|\alpha_1^\infty)$, because $\beta^{(n)}$ increase to α (see e.g. [P], theorem 2.2 (ii)). By the monotone convergence theorem $H_\mu(\beta^{(n)}|\alpha_1^\infty)\uparrow H_\mu(\alpha|\alpha_1^\infty)$, and we conclude that $h_\mu(T,\beta^{(n)})\geq H_\mu(\beta^{(n)}|\alpha_1^\infty)\xrightarrow[n\to\infty]{} H_\mu(\alpha|\alpha_1^\infty)$. Since $h_\mu(T)\geq h_\mu(T,\beta^{(n)})$, the proof is completed. □

Proposition 8.1. *Suppose ϕ has summable variations, has finite Gurevich pressure, and $\sup\phi<\infty$. Suppose further that $h>0$ is positive continuous, ν is positive and finite on cylinders, $L_\phi h=\lambda h$, $L_\phi^*\nu=\lambda\nu$, and $\int h d\nu=1$. If $d\mu=hd\nu$ has finite entropy, then it is an equilibrium measure of ϕ.*

Proof. One can show, as in [L], that $I_\mu(\alpha|\alpha_1^\infty)=-\ln\frac{d\mu}{d\mu\circ T}\equiv-[\phi+\ln h-\ln h\circ T-P_G(\phi)]$, so

$$\int(I_\mu(\alpha|\alpha_1^\infty)+\phi+\ln h-\ln h\circ T)d\mu = P_G(\phi).$$

By lemma 8.4, $\int I_\mu(\alpha|\alpha_1^\infty)d\mu = H_\mu(\alpha|\alpha_1^\infty)\leq h_\mu(T)<\infty$, so I_μ is absolutely integrable (it is a non-negative function). Since $\phi+\ln h-\ln h\circ T$ is bounded from above (by $P_G(\phi)$), it is also absolutely integrable, and

$$h_\mu(T)+\int[\phi+\ln h-\ln h\circ T]d\mu \geq \int[I_\mu+\phi+\ln h-\ln h\circ T]d\mu = P_G(\phi). \tag{8.3}$$

We claim that $\phi\in L^1(\mu)$, and $\int\phi d\mu=\int[\phi+\ln h-\ln h\circ T]d\mu$. The following holds for almost every $x\in X$:

(a) $\phi_n(x)/n\xrightarrow[n\to\infty]{}\int\phi d\mu$ (because $\sup\phi<\infty$ and μ is ergodic);
(b) $[\phi_n(x)+\ln h(x)-\ln h(T^n x)]/n \xrightarrow[n\to\infty]{} \int[\phi+\ln h-\ln h\circ T]d\mu$ (because $\phi+\ln h-\ln h\circ T\in L^1(\mu)$);

(c) $\exists n_k(x) \uparrow \infty$ s.t. $|\ln h(x) - \ln h(T^{n_k(x)}x)| \leq 1$ (because of the Poincaré recurrence theorem, and the continuity of h).

Choose one such x, then

$$\int \phi d\mu = \lim_{k\to\infty} \frac{1}{n_k(x)} \phi_{n_k(x)} = \lim_{k\to\infty} \frac{1}{n_k(x)} \Big(\phi_{n_k(x)} + \ln h(x) - \ln h(T^{n_k(x)}x) \Big)$$
$$= \lim_{n\to\infty} \frac{1}{n} \left(\phi_n + \ln h(x) - \ln h(T^n x) \right) = \int \left(\phi + \ln h - \ln h \circ T \right) d\mu.$$

By (8.3) $h_\mu(T) + \int \phi d\mu \geq P_G(\phi)$. The proposition now follows from the variational principle (appendix A, theorem 7.1). □

8.3. **Proof of theorem 1.1.** Suppose that X is topologically mixing, and $\phi \in \Phi$ has the SGP and satisfies $\sup \phi < \infty$. Let λ, P, N be as in definition 1.1.

