

# Identifiability Conditions for Compressive Multichannel Blind Deconvolution

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**Abstract**—In applications such as multi-receiver radars and ultrasound array systems, the observed signals are often modeled as the convolution of the transmit pulse signal and a set of sparse filters representing the sparse target scenes. A sparse multichannel blind deconvolution (MBD) problem simultaneously identifies the unknown signal and sparse filters, which is in general ill-posed. In this paper, we consider the identifiability problem of sparse-MBD and show that, similar to compressive sensing, it is possible to identify the sparse filters from compressive measurements of the output sequences. Specifically, we consider compressible measurements in the Fourier domain and derive deterministic identifiability conditions. Our main results demonstrate that  $L$ -sparse filters can be identified from  $2L^2$  Fourier measurements from two or more coprime channels. We also show that  $2L$  measurements per channel are necessary. The sufficient condition sharpens as the number of channels increases and is asymptotically optimal, i.e., it suffices to acquire on the order of  $L$  Fourier samples per channel. We also propose a kernel-based sampling scheme that acquires Fourier measurements from a commensurate number of time-domain samples. The gap between the sufficient and necessary conditions is illustrated through numerical experiments including comparing practical reconstruction algorithms. The proposed compressive MBD results require fewer measurements and fewer channels for identifiability compared to previous results, which aids in building cost-effective receivers.

**Index Terms**—Sparse multichannel blind deconvolution, identifiability, deterministic sparsity model, subsampling, blind gain and phase calibration.

## I. INTRODUCTION

IN A wide range of applications, an unknown signal is observed through multiple channels. The output signal in each channel is given as the linear convolution of the unknown signal and the filter corresponding to the impulse response of the channel. The problem of identifying both the unknown signal

and the filters is known as multichannel blind deconvolution (MBD). In general, this problem is ill-posed. It can often be solved by imposing structural constraints on the source and the filters. In this paper, we consider the sparse-MBD problem, where the filters are assumed to be sparse.

Sparse-MBD models ubiquitously arise in many practical applications such as radar imaging [1], [2], seismic signal processing [3], room impulse response modeling [4], sonar imaging [5], and ultrasound imaging [6], [7], where a transmit signal is observed through multiple receivers after reflecting from sparsely located targets. The filters indicate the locations of the targets relative to the position of the receivers. Typically, the transmit signal is assumed to be known, however, in practice, it is often distorted during transmission and propagation [8]. Hence, the output signals from the receivers can be modeled well by the sparse MBD framework.

In the aforementioned applications, the implementation cost is determined by the number of receivers (or equivalently the number of channels) and the computational cost is governed by the length of the output sequences. Hence, it is desirable to solve the MBD problem from a minimal number of channels and minimal number of samples per channel. We study the problem of identifying the sparse filters from fewer measurements of the output sequences compared to their ambient dimension, which we call *compressive MBD*. We show that compressive MBD is possible by combining the deterministic MBD approach developed for the non-sparse case [9] and the sparse signal identifiability results from the compressive sensing framework [10], [11].

Blind deconvolution of linear measurements suffers from shift and scaling ambiguity. Despite redundant observations from multiple channels, this fundamental ambiguity still remains in MBD. Therefore, even in the absence of measurement noise, the source and filters can be exactly determined only up to this fundamental ambiguity class, which is referred to as the identifiability of MBD. Determining a set of conditions for identifiability is fundamental since it governs the performance of any method regardless of noise. In the literature, identifiability has been studied by imposing structural assumptions on the source and filters in MBD.

