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LASSO-BASED FAST RESIDUAL RECOVERY FOR MODULO SAMPLING

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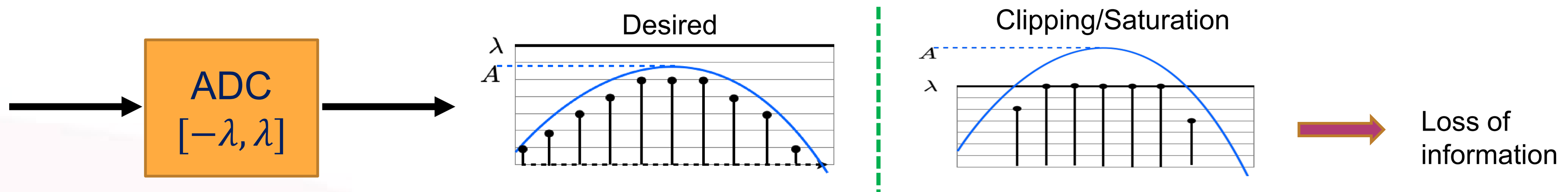
1. Motivation and Contributions

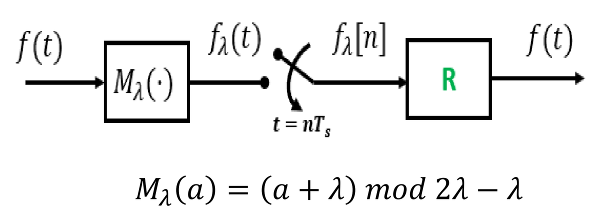
- Analog-to-digital-converters (ADCs) play a vital role in modern information processing systems
- Ideally, the dynamic range (DR) of an ADC should be greater than the DR of the input signal, otherwise, it leads to loss of information

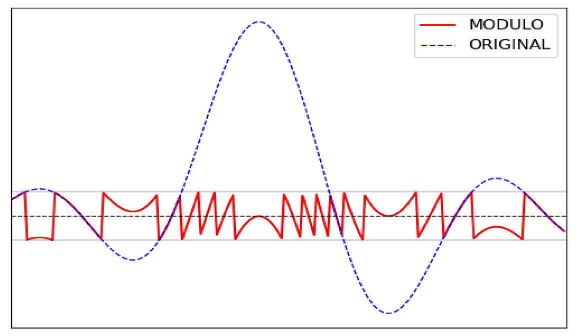


- Sampling is an important building block in an ADC. Typically, a band-limited signal is sampled at Nyquist rate, which is greater than twice the signal's maximum frequency component
- High sampling rate → Expensive and power consuming ADCs
- **Desirable**: High-DR-ADCs that operate at lower sampling rate
- This paper addresses the high DR issue of an ADC using the non-linear modulo operator
- We propose a fast and robust algorithm called LASSO-B^2R^2 for recovering bandlimited true samples from folding samples
- We show that the first-order difference of the residual samples is sparse and can be recovered from its partial Fourier measurements by formulating a sparse recovery problem
- We demonstrate that the proposed algorithm is robust to noise and computationally efficient compared to the existing methods

2. Problem Statement

Modulo Sampling:





$M_\lambda(a) = (a + \lambda) \bmod 2\lambda - \lambda$

$f_\lambda[n] = f[n] + z[n],\ z[n] \in 2\lambda\mathbb{Z}$

Existing Literature:

- **Higher-Order Differences (HODs) Based** [1]: Sensitive to noise and requires a sampling rate of $2\pi e$ times Nyquist rate
- **Prediction-Based** [2]: Improved upon HODs and required a sampling rate that is greater than Nyquist rate
- **Beyond Bandwith Residual Recovery (B^2R^2)** [3]: Improved upon prediction-based; requires a sampling rate slightly higher than Nyquist. Computationally expensive and not fast enough

[1] A. Bhandari et al. IEEE TSP 2020. [2] E. Romanov et al. IEEE SPL 2019. [3] E. Azar et al. ICASSP 2022.

R: Robust, fast, and operates at low sampling rate

3. Properties of Finite-Energy BL Signals

$f(t) \in \boldsymbol{L^2}(\mathbb{R}) \cap \boldsymbol{B}_{w_m}$: Finite-energy band-limited signal