Proof of (a). Lemma 8.1 says that P is of the form $Pf = h \int f d\nu$, where $L_\phi h = \lambda h$, $L_\phi^* \nu = \lambda \nu$, and $\int h d\nu = 1$. Proposition 8.1 says that if $dm_\phi = h d\nu$ has finite entropy, then m_ϕ is an equilibrium measure for ϕ. By [BS], there is at most one such measure, so m_ϕ is unique.

Proof of (b). Let $\rho(N)$ denote the spectral radius of $N : \mathcal{L} \to \mathcal{L}$. By the SGP, $\exists \kappa \in (\rho(N)/\lambda, 1)$. If f is (bounded and) Hölder continuous, then $L_\phi(fh) \in \mathcal{L}$ (lemma 8.3). If $g \in L^\infty(m_\phi)$, then the identities $dm_\phi = h d\nu$, $L_\phi^* \nu = \lambda \nu$, and $PL_\phi(fh) = \lambda h \int f dm_\phi$ imply

$$\left| \int f \cdot g \circ T^n dm_\phi - \int f dm_\phi \int g dm_\phi \right| = \left| \int \lambda^{-n} [L_\phi^n(fh) - \lambda^{n-1} PL_\phi(fh)] g d\nu \right|$$
$$\leq \|g\|_\infty \left\| \lambda^{-n} N^{n-1} L_\phi(fh) \right\|_{L^1(\nu)} \leq C_0 \|g\|_\infty \left\| \lambda^{-n} N^{n-1} L_\phi(fh) \right\|_{\mathcal{L}}$$
$$\leq C_0 \lambda^{-1} \|g\|_\infty \lambda^{-(n-1)} \|N^{n-1}\| \|L_\phi(fh)\|_{\mathcal{L}} \leq \text{const}\, \|g\|_\infty \|L_\phi(fh)\|_{\mathcal{L}} \kappa^n.$$

Part (c). We assume without loss of generality that

$$\lambda = 1 \ , \ \mathbb{E}_{m_\phi}[\psi] = 0.$$

To arrange this, replace ϕ by $\phi - \log \lambda$ and ψ by $\psi - \mathbb{E}_{m_\phi}[\psi]$.

Part (e) of the SGP is stable under perturbation in $\mathrm{Hom}(\mathcal{L}, \mathcal{L})$ [K]: There exists a neighborhood U of L_ϕ in $\mathrm{Hom}(\mathcal{L}, \mathcal{L})$ and analytic maps $P, N : U \to \mathrm{Hom}(\mathcal{L}, \mathcal{L})$, $\lambda : U \to \mathbb{C}$ such that for all $L \in U$,

$$L = \lambda(L)P(L) + N(L), P(L)N(L) = N(L)P(L) = 0, P(L)^2 = P(L), \dim \mathrm{Im}\, P(L) = 1.$$

If U is sufficiently small, then there is some $\varepsilon_0 > 0$ s.t. for all $L \in U$, the spectral radius of $N(L)$ is less that $1 - 2\varepsilon_0$ and the spectral radius of L (equal to $|\lambda(L)|$) is more than $1 - \varepsilon_0$.

By the SGP, $t \mapsto L_t := L_{\phi + it\psi}$ is analytic on a neighborhood of zero. The maps $\lambda_t = P(L_t)$, $P_t = P(L_t)$, $N_t = N(L_t)$ must also be analytic in t on a small neighborhood I of zero.

Recall that there is a constant C_0 s.t. $\|\cdot\|_{L^1(\nu)} \leq C_0 \|\cdot\|_{\mathcal{L}}$. For t in I,

$$\begin{aligned}\mathbb{E}_{m_\phi}[e^{it\psi_n}] &= \int \lambda^{-n} L_\phi^n(e^{it\psi_n}h)d\nu = \int L_t^n h d\nu \qquad\qquad (\because \lambda = 1)\\ &= \lambda_t^n \int [P_t h + \lambda_t^{-n} N_t^n h]d\nu\\ &= \lambda_t^n[1 \pm C_0(\|P_t - P\| + |\lambda_t|^{-n}\|N_t^n\|)\|h\|_{\mathcal{L}}].\end{aligned}$$