During the 90s various practical reconstruction algorithms for MBD have been studied, largely in the context of blind channel identification, where the goal is to uniquely identify the filters [9], [12]–[17]. These methods are classified as statistical [12]–[15] or deterministic [9], [16] (cf. [17] for a comprehensive review of classical MBD approaches). The statistical

Manuscript received December 31, 2019; revised May 27, 2020 and July 11, 2020; accepted July 21, 2020. Date of publication July 24, 2020; date of current version August 20, 2020. The associate editor coordinating the review of this manuscript and approving it for publication was Dr. Y. Xia. This work was supported in part by the Benoziyo Endowment Fund for the Advancement of Science, Estate of Olga Klein – Astrachanthe; European Union's Horizon 2020 research and innovation program under Grant 646804-ERC-COG-BNYQ and in part by the Israel Science Foundation under Grant 0100101. The work of Kiryung Lee was supported in part by the NSF under Grant CCF 17-18771. (Corresponding author: Satish Mulleti.)

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Digital Object Identifier 10.1109/TSP.2020.3011804

approaches estimate the filters from second-order statistics of the output sequences. The performance critically depends on accurate estimate of the second-order statistics, which is not available when the source varies fast over time or the observation period is limited. In contrast, the deterministic approaches are free from the need of this expensive estimate.

Our study of compressive-MBD is motivated by the deterministic approach of Xu *et al.* [9], who considered least-squares approach to MBD from the outputs of a deterministic source through multiple FIR channels of order  $M_x$  over a finite interval. They derived identifiability assuming at least  $3M_x$  consecutive time samples under the following assumptions: i) the filters are coprime, i.e., their  $z$ -transform do not share any common zeros except the zeros at  $z = 0$ ; ii) the linear complexity of the source, within the observation interval, is greater than  $2M_x$ . The linear complexity of any sequence is a measure of its predictability and is given by the minimum number of exponentials which the signal consists of. We provide an improvement over their result so that identifiability is achieved from fewer time samples ( $2M_x - 1$ ) and under a less stringent condition on the source.

During the last decade, there has been renewed interest in MBD and, particularly in blind gain and phase calibration (BGPC) problems with sparsity and subspace constraints (e.g., [18]–[30]). BGPC is a bilinear inverse problem arising in multi-sensor systems, where the objective is to determine the unknown gains and phases of the sensors as well as the unknown signals from the sensor outputs. Multichannel blind deconvolution is a special case of BGPC in the Fourier domain. In this case, the unknown gains and phases correspond to the Fourier coefficients of the common source sequence, and the unknown signals are the Fourier transforms of the filters. Recent results analyzed the case of uniform samples in the Fourier domain under the assumption that the sparse filters are random or generic [20]–[23]. In these works, it is assumed that the output sequences are obtained by the circular convolution between the source and the sparse filters. Assuming that all the output samples are available and the filters are random and sparse, identifiability results have been derived in terms of a sufficient number of channels. Specifically, in [22] and [23], it is shown that the sparse MBD problem is uniquely identifiable provided that the filter coefficients are modeled as independent and identically distributed Bernoulli-Gaussian random variables and all  $M$  output samples are available from  $N = \mathcal{O}(M \log^4 M)$  channels. In addition, they have shown that sparse MBD and the corresponding BGPC problem can be solved by a practical algorithm [22], [26]. However, there are limitations in existing identifiability results on sparse-MBD [21]–[23] from the following perspectives:

- The methods therein require access to the entire output sequences whereas only partial observation is available in compressive-MBD.
- The identifiability result holds under a conservative sufficient condition that the number of channels is at least proportional to the signal length. Empirically sparse-MBD can be solved with fewer channels (even from only two channels).

- The specific stochastic models therein do not describe the filters arising in practical applications.

Our main results provide a set of identifiability conditions for compressive-MBD that overcome the aforementioned limitations. In particular we show that the simultaneous recovery of the sparse filters and the source is possible from a small number of linear measurements. Upper and lower bounds on a sufficient number of measurements for identifiability are derived under a set of deterministic conditions, which are satisfied by instances of several practical applications. Importantly, our identifiability results are purely deterministic in the sense that they apply to any instance satisfying given conditions. This is in contrast to generic identifiability results in the literature apply except for a null set, which is of zero measure. Unfortunately, this null set of undesirable instances is typically not identified and can include signals of interest.

