CORRECTIONS TO *INVARIANT MEASURES AND ASYMPTOTICS FOR SOME SKEW PRODUCTS*

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Corollaries 2.7 and 2.8 in [ANSS] need correction. In the proof of 2.7, it is wrongly stated a) that ν must be non-atomic, and b) that m is locally finite. The following gives correct versions. References not listed here are listed in [ANSS].

2.7 Corollary. *Suppose that Σ is a mixing SFT and that $f : \Sigma \to \mathbb{Z}^d$ $(d \geq 1)$ has finite memory. If $\nu \in \mathcal{P}(\Sigma)$ is S_A^f-invariant, ergodic, and with $U := \operatorname{supp} \nu \subseteq \Sigma$ clopen and $\nu \circ \tau_U \sim \nu$, then $\nu \propto \mu_\alpha|_U$ for some homomorphism $\alpha : \mathbb{Z}^d \to \mathbb{R}$. If (in addition) $f\Sigma \to \mathbb{Z}^d$ is aperiodic, then $U = \Sigma$.*

Proof Since ν is finite and τ_U-non-singular, it is non-atomic and the unique τ_{ϕ_f}-ergodic, invariant measure m on $\Sigma \times \mathbb{Z}^d$ so that $m(A \times \{0\}) = \nu(A)$ is locally finite.

In case f is aperiodic, theorem 2.2 shows that $m = m_\alpha$, $U = \Sigma$ and $\nu = \mu_\alpha$ for some homomorphism $\alpha : \mathbb{Z}^d \to \mathbb{R}$.

Otherwise (see e.g. proposition 5.1 in [Pa-S] for $d = 1$) there is a subgroup $\mathbb{F} \subset \mathbb{Z}^d$ and

$$(\spadesuit) \qquad f = a + g - g \circ T + \overline{f}.$$

where $a \in \mathbb{Z}^d$, $\overline{f} : \Sigma \to \mathbb{F}$ aperiodic and $g : \Sigma \to \mathbb{Z}^d$ both with memory no longer than that of f.

Thus $m \circ \pi^{-1}$ (where $\pi(x, z) := (x, z - g(x))$) is locally finite, $\tau_{\phi_{\overline{f}}}$-ergodic, invariant and supported on $\Sigma \times (z_0 + \mathbb{F})$ (some $z_0 \in \mathbb{Z}^d$). The result now follows from the aperiodic case. $\square$

Remarks

1) In the situation of corollary 2.7, the ergodic decomposition of μ_α ($\alpha : \mathbb{Z}^d \to \mathbb{R}$ a homomorphism) is $\{[g \in b + \mathbb{F}] : \ b \in \mathbb{Z}^d\}$ (where g is as in ($\spadesuit$)). The proof of this uses [G-H] as in the proof of ergodicity in the aperiodic case.

2) Suppose that Σ is a mixing SFT and that $\nu \in \mathcal{P}(\Sigma)$ is S_A^+-invariant and ergodic, then (for $s \in S$) $N_s := \sum_{n=0}^{\infty} 1_{[s]} \circ T^n$. is S_A^+-invariant, whence constant ν-a.e.. Call $s \in S$ *ephemeral* if $1 \leq N_s < \infty$ ν-a.e., and *recurrent* if $N_s = \infty$ a.e.. Let S_∞ and S_e denote the collections of recurrent and ephemeral states (respectively). Evidently $S_\infty \neq \varnothing$ and $N := \sum_{f \in S_e} N_f$ is constant and finite ($N := 0$ if $S_e = \varnothing$).

2.8 Corollary *Suppose Σ is a mixing SFT. If $\nu \in \mathcal{P}(\Sigma)$ is S_A^+-invariant and ergodic, then $\exists$ a cylinder $f = [f_1, \ldots, f_N] \subset \Sigma$ with $f_1, \ldots f_N \in S_e$; a mixing SFT $\Sigma' = \Sigma_{A'} \subset \Sigma \cap S_\infty^{\mathbb{N}}$; a clopen, $S_{A'}^+$-invariant subset $U \subset \Sigma'$; a homomorphism*

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$\alpha : \mathbb{Z}^{S_\infty} \to \mathbb{R}$ *so that*

$$\nu = c \sum_{\pi \in S_N,\ \pi f \cap U \neq \varnothing} \delta_{\pi f} \times \mu|_{T^N \pi f \cap U}$$

where $\pi f := [f_{\pi(1)}, \ldots, f_{\pi(N)}],\ \mu \in \mathcal{P}(\Sigma'),\ \frac{d\mu\circ T}{d\mu} = c' e^{\alpha \circ F^{\#}}$ *and* $c, c' > 0$.

Proof

Suppose first that $S_e = \varnothing$. We claim first that ν is the restriction of a Markov measure to a union of initial states. To see this, choose $s \in S_\infty$ with $\nu([s]) > 0$, then $\nu|_{[s]}$ is ergodic, invariant under finite permutations of inter-arrival words, whence by de Finetti's theorem (see e.g. [D-F]) a product measure. By Proposition 15 of [D-F], $\nu|_{[s]}$ is the restriction of a stationary Markov measure to $[s]$, and by S_A^+-invariance and ergodicity, the transition matrix p does not depend on $s,\ \nu([s]) > 0$. It follows that $U := \text{supp}\nu = \bigcup_{s\in S_\infty,\ \nu([s])>0}[s]$ is clopen in $\Sigma_{A'}$ $(A'_{s,t} = 1_{[p_{s,t}>0]})$, and that $\nu \circ \tau_U \sim \nu$ where τ is the adic transformation on $\Sigma_{A'}$. The result in case $S_e = \varnothing$ now follows from corollary 2.7.

In general,

$$x_n \in \begin{cases} S_e & 1 \le n \le N := \sum_{f \in S_e} N_f, \\ S_\infty & \quad n > N, \end{cases}$$

$\nu \circ T^{-N}$ is as above. The result follows from this.

□

Remarks

Examples illustrating the various cases of corollary 2.8 can be extracted from [P-S]. Corollary 2.8 now extends the one-sided version of theorem 6.2 in [P-S]. Theorems 2.9 and 2.11 there follow from it (by identification of possible Σ').

References

[ANSS] J. Aaronson, H. Nakada, O. Sarig, R. Solomyak, Invariant measures and asymptotics for some skew products. *Israel J. Math.* **128** (2002), 93–134.

[D-F] P. Diaconis, D. Freedman, De Finetti's theorem for Markov chains, (French) *Annals Probab.* **8**, no. 8, (1980), 115-130.

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