The spectral radius of N_t is less than $1 - 2\varepsilon_0$, and $|\lambda_t| \geq 1 - \varepsilon_0$, so this gives

$$\mathbb{E}_{m_\phi}[e^{it\psi_n}] = \lambda_t^n[1 + \varepsilon_n(t)] \text{ for all } n, \text{ where } \varepsilon_n(t) \xrightarrow[t\to 0, n\to\infty]{} 0.$$

Later we will see that if ψ is not cohomologous to a constant, then

$$\lambda_t = 1 - \frac{\sigma^2}{2}t^2 + o(t^2) \text{ as } t \to 0, \tag{8.4}$$

where $\sigma > 0$. It will then follow that $\mathbb{E}_{m_\phi}[\exp(it\psi_n/\sqrt{n})] \xrightarrow[n\to\infty]{} \exp(-\sigma^2 t^2/2)$, which means that $\frac{1}{\sqrt{n}}\psi_n$ converges in distribution (w.r.t m_ϕ) to a normal law with mean zero and standard deviation σ.

To prove (8.4), we expand λ_t as in [GH]. Define for this purpose $h_t := P_t 1/\int P_t 1 d\nu$ (the denominator approaches 1 as $t \to 0$ so it is not zero for all $|t|$ sufficiently small), and write $L := L_0 = L_\phi$. Then $L_t h_t = \lambda_t h_t$ and so

$$\begin{aligned}\lambda_t &= \int L_t h_t d\nu = \int (L_t - L)(h_t - h)d\nu + \int (L_t - L)h d\nu + \int L h_t d\nu\\ &= \int (L_t - L)(h_t - h)d\nu + \mathbb{E}_\nu[L((e^{it\psi} - 1)h)] + \int h_t d\nu \qquad (\because L^*\nu = \lambda\nu = \nu)\\ &= \int (L_t - L)(h_t - h)d\nu - \frac{t^2}{2}\int \psi^2 h d\nu + o(t^2) + 1,\end{aligned} \tag{8.5}$$

where we have used the fact that ψ is bounded to expand $e^{it\psi} = 1 + it\psi - \frac{t^2}{2}\psi^2 + o(t^2)$, and the assumption that $\mathbb{E}_{m_\phi}[\psi] = 0$ to note that $\int \psi h d\nu = 0$. (The assumption that ψ is bounded is an overkill.)

The analyticity of $t \mapsto L_t, P_t$ and the estimate $\|\cdot\|_{L^1(\nu)} \leq C_0\|\cdot\|_{\mathcal{L}}$ can be used to show that $\int (L_t - L)(h_t - h)d\nu = o(t)$ as $t \to \infty$. Thus $\lambda_t = 1 + o(t)$.

Next we study the difference $h_t - h$, as in [G1]. In what follows, $o(1)$ means an element of $\mathcal{L}$ whose $\mathcal{L}$–norm is $o(1)$:

$$\begin{aligned}\frac{h_t - h}{t} &= \frac{\lambda_t h_t - h}{t} + o(1) \text{ (because } \lambda_t = 1 + o(t) \text{ and } \|h_t\|_{\mathcal{L}} \text{ is bounded near zero)}\\ &= \frac{L_t h_t - Lh}{t} + o(1) = (L_t - L)\frac{h_t}{t} + L\left(\frac{h_t - h}{t}\right) + o(1).\end{aligned}$$

Subtracting the second summand from both sides, we obtain

$$(1 - L)\left(\frac{h_t - h}{t}\right) = (L_t - L)\frac{h_t}{t} + o(1) = L\left[\left(\frac{e^{it\psi} - 1}{t}\right)h_t\right] + o(1). \tag{8.6}$$

The left side of (8.6) converges in $\mathcal{L}$, whence in $L^1(\nu)$, to $(1 - L)a$, where

$$a := \left.\frac{d}{dt}\right|_{t=0} h_t.$$

The right side of (8.6) converges in $L^1(\nu)$ to $iL(\psi h)$. To see this, note the following:

(a) ψ is bounded, so $\exists M$ s.t. $|(e^{it\psi}-1)/t| \leq M$ for all $|t|<1$;

(b) $\left(\frac{e^{it\psi}-1}{t}\right) h \xrightarrow[t\to 0]{L^1(\nu)} i\psi h$, because of the dominated convergence theorem and the bound $\|h\|_{L^1(\nu)} \leq C_0\|h\|_{\mathcal{L}} < \infty$;

(c) $\left\|\left(\frac{e^{it\psi}-1}{t}\right)(h_t-h)\right\|_{L^1(\nu)} \leq C_0 M\|h_t-h\|_{\mathcal{L}} \xrightarrow[t\to 0]{} 0$.

Thus $\left(\frac{e^{it\psi}-1}{t}\right)h_t \xrightarrow[t\to 0]{} i\psi h$ in $L^1(\nu)$. Now L extends to a bounded operator on $L^1(\nu)$ s.t. $\|L\|\leq 1$ (the transfer operator of ν), so $L\left[\left(\frac{e^{it\psi}-1}{t}\right)h_t\right] \xrightarrow[t\to 0]{L^1(\nu)} iL(\psi h)$.

Passing to the limit $t\to 0$ in (8.6), we see that $(1-L)a = iL(\psi h)$ ν–a.e. Since all elements of $\mathcal{L}$ are assumed to be continuous, and since ν is globally supported, $(1-L)a = iL(\psi h)$.

Apply L^k to both sides: $L^k a - L^{k+1}a = iL^k(\psi h)$. The norm of the right hand side is summable:

$$\begin{aligned}\|L^k(\psi h)\|_{\mathcal{L}} &= \|P(L(\psi h)) + N^{k-1}L(\psi h)\|_{\mathcal{L}} \\ &= \|N^k\|\|L(\psi h)\|_{\mathcal{L}} \quad (\because P[L(\psi h)] = h\int L[\psi h]d\nu = h\int \psi dm_\phi = 0),\end{aligned}$$

and $\sum \|N^k\| < \infty$. Summing over $k\geq 0$, we obtain $a = i\sum_{k=1}^\infty L^k(\psi h)$.

Returning to the expansion (8.5) of λ_t, we see that

$$\begin{aligned}\lambda_t &= \int (L_t - L)(at + o(t))d\nu - \frac{t^2}{2}\int \psi^2 h d\nu + o(t^2) + 1 \\ &= t^2 \int \left(\frac{e^{it\psi}-1}{t}\right)(a+o(1))d\nu - \frac{t^2}{2}\int \psi^2 h d\nu + o(t^2) + 1 \\ &= 1 - \frac{t^2}{2}\int \psi^2 h d\nu - t^2\int \psi \sum_{k=1}^\infty L^k(\psi h) d\nu + o(t^2),\end{aligned}$$

and we obtain (8.4) with

$$\sigma^2 := \int \left[\psi^2 + \frac{2}{h}\psi\sum_{k=1}^\infty L^k(\psi h)\right] dm_\phi.$$

But it is not yet clear that σ^2 is strictly positive. To see this we follow [G1] and rewrite the integrand in terms of the function $u := \sum_{k=0}^\infty L^k(\psi h)$, noting that $\psi h = u - Lu$:

$$\begin{aligned}\sigma^2 &= \int \frac{1}{h^2}\left[(\psi h)^2 + 2\psi h(u-\psi h)\right] dm_\phi = \int \frac{1}{h^2}\left[(u-Lu)^2 + 2(u-Lu)Lu\right] dm_\phi \\ &= \int \frac{1}{h^2}\left[u^2 - (Lu)^2\right] dm_\phi = \int \left[(u/h)^2 - \left(\frac{1}{h}L(h\cdot u/h)\right)^2\right] dm_\phi.\end{aligned}$$

The operator $\widehat{T}: v \mapsto h^{-1}L(hv)$ preserves m_ϕ: $\widehat{T}^* m_\phi = m_\phi$ (it is the transfer operator of $m_\phi = hd\nu$). Thus we get

$$\sigma^2 = \int \left[\widehat{T}[(u/h)^2] - (\widehat{T}(u/h))^2\right] dm_\phi.$$