We consider the following two scenarios of compressive-MBD. The first scenario aims to identify only the sparse filters in the context of radar and sonar applications, where the filters contain useful information about the targets. We combine the ideas of Xu *et al.* [9] and compressive sensing to derive identifiability results. Via a cross-convolution approach, the identifiability problem reduces to recovery of a superposition of convolutions of sparse filters from their partial Fourier measurements. Our main results follow by applying the full spark property of partial Fourier matrices and the coprimeness of the filters. We show that  $2L^2$  Fourier measurements per channel are sufficient for the identifiability whenever the number of channels  $N$  is at least 2. We also show that  $2L$  measurements per channel is necessary for identifiability. Moreover, asymptotically in  $N$ , i.e., with a very large number of distinct channels, we show that on average  $\mathcal{O}(L)$  measurements per channel are sufficient for identifiability.

Second, we consider the simultaneous identification of both the source and filters. We propose a pairwise measurement strategy, where we consider different Fourier measurements from each unique pair of channels. The identifiability of this approach follows from the results in the first scenario. We show that  $2L^2$  Fourier measurements are sufficient from  $\max\{\frac{M_s - L^2 - 1}{L^2 - 1}, 2\}$  channels and  $2L$  from the rest, where  $M_s$  is the source length. Additionally, we provide analogous identifiability results from time-domain observations based on a sampling-kernel-based technique [6], [31].

In the special case where the filters are FIR without further structure, our results provides an improvement over the prior identifiability result by Xu *et al.* [9]. First, our frequency-domain approach requires fewer measurements. Second, our approach is guaranteed when the Fourier transform of the source signal does not vanish at the observed frequencies, which is a milder condition than the condition on the linear complexity of the source [9].

In addition to the theoretical guarantees, we also discuss practical algorithms to identify the source and the filters in the two-channel case. In particular, we consider the truncated power iteration method [26] and blind dictionary calibration (BDC) methods [30] to discuss the gap between the necessary and sufficient conditions through simulations. In addition, for completion

of the experimental analysis, we also consider identifiability in the presence of measurement noise. In the presence of noise, several algorithms are presented. For example, Rietsch [32] proposed an algorithm based on combination of eigenvectors related to homogeneous equation in the FIR case, in [33], [34], the authors considered sparsity priors for sparse MBD problem, and in [35], [36] structures total least squares is applied for a single channel blind deconvolution. However, since the main objective of this paper is to establish the identifiability results for sparse MBD problem from compressive measurements in the absence of measurement noise and not analyzing different algorithm, we analyzed the noise case through algorithms in [26] and [30].

The rest of the paper is organized as follows. In the next section, we present the problem formulation along with relevant mathematical preliminaries. In Section III, we show how to achieve compressive Fourier measurements from a finite set of time-domain samples. Identifiability results for compressive MBD are discussed in Section IV. In Section V, we present a detailed comparison of the proposed results with the recent sparse and classical non-sparse results. Simulations are provided in Section VI followed by the proof of the main results in Section VII.

Throughout the paper, we use the following notations. For a positive integer  $M$ , let  $[M]$  denote the set  $\{0, 1, \dots, M-1\}$ . For a sequence  $x$ , its support denoted by  $\text{supp}(x)$  is defined by  $\{k \in \mathbb{Z} | x[k] \neq 0\}$ . The  $\ell_0$  pseudo-norm of  $x$ , denoted  $\|x\|_0$ , counts the number of its nonzero elements. The  $z$ -transform and the discrete-time Fourier transform (DTFT) of  $x$  are written as  $X(z)$  and  $X(e^{j\omega})$  respectively. Pointwise multiplication is denoted as  $\odot$ .

## II. COMPRESSIVE MBD FROM FOURIER MEASUREMENTS

In this section, we formulate the compressive MBD problem and discuss the assumptions made in deriving the main results.