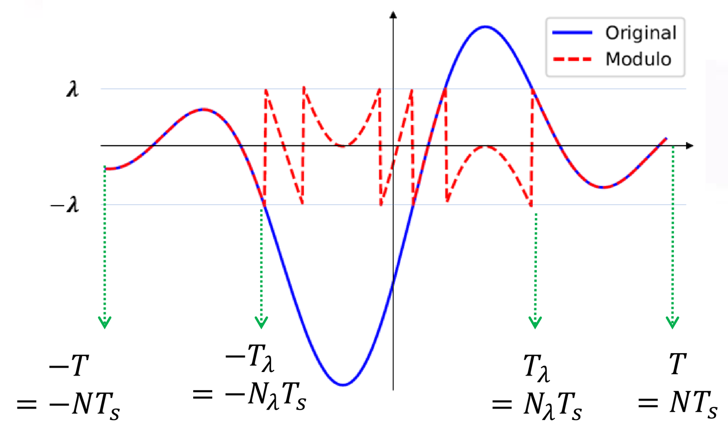
$f_\lambda(t) = f(t) + z(t)$ → ($t = nT_s$) → $f_\lambda[n] = f[n] + z[n],\ z[n] \in 2\lambda\mathbb{Z}$

Time-Domain Separation:

Riemann-Lebesgue Lemma: $\lim_{|t|\to\infty} f(t) = 0$

- $T = NT_s$: Covers 98 – 99% of energy of $f(t)$
- $T_\lambda = N_\lambda T_s$: $f_\lambda(t) = f(t), \forall\, t > T_\lambda$ and $f_\lambda[n] = f[n], \forall\, n > N_\lambda$

$z(t)/z[n]$: Finite T-duration/N-length signal

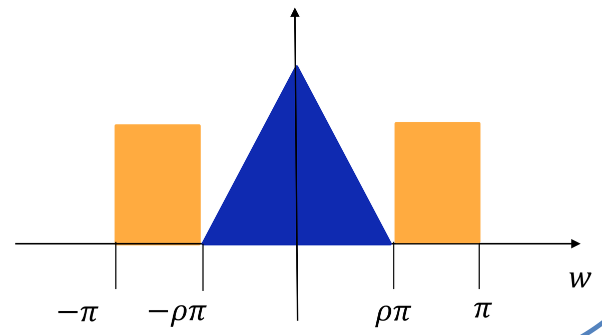


Frequency-Domain Separation:

$$Z(e^{jw}) = F_\lambda(e^{jw}), \forall\, \rho\pi < |w| < \pi$$

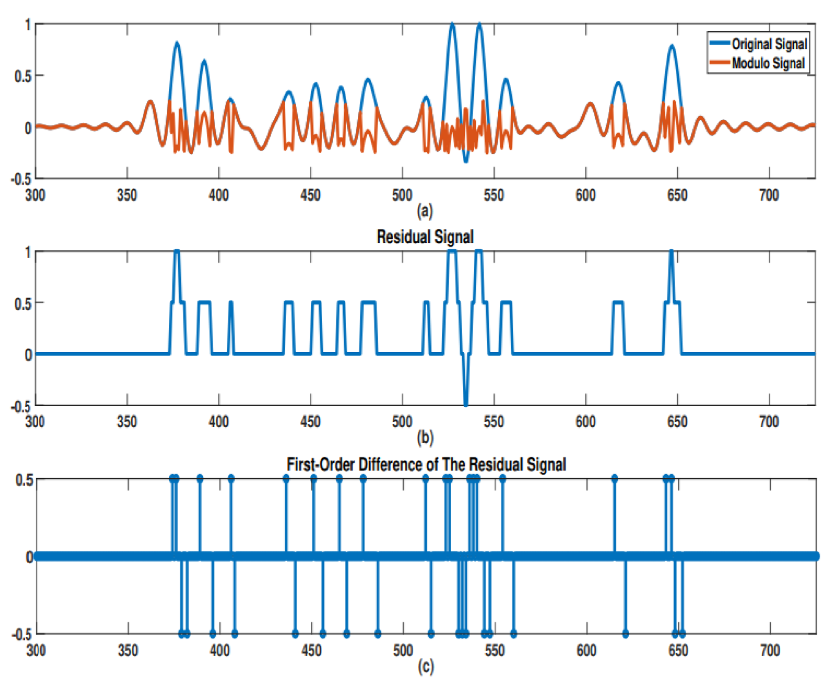
where $\rho = \frac{2w_m}{w_s} = \frac{1}{OF}$

OF : Over-sampling Factor



4. Observation

$\hat{f}_\lambda[n] = \hat{f}[n] + \hat{z}[n],\ \hat{z}[n] = \Delta z[n]$ **is sparse**



$$L \le \min\left(4K + 4K\left\lfloor\frac{\|f(t)\|_\infty - \lambda}{2\lambda}\right\rfloor, N\right),\ K = \left\lfloor\frac{N}{2OF}\right\rfloor$$

λ	OF = 4		OF = 6		OF = 8	
	$2K$	Upper bound on L	$2K$	Upper bound on L	$2K$	Upper bound on L
0.75	256	512	170	340	128	256
0.5	256	512	170	340	128	256
0.25	256	1024	170	680	128	512
0.05	256	1024	170	1024	128	1024