It is not difficult to see that $\widehat{T}$ takes the form $\widehat{T}f = \sum_{Ty=x} g(y)f(y)$ where $g = e^{\phi}h/h\circ T$. We have $\sum_{Ty=x} g(y) \equiv 1$. Since $t \mapsto t^2$ is convex, $\widehat{T}[(u/h)^2] \geq (\widehat{T}(u/h))^2$ and we get that $\sigma^2 \geq 0$, with equality iff $\widehat{T}[(u/h)^2] = [\widehat{T}(u/h)]^2$ m_ϕ–a.e.

By the strict convexity of $t \mapsto t^2$, u/h must be constant on $\{y : Ty = x\}$ for a.e. x. Since $m_\phi \sim m_\phi \circ T$ ($\because dm/dm \circ T = e^{\phi}h/h \circ T > 0$), this means that there is a function φ s.t. $u/h = \varphi \circ T$ almost everywhere. Thus

$$\psi = \frac{1}{h}(u - Lu) = \varphi \circ T - \frac{1}{h}L(h\varphi \circ T) = \varphi \circ T - \varphi \text{ a.e.}$$

It follows that ψ is an almost everywhere coboundary w.r.t m_ϕ. By the Livsic theorem of Gouëzel [G2], ψ is a coboundary with a continuous transfer function. But part (c) assumes that ψ is not like that.

Part (d). Suppose ψ is a (bounded) Hölder continuous function, and let L_t, λ_t, P_t, N_t be as above. We saw that $t \mapsto \lambda_t, P_t, N_t$ are analytic on some complex neighborhood of 0, and that for all $|t|$ sufficiently small, $\rho(N_t) < |\lambda_t|$.

We claim that $\lambda_t = \exp P_G(\phi + t\psi)$ on some real neighborhood of $t = 0$. This is because of the estimates

$$Z_n(\phi + t\psi, a) \asymp L_t^n 1_{[a]}(x) = \lambda_t^n P_t 1_{[a]}(x) + N_t^n 1_{[a]}(x) \asymp \lambda_t^n$$

which hold uniformly in x on $[a]$ provided t is small enough that $\rho(N_t) < |\lambda_t|$ (see appendix A, remark 7.1).

In particular $\lambda_0 = \exp P_G(\phi) \neq 0$, and $P_G(\phi + t\psi) = \log \lambda_t$ is real analytic on a neighborhood of zero. □

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Note added in proof. The first author has recently found a combinatorial characterization of the topologically mixing CMS for which $\{\phi \in \Phi : \phi \text{ does } \textit{not} \text{ have SGP}\}$ is not empty. As it turns out, "most" infinite state CMS are like that, e.g. all CMS whose associated transition graph $\mathcal{G}$ contains an infinite ray. Complete statements and detailed proofs will appear elsewhere.

References

[ADU] J. Aaronson, M. Denker, and F. Urbanski: *Ergodic theory for Markov fibred systems and parabolic rational maps.* Trans. Amer. Math. Soc. **337** (1993), no. 2, 495–548.

[A] J. Aaronson: An introduction to infinite ergodic theory, *Math. Surv. and Monog.* **50**, *Amer. Math. Soc. (1997). xii+284pp.*

[AD] J. Aaronson, and M. Denker: *Local limit theorems for Gibbs–Markov maps.* Stochastics Dyn. **1**, 193–237 (2001).

[B] V. Baladi: Positive Transfer Operators and Decay of Correlations, *Advanced Series in Nonlinear Dynamics,* **16**, World Scientific, Singapore (2000)

[BS] J. Buzzi and O. Sarig: *Uniqueness of equilibrium measures for countable Markov shifts and multi-dimensional piecewise expanding maps.* Ergodic Thy. Dynam. Sys. **23** 1383-1400 (2003).

[DF] W. Doeblin and R. Fortet: *Sur des chaînes à liasions complètes*, Bull. Soc. Math. France **65**, 132–148 (1937).