### A. Problem Statement

Let  $y_1, \dots, y_N$  denote multichannel output sequences from a common source sequence  $s$ . Let  $x_1, \dots, x_N$  denote the filters corresponding to the impulse responses of the channels. Then

$$y_n = s * x_n, \quad n = 1, \dots, N, \quad (1)$$

where  $*$  denotes linear convolution. The MBD problem is to identify  $s$  and  $\{x_n\}_{n=1}^N$  from the output sequences  $\{y_n\}_{n=1}^N$ .

Let

$$\mathcal{C}(s, \{x_n\}_{n=1}^N) = \left\{ \left( \alpha^{-1} \mathcal{S}_{-m}(s), \{\alpha \mathcal{S}_m(x_n)\}_{n=1}^N \right) \mid \alpha \neq 0, m \in \mathbb{Z} \right\} \quad (2)$$

denote the orbit of  $(s, \{x_n\}_{n=1}^N)$  by the actions of shift and scaling, where  $\mathcal{S}_m$  denotes a shift operator that maps a sequence to its shifted version by  $m$  samples. Then any element in  $\mathcal{C}(s, \{x_n\}_{n=1}^N)$  generates the same output sequences. Therefore, we aim to identify  $(s, \{x_n\}_{n=1}^N)$  up to the *fundamental-ambiguity class* given by (2), that is, to find any element in

$\mathcal{C}(s, \{x_n\}_{n=1}^N)$  that satisfies (1). Unique identification hereafter will be referred to as this case.

When all time-samples of  $y_n$  are available, one can uniquely identify the filters (resp. both the source and the filters) by the method of Xu *et al.* [9] (resp. recent sparse MBD methods by [21]–[23]) provided that the assumed conditions on  $s$  and  $\{x_n\}_{n=1}^N$  therein are satisfied (cf. Section V for details).

The main question of our interest is whether one can deconvolve  $s$  and  $\{x_n\}_{n=1}^N$  from compressive measurements of  $\{y_n\}_{n=1}^N$ , which will be referred to as compressive MBD.

We specifically consider partial Fourier measurements given as the DTFT of  $y_n$  at selected frequencies, that is, the set of linear measurements is written as

$$\{Y_n(e^{jk\omega_0}) \mid k \in \mathcal{K}_n, n = 1, \dots, N\}, \quad (3)$$

where  $\mathcal{K}_n$  denotes the index set  $\{m_{n,1}, m_{n,2}, \dots, m_{n,K_n}\} \subset \mathbb{Z}$  for  $n = 1, \dots, N$ . We are particularly interested in the setting where  $|\mathcal{K}_n| \ll M$  where  $M$  is the length of  $y_n$ .

Our choice of partial Fourier measurements is motivated by the following two reasons. First, in the Fourier-domain, the measurements in (3) are written as the product of the Fourier transforms of the source and the filters, that is,

$$Y_n(e^{jk\omega_0}) = S(e^{jk\omega_0})X_n(e^{jk\omega_0}), \quad k \in \mathcal{K}_n, n = 1, \dots, N. \quad (4)$$

The entrywise product form in (4) allows the flexibility to design sampling patterns. For any  $n_1 \neq n_2$  and  $k \in \mathcal{K}_{n_1} \cap \mathcal{K}_{n_2}$ , we have

$$Y_{n_2}(e^{jk\omega_0})X_{n_1}(e^{jk\omega_0}) = Y_{n_1}(e^{jk\omega_0})X_{n_2}(e^{jk\omega_0}),$$

which enables applying the cross-convolution approach by Xu *et al.* [9] even with sampling in the Fourier domain for any  $\{\mathcal{K}_n\}_{n=1}^N$ . Second, due to the uncertainty principle, the Fourier transforms of the filters are well spread in the Fourier domain as they are sparse in the time domain. This helps recover the filters from fewer Fourier coefficients.

