CORRECTIONS TO 'CONTINUOUS PHASE TRANSITIONS FOR DYNAMICAL SYSTEMS'

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Proposition 1 in [**S**] should read

Proposition 1. *Let X, X_n be random variables such that for some $\omega > 0$, $C > 0$, $\mathbb{E}(e^{tX_n}), \mathbb{E}(e^{tX}) \le C$ for all $0 \le t \le \omega$. The following are equivalent:*

(1) $\mathbb{E}(e^{tX_n}) \xrightarrow[n\to\infty]{} \mathbb{E}(e^{tX})$ *for all $0 \le t \le t_0$ and some $t_0 > 0$;*

(2) $X_n \xrightarrow[n\to\infty]{dist} X$.

The difference with [**S**] is that there, the proposition is stated erroneously under the weaker assumption that $\mathbb{E}(e^{tX_n}), \mathbb{E}(e^{tX})$ are finite for all $0 \le t \le \omega$, with [**ML**] as the reference.

The implication (1) $\Rightarrow$ (2) does hold under this weaker assumption, as shown in [**ML**]. But the implication (2) $\Rightarrow$ (1) (which is not asserted in [**ML**]) is false, as demonstrated by the example $X := N(0,1)$, $X_n := X + n^3 1_{[n \le X \le n+1]}$.

To see that (2) $\Rightarrow$ (1) when $\sup_n \mathbb{E}(e^{tX_n}) \le C$ for all $t \in [0, \omega]$, observe that (2) implies that for all $t \in [0, \omega/2]$, $Y_n := e^{tX_n} \xrightarrow[n\to\infty]{\text{dist}} e^{tX}$, and $\mathbb{E}(Y_n^2) \le C$. Thus $\mathbb{E}(Y_n) \xrightarrow[n\to\infty]{} \mathbb{E}(Y)$ and we obtain (1) with $t_0 := \omega/2$.

The implication (2)$\Rightarrow$(1) is used once in [**S**], on page 644. The setting is as follows (see [**S**] for notation and terminology): ϕ, ψ are real valued locally Hölder continuous functions on a topologically mixing countable Markov shift with the BIP property, and

$$X_n := \frac{\psi_n}{B_n}, \text{ where } \psi_n := \sum_{k=0}^{n-1} \psi \circ T^k \text{ and } T \text{ the left shift map, and } B_n \to \infty$$

$$X := G_\alpha, \text{ where } G_\alpha \text{ is the distribution s.t. } \mathbb{E}(e^{tG_\alpha}) = e^{\mathrm{sgn}(\alpha-1)t^\alpha} \quad (0 < \alpha \le 2).$$

It is assumed that $\sup \phi, \sup \psi < \infty$, $P_{\mathrm{top}}(\phi) = 0$, and that $X_n \xrightarrow[n\to\infty]{\text{dist}} X$ w.r.t. μ_ϕ, the equilibrium (or Gibbs) measure of ϕ.

We claim that in this context it is always the case that $\mathbb{E}(e^{tX_n}) \le C$ on some non-trivial interval $[0, \omega]$, so that the proofs done in [**S**] remain valid.

The key is proposition 3 in [**S**], which says that $\exists \epsilon(t) \xrightarrow[t\to 0^+]{} 0$ and $\epsilon_0 > 0$ such that for all $0 \le t \le \epsilon_0$ the following holds uniformly for all n:

$$\mathbb{E}_{\mu_\phi}[e^{t\psi_n}] = [1 + O(\epsilon(t))] \exp[n P_{\mathrm{top}}(\phi + t\psi)]. \tag{1}$$

If $P_{\mathrm{top}}(\phi + t\psi)$ vanishes on some interval $[0, \omega]$, then (1) implies that $\mathbb{E}_{\mu_\phi}[e^{t\psi_n}] \le 1 + \mathrm{const} \cdot \epsilon(t)$ for all n so large that $t/B_n \le \min\{\omega, \epsilon_0\}$, and we are done.