[GM] G. Gallavotti and S. Miracle-Sole: *Statistical mechanics of lattice systems*, Comm. Math. Phys. **5**, Number 5 (1967), 317-323.

[G1] S. Gouëzel: *Central limit theorem and stable laws for intermittent maps.* Probab. Theory Related Fields **128** (2004), no. 1, 82–122.

[G2] S. Gouëzel: *Regularity of coboundaries for nonuniformly expanding Markov maps.* Proc. Amer. Math. Soc. **134** (2006), no. 2, 391–401.

[GH] Y. Guivarc'h and J. Hardy: *Théorèmes limites pour une classe de chaînes de Markov et applications aux difféomorphismes d'Anosov.* (French. English summary) [Limit theorems for a class of Markov chains and applications to Anosov diffeomorphisms] Ann. Inst. H. Poincar Probab. Statist. **24** (1988), no. 1, 73–98.

[GS] B.M. Gurevich and S.V. Savchenko: *Thermodynamics formalism for countable Markov chains.* Uspekhi Mat. Nauk **53**:2 3–106. *Engl. transl.* in Russian Math. Surveys **53**:2 3–106.

[HH] H. Hennion and L. Hervé: Limit theorems for Markov chains and stochastic properties of dynamical systems by quasi–compactness, *LNM* **1766**, *Springer, 144 pp (2001)*

[K] T. Kato: Perturbation theory for linear operators. *Reprint of the 1980 edition. Classics in Mathematics. Springer-Verlag, Berlin, 1995. xxii+619 pp.*

[L] F. Ledrappier: *Principe variationnel et systemes dynamiques symboliques*, Z. Wahrscheinlichkeitstheorie und Verb. Gebiete **30** (1974), 185–202.

[Li] C. Liverani: *Central limit theorem for deterministic systems.* International Conference on Dynamical Systems (Montevideo, 1995), 56–75, Pitman Res. Notes Math. Ser., **362**, Longman, Harlow, 1996.

[Lo] A. Lopes: *The zeta function, nondifferentiability of pressure, and the critical exponent of transition.* Adv. Math. **101** (1993), no. 2, 133–165.

[P] W. Parry: Entropy and generators in ergodic theory, *W.A. Benjamin Inc.* xii+123 pages (1969)

[PP] W. Parry and M. Pollicott: Zeta functions and the periodic orbit structure of hyperbolic dynamics. *Astrisque* **187 - 8** *(1990).*

[PrS] T. Prellberg and J. Slawny: *Maps of intervals with indifferent fixed points: thermodynamic formalism and phase transitions.* J. Statist. Phys. **66** (1992), no. 1-2, 503–514.

[R] D. Ruelle: Thermodynamic formalism. The mathematical structures of equilibrium statistical mechanics. Second Edition, *Cambridge Univ. Press, xx+174pp* (2004)

[Rt] S. Ruette. *On the Vere-Jones classification and existence of maximal measures for countable topological Markov chains.* Pacific J. Math., **209** No. 2, 365-380 (2003).

[S1] O. Sarig: *Thermodynamic formalism for countable Markov shifts*, Ergodic Th. Dynam. Syst. **19**, 1565–1593 (1999).

[S2] O. Sarig: *Thermodynamic formalism for null recurrent potentials*, Israel J. Math. **121**, 285–311 (2001).

[S3] O. Sarig: *Phase transitions for countable Markov shifts*, Commun. Math. Phys. **217**, 555–577 (2001).

[S4] O. Sarig: *Subexponential decay of correlations.* Invent. Math. **150**, 629–653 (2002).

[S5] O. Sarig: *Critical exponents for dynamical systems.* Commun. Math. Phys. **267**, 631-667 (2006).

[VJ] D. Vere–Jones: *Geometric ergodicity in denumerable Markov chains*, Quart. J. Math. Oxford (2) **13**, 7–28 (1962).

[Y] L.-S. Young: *Statistical properties of dynamical systems with some hyperbolicity*, Ann. of Math. (2) **147**, 585–650 (1998).

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