In order to uniquely identify the solution to compressive MBD, we impose the following structural assumptions on the filters  $\{x_n\}_{n=1}^N$  and the source  $s$ :

- A1) **Sparse filters:**  $\|x_n\|_0 \leq L$  and  $\text{supp}\{x_n\} \subset [M_x]$  for  $n = 1, \dots, N$ .
- A2) **Finite-length source:**  $s$  is supported within  $[M_s]$ .
- A3) **Coprime filters:**  $X_1(z), \dots, X_N(z)$  do not share any common zeros except at  $z = 0$ .
- A4) **Non-vanishing source:**  $S(e^{jk\omega_0}) \neq 0$  for all  $k \in \bigcup_{n=1}^N \mathcal{K}_n$ .
- A5) **Universal sampling:** The sampling interval  $\omega_0$  and index sets  $\{\mathcal{K}_n\}_{n=1}^N$  form a universal sampling set. Such sets are defined in Section II-B.

A few remarks on these assumptions are in order.

- (A1) and (A2) imply that each  $y_n$  is supported on  $[M]$ , where  $M = M_x + M_s - 1$ .
- (A2) is not necessary if one only concerns the identification of the filters.
- (A3) is a necessary condition for unique identifiability without (A1) and (A2). See Section II-C for more details.
- (A4) avoids the case where the Fourier measurements at a sampled frequency are zero for all channels. The condition



is also satisfied for a bandlimited source. For example, consider a source that is bandlimited to a frequency interval  $[-\omega_s, \omega_s]$ . If we choose  $\bar{\mathcal{K}} = \cup_{n=1}^N \mathcal{K}_n$  as a set of consecutive integers and set  $\omega_0$  to  $\min\{2\omega_s/|\bar{\mathcal{K}}|, 2\pi/\bar{M}\}$  so that  $k\omega_0 \in [-\omega_s, \omega_s]$  for  $k \in \bar{\mathcal{K}}$  then (A4) is satisfied.

- (A5) enables to solve sparse MBD from compressive Fourier measurements.

Unique identification in compressive MBD is then defined as follows.

**Definition 1 (Identifiability of Compressive MBD):** Compressive MBD is uniquely identifiable if any feasible solution  $(\hat{s}, \{\hat{x}_n\}_{n=1}^N)$ , which satisfies (A1)–(A4) and is consistent with the measurements in (3), belongs to the fundamental ambiguity class  $\mathcal{C}(s, \{x_n\}_{n=1}^N)$  of the ground-truth signals  $(s, \{x_n\}_{n=1}^N)$  defined in (2).

Our objective is to derive necessary and sufficient conditions on the index sets  $\{\mathcal{K}_n\}_{n=1}^N$  such that either only the filters  $\{x_n\}_{n=1}^N$  or both the source  $s$  and the filters are uniquely identifiable according to Definition 1 under (A5).

Simultaneous identification of both the source and the filters requires extra conditions, which go beyond universal sampling (see Section IV-B). We first derive conditions for the unique identification of the filters followed by those extra conditions for the identification of the source signal. Note that recovery of the filters is equivalent to the simultaneous recovery of both the filters and the DTFT of the source signal at the observed frequencies. Therefore partial identifiability is as follows.

**Definition 2 (Partial Identifiability Only for Filters):** Compressive MBD is partially identifiable if for any feasible solution  $(\hat{s}, \{\hat{x}_n\}_{n=1}^N)$ , there exists  $\tilde{s}$  such that

- 1)  $\hat{S}(e^{jk\omega_0}) = \hat{S}(e^{jk\omega_0})$  for  $k \in \cup_{n=1}^N \mathcal{K}_n$ .
- 2)  $(\tilde{s}, \{\hat{x}_n\}_{n=1}^N) \in \mathcal{C}(s, \{x_n\}_{n=1}^N)$ .