Otherwise, since $t \mapsto P_{\mathrm{top}}(\phi + t\psi)$ is convex, there is some $\omega_1 > 0$ such that $t \mapsto P_{\mathrm{top}}(\phi + t\psi)$ is finite, strictly monotonic, and continuous on $[0, \omega_1]$. Let σ

denote the sign of this function on $(0,\omega_1]$. Define B_n^* as the (unique) solution of

$$nP_{\rm top}(\phi+\frac{\omega_1}{B_n^*}\psi)=\sigma$$

(such a solution exists for all n large enough). Obviously $B_n^*\to\infty$.

By (1), $\sup_n \mathbb{E}_{\mu_\phi}(e^{t\psi_n/B_n^*})$ is uniformly bounded on $[0,\omega_1]$. We will show that $M:=\sup[B_n^*/B_n]<\infty$. This implies that $\sup_n \mathbb{E}_{\mu_\phi}(e^{t\psi_n/B_n})$ is uniformly bounded on $[0,\omega_1/M]$, and again we are done.

Suppose by way of contradiction that $\exists n_k$ such that $B_{n_k}^*/B_{n_k}\to\infty$, whence $M:=\sup[B_n^*/B_n]<\infty$.

The functions $f_{n_k}(t):=nP_{\rm top}(\phi+t\psi/B_{n_k}^*)$ are convex, and uniformly bounded on $[0,\omega_1]$(with values between 0 and α). Choose a subsequence $f_{n_{k_\ell}}$ which converges on every rational point in $[0,\omega_1]$. By convexity, the sequence $f_{n_{k_\ell}}$ must converge everywhere on $[0,\omega_1)$. The limit $f(t)$ is convex, finite, continuos, monotonic, and non-constant, because $f(0)=0$ and $f(\omega_1)=\sigma$.

By (1), $\mathbb{E}_{\mu_\phi}[e^{t\psi_{n_{k_\ell}}/B^*_{n_{k_\ell}}}]\to\exp[f(t)]$ on $[0,\omega_1]$, and $\sup_n \mathbb{E}_{\mu_\phi}[e^{t\psi_{n_{k_\ell}}/B^*_{n_{k_\ell}}}]$ is uniformly bounded on $[0,\omega_1]$. It follows that $\psi_{n_{k_\ell}}/B^*_{n_{k_\ell}}$ converges in distribution to a distribution with Laplace transform $\exp[f(t)]$ (see e.g. [**ML**], Lemma C1). Since $f(t)$ is non-constant, the limiting distribution is not degenerate. Call it F, and choose some $x\neq 0$ such that $0<F(x)<1$. Since $\frac{1}{B_n}\psi_n\to G_\alpha$ and $B_{n_k}^*/B_{n_k}\to\infty$,

$$G_\alpha(\infty) \text{ or } G_\alpha(-\infty) \quad \underset{\ell\to\infty}{\longleftarrow} \quad \mu_\phi\left[\frac{\psi_{n_{k_\ell}}}{B_{n_{k_\ell}}}\leq\frac{B^*_{n_{k_\ell}}}{B_{n_{k_\ell}}}x\right]=\mu_\phi\left[\frac{\psi_{n_{k_\ell}}}{B^*_{n_{k_\ell}}}\leq x\right]\underset{\ell\to\infty}{\longrightarrow}F(x).$$

But $F(x)\neq 0,1$, so it cannot equal $G_\alpha(\infty)$ or $G_\alpha(-\infty)$. This contradiction shows that there is no subsequence n_k such that $B_{n_k}^*/B_{n_k}\to\infty$.

References

[**S**] O. Sarig: *Continuous phase transitions for dynamical systems.* Commun. Math. Phys. **267**, 631–667 (2006).

[**ML**] A. Martin–Löf: *Mixing properties, differentiability of the free energy and the central limit theorem for a pure phase in the Ising model at low temperature.* Comm. Math. Phys. **32** (1973), 75–92.

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