- In practice, the actual L value is far less than its upper bound

λ decreases → L increases

5. LASSO-B^2R^2

Time-Domain Separation

$\hat{z}[n]$ is a finite N-length signal

Fourier-Domain Separation:

$$\hat{F}_\lambda\left(e^{\frac{j2\pi k}{N}}\right) = \sum_{n=0}^{N-1} \hat{z}(n) e^{\frac{-j2\pi kn}{N}},\ \frac{2\pi k}{N} \in (\rho\pi, 2\pi - \rho\pi)$$

$$\left[\hat{\mathbf{F}}_\lambda\right]_{M\times 1} = [\mathbf{V}]_{M\times N}[\hat{\mathbf{z}}]_{N\times 1}$$

Observation

$\hat{z}[n]$ is a sparse signal

Algorithm 1 LASSO-B^2R^2 Algorithm

Input: $f_\lambda(n)$, λ, ρ, maxItr ▷ maxItr denotes the maximum number of iterations
Compute N
Construct $[\mathbf{V}]_{M\times N}$ using (4)
Compute $\hat{F}_\lambda(e^{\frac{j2\pi k}{N}}), \forall\, k \in \mathbb{Z}$ and $\frac{2\pi k}{N} \in (\rho\pi, 2\pi - \rho\pi)$
Initialize: $\gamma = 0.1\|\mathbf{V}^H\hat{\mathbf{F}}_\lambda\|_\infty$, $\tau = \frac{1}{\|\mathbf{V}\|_2^2}$, maxItr = 1000, and $\hat{\mathbf{z}}^{(0)} \in \mathcal{N}(0,1)$
for $i = 0$: maxItr **do**
  $\hat{\mathbf{z}}^{(i+1)} = S_{\gamma\tau}\left(\hat{\mathbf{z}}^{(i)} - \tau\mathbf{V}^H\left(\mathbf{V}\hat{\mathbf{z}}^{(i)} - \hat{\mathbf{F}}_\lambda\right)\right)$
  if $\|\hat{\mathbf{z}}^{(i+1)} - \hat{\mathbf{z}}^{(i)}\|_2 < 10^{-4}$ **then**
    $\hat{\mathbf{z}} = \hat{\mathbf{z}}^{(i+1)}$
    exit
  end if
end for
$\hat{\mathbf{z}} \leftarrow \lceil \frac{\lfloor \hat{\mathbf{z}}/\lambda \rfloor}{2} \rceil$ ▷ Rounding the residual to $2\lambda\mathbb{Z}$
$\mathbf{z} \leftarrow \text{cumsum}(\hat{\mathbf{z}})$ ▷ Cumulative summation operator on $\hat{\mathbf{z}}$
$f(n) \leftarrow f_\lambda(n) - z(n)$
Output: $f(n)$

LASSO-B^2R^2:

$$\min_{\hat{\mathbf{z}}} \frac{1}{2}\|\hat{\mathbf{F}}_\lambda - \mathbf{V}\hat{\mathbf{z}}\|_2^2 + \gamma\|\hat{\mathbf{z}}\|_1$$

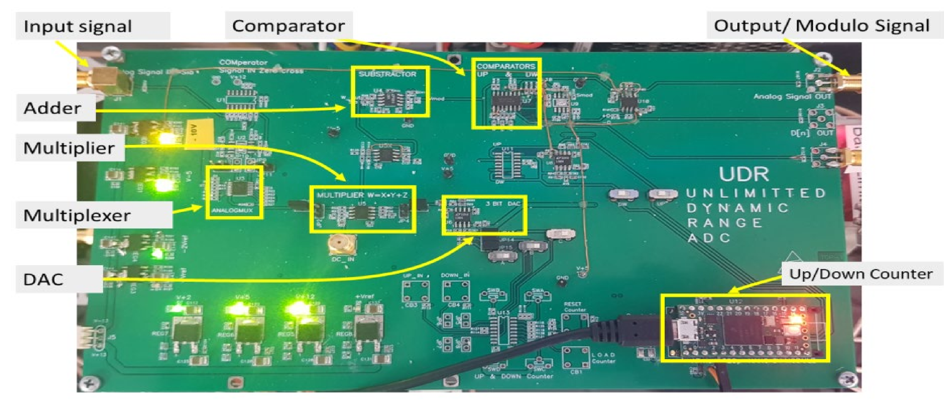
ISTA Algorithm:

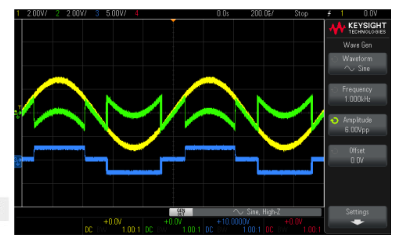
$$\hat{\mathbf{z}}^{(i+1)} = S_{\gamma\tau}\left(\hat{\mathbf{z}}^{(i)} - \tau\mathbf{V}^{\mathbf{H}}\left(\mathbf{V}\hat{\mathbf{z}}^{(i)} - \hat{\mathbf{F}}_\lambda\right)\right)$$

where $S_{\gamma\tau}(x) = \text{sign}(x)\max(|x| - \gamma\tau, 0)$

6. Results

Modulo Hardware Board →

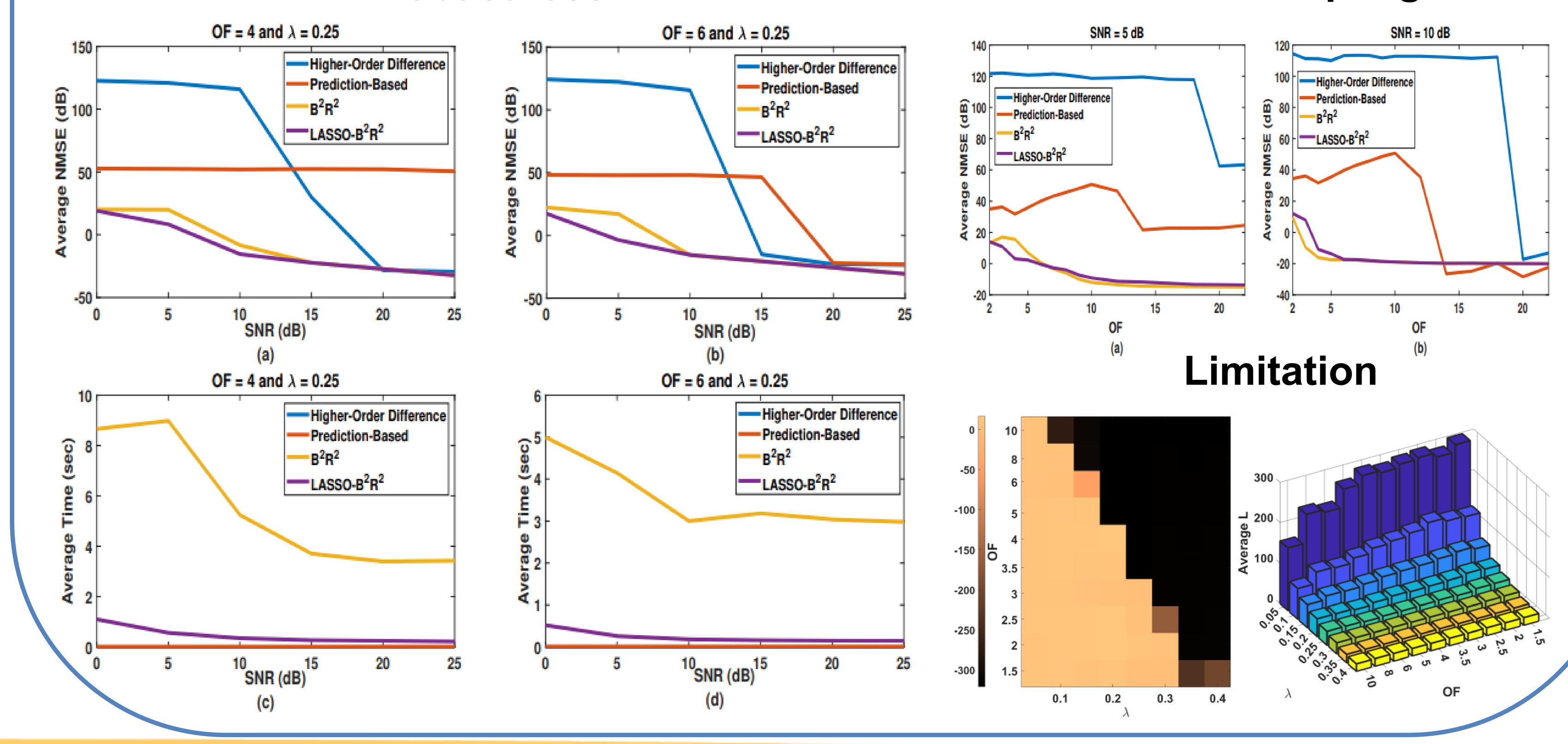




[4] S. Mulleti et al. IET 2023

Simulation:

Robustness **Over-Sampling** **Limitation**



7. Conclusions

- We proposed a fast and robust algorithm to recover the residual signal in modulo sampling
- We showed that the residual signal's first-order difference is a sparse vector
- Then, the problem of recovering the residual signal is formulated as a sparse recovery problem
- Through simulations, we demonstrated that the proposed algorithm is fast and robust compared to existing methods