### B. Universal Sets

Universal sampling sets have been introduced for compressed sensing from partial Fourier measurements (e.g., see [10, Def. 14.1]). The partial Fourier measurement matrix corresponding to the  $n$ th channel measurements in (3) is given as a  $|\mathcal{K}_n| \times \bar{M}$  Vandermonde matrix  $\mathbf{V}_n$  whose  $(k, m)$ th element is  $e^{jk m \omega_0}$ . In our settings,  $\bar{M} \geq |\mathcal{K}_n|$ , where  $\bar{M} = \max\{2M_x - 1, M_s\}$  (see Section IV-B for details). To avoid aliasing,  $\omega_0$  is chosen such that the elements of the set  $\{e^{jm\omega_0}\}_{m=0}^{\bar{M}-1}$  are distinct. For each  $n \in \{1, \dots, N\}$ , the index set  $\mathcal{K}_n$  is called *universal* if every submatrix of  $\mathbf{V}_n$  obtained by taking  $|\mathcal{K}_n|$  columns has full rank [10, Def. 14.1], that is,  $\mathbf{V}_n$  has full spark [11]. Note that the universal sets depend on the frequency interval  $\omega_0$ .

For example, if  $\mathcal{K}_n$  is a set of consecutive integers and  $\omega_0 = 2\pi/\bar{M}$  then each index set  $\mathcal{K}_n$  is universal. Various alternative constructions of universal sets have been studied (e.g., [37]–[40], also see [10]).

### C. Coprimeness of the Filters

In an MBD framework, unless any further restrictions are imposed on the supports of the source and the filters, the coprime condition on the filters is necessary for unique identification of the solution. To elaborate, let the filters  $\{x_n\}_{n=1}^N$  share nontrivial

common zeros in the  $z$ -domain. Specifically, each filter can be decomposed as

$$x_n = h_0 * \hat{x}_n, \quad (5)$$

where the sequence  $h_0$  contains the common zeros except at  $z = 0$  and  $\hat{x}_n$  is the novel factor. As a result, the outputs  $\{y_n = s * x_n\}_{n=1}^N$  can also be decomposed as

$$y_n = s * x_n = s * h_0 * \hat{x}_n = \hat{s} * \hat{x}_n, \quad (6)$$

where  $\hat{s} = s * h_0$  and  $\{\hat{x}_n\}_{n=1}^N$  provide an alternative solution that produces the same outputs. Hence, without any assumptions on the source and the filters, coprimeness is a necessary condition. With (A1) and (A2), it is no longer a necessary condition as the alternative solution may not satisfy these assumptions. However, we keep the coprimeness assumption in our settings to derive the identifiability conditions.

## III. COMPRESSIVE MBD FROM TIME DOMAIN MEASUREMENTS

In this section we propose a method that acquires the compressive Fourier measurements of an FIR sequence without explicitly observing the entire sequence. The number of time samples can be as small as the number of Fourier measurements. Our approach is inspired by the kernel-based sampling and reconstruction approach for finite-rate-of-innovation (FRI) signals [6], [31]. It has been shown that Fourier measurements of FRI signals can be computed from time samples at a sub-Nyquist rate by applying a suitable sampling kernel. Then the parameters of FRI signals can be computed from Fourier measurements on a grid.

Let  $y$  be an FIR signal supported on  $[M]$  and  $\mathcal{K} \subset \mathbb{Z}$  be a finite set. The DTFT coefficients  $Y(e^{jk\omega_0})$  of  $y$  for  $k \in \mathcal{K}$  are computed from  $|\mathcal{K}|$  consecutive time samples of  $y$  with an appropriate sampling kernel.

Let  $h$  be an FIR filter supported on  $[M_h]$ , where  $M_h > M$ , which satisfies

$$h[m] = \sum_{k \in \mathcal{K}} e^{jk\omega_0 m}, \quad m \in [M_h]. \quad (7)$$

The filter  $h$  in (7) is a discrete-time analog of the sum-of-sincs (SOS) filter proposed by Tur *et al.* [6]. The filtered version of  $y$  by  $h$ , denoted  $\bar{y}$ , satisfies

$$\bar{y}[m] = \sum_{p=0}^{M-1} y[p] h[m-p] = \sum_{k \in \mathcal{K}} e^{jk\omega_0 m} Y(e^{jk\omega_0}) \quad (8)$$

for  $M-1 \leq m \leq M_h-1$ . Then the vectors respectively containing  $\{Y(e^{jk\omega_0})\}_{k \in \mathcal{K}}$  and  $\{\bar{y}[m]\}_{m=M-1}^{M_h-1}$  are related through a Vandermonde matrix  $\mathbf{V}$  of size  $(M_h - M + 1) \times |\mathcal{K}|$  with its  $(m, k)$ th element given as  $e^{jm k \omega_0}$ . Therefore if  $\omega_0$  and  $\mathcal{K}$  form a universal set then the condition  $M_h \geq M + |\mathcal{K}| - 1$  ensures that  $\{Y(e^{jk\omega_0})\}_{k \in \mathcal{K}}$  are computed uniquely from  $\{\bar{y}[m]\}_{m=M-1}^{M_h-1}$ . In other words,  $|\mathcal{K}|$  observations of the  $\bar{y}$  are sufficient to compute  $|\mathcal{K}|$  Fourier measurements of  $y$  without observing the entire sequence.

A schematic of the sampling mechanism is shown in Fig. 1. The switch is closed for samples at  $m \in \{M-1, \dots, M_h-1\}$ .

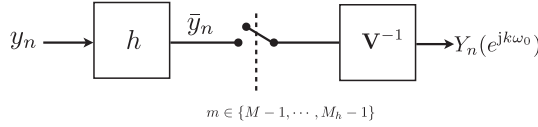


Fig. 1. Compressive measurements of Fourier samples of  $y_n$  by using discrete-SOS filter  $h$ : the sequence  $y_n$  of dimension of  $M$  is passed through an FIR kernel of length  $M_h$  defined as in (7). The output sequence  $\bar{y}_n$  has length  $M_h + M - 1$  out of which few measurements are taken by closing the switch at sample indices  $m \in \{M-1, \dots, M_h-1\}$ . From the truncated samples  $Y_n(e^{jk\omega_0})$  is computed by inverting a linear system of equations governed by matrix  $V$  considered in Section II-B.

If we further assume that  $\mathcal{K}$  is a universal set, any  $|\mathcal{K}|$  time samples of  $\bar{y}$  from  $[M-1, M_h-1]$  will make the resulting matrix  $V$  full rank. This enables optimizing the sampling pattern to improve the condition number of  $V$ .

#### IV. IDENTIFIABILITY OF COMPRESSIVE MBD

We consider a two-step approach to the compressive MBD problem. The first step identifies only the filters corresponding to the impulse responses of the channels, similarly to blind channel estimation in communications (e.g., [9], [16]). Once the filters are identified, the second step reconstructs the common input source to the channels.

##### A. Identifying Sparse Filters

We identify the filters from the Fourier measurements in (4) under assumptions (A1)–(A5) except (A2). The following theorem presents the result in the two-channel case.

**Theorem 1 (Partial Identifiability of Compressive MBD):** Suppose that (A1), (A3), (A4), and (A5) hold with  $\mathcal{K}_1 = \mathcal{K}_2 = \mathcal{K}$  and  $L < \sqrt{M_x}$ .

- 1) If  $|\mathcal{K}| \geq 2L^2$ , then compressive MBD is partially identifiable from the Fourier measurements according to Definition 2.
- 2) If  $|\mathcal{K}| < 2L$ , then compressive MBD is not partially identifiable.

Recall that  $\{S(e^{j\omega_0 k})\}_{k \in \mathcal{K}}$ ,  $x_1$ , and  $x_2$  were arbitrary in Definition 2. Therefore, the partial identifiability in Theorem 1 implies that for any instance within the assumed model, the filters are uniquely identified up to the ambiguity class. When it is not partially identifiable, there exists an instance where the filters are not uniquely determined.

Next, by combining the Fourier domain identifiability results of Theorem 1 with the kernel-based measurement scheme proposed in Section III, we obtain the following corollary, which provides the analogous results for compressive MBD from time-domain measurements.

**Corollary 1 (Time-Domain Compressive MBD):** Suppose that the hypotheses of Theorem 1 and (A2) hold. Let  $h$  be an FIR filter of length  $M + |\mathcal{K}| - 1$  with impulse response defined in (7). Then compressive MBD is partially identifiable from  $|\mathcal{K}|$  consecutive time-samples of  $y_1 * h$  and  $y_2 * h$ , where the samples are indexed by the set  $\{M-1, \dots, M + |\mathcal{K}| - 2\}$ , if

$|\mathcal{K}| \geq 2L^2$ . On the other hand, compressive MBD is not partially identifiable if  $|\mathcal{K}| < 2L$ .

The time-domain results are similar to the result in the context of FRI signal sampling where the number of Fourier coefficients needed for identification governs the desired number of time samples and the sampling rate [6].

The proof of Theorem 1 directly follows from the results on the following feasibility problem:

$$\begin{aligned} & \text{find } \{\tilde{S}(e^{j\omega_0 k})\}_{k \in \cup_n \mathcal{K}_n} \text{ and } (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_N) \\ & \text{s.t. } Y_n(e^{j\omega_0 k}) = \tilde{S}(e^{j\omega_0 k}) \tilde{X}_n(e^{j\omega_0 k}), k \in \mathcal{K}_n, \forall n, \end{aligned} \quad (9)$$

(A1), (A3), (A4) and (A5) are satisfied,

where the tilde is used to distinguish the variables of the problem from the corresponding ground-truth signals. If the solution to (9) is unique up to a scaling and shift ambiguity, then  $(\{S(e^{j\omega_0 k})\}_{k \in \cup_{n=1}^N \mathcal{K}_n}, \{x_n\}_{n=1}^N)$  is uniquely identifiable, up to a shift and scaling ambiguity, from the measurements in (4). The following lemma, whose proof is deferred to Section VII, provides necessary and sufficient conditions for the uniqueness of the feasibility problem in (9) when  $N = 2$ .

**Lemma 1:** Let  $\omega_0, \mathcal{K}_1, \mathcal{K}_2$ , and  $\mathcal{K}$  be as in Theorem 1. If  $|\mathcal{K}| \geq \min\{2L^2, 2M_x - 1\}$ , then the feasibility problem in (9) has a unique solution. On the other hand, if  $|\mathcal{K}| < \min\{2L, M_x\}$ , then the problem in (9) is not uniquely identifiable.

The ground-truth signals  $(\{S(e^{j\omega_0 k})\}_{k \in \mathcal{K}_1 \cup \mathcal{K}_2}, \{x_1, x_2\})$  are feasible to (9). Let

$$q = x_1 * \hat{x}_2 - x_2 * \hat{x}_1, \quad (10)$$

where  $(\{\hat{S}(e^{j\omega_0 k})\}_{k \in \mathcal{K}}, \hat{x}_1, \hat{x}_2)$  is another feasible solution to (9). In the proof of Lemma 1, we have shown that the number of Fourier measurements for the unique identification is determined as the worst-case  $\|q\|_0$  maximized over all feasible  $(\hat{x}_1, \hat{x}_2)$ . Since  $x_1, x_2, \hat{x}_1$ , and  $\hat{x}_2$  are  $L$ -sparse vectors with support over  $[M_x]$ , in general, the worst case support of  $q$  is  $\min\{2L^2, 2M_x - 1\}$ . For high sparse signals, that is, when  $L \ll M_x$  or  $L < \sqrt{M_x}$ , we have that  $\|q\|_0 = 2L^2$ . Therefore by modifying constraints on  $\tilde{x}_1$  and  $\tilde{x}_2$  in (9), we obtain similar results in a different scenario as an immediate corollary.

**Corollary 2 (Sufficient Conditions for General Sparsity Case):** Let  $\omega_0, \mathcal{K}_1, \mathcal{K}_2$ , and  $\mathcal{K}$  be as in Theorem 1. Let  $L < M_x$  without the restriction  $L < \sqrt{M_x}$ . If  $|\mathcal{K}| \geq \min\{2L^2, 2M_x - 1\}$ , then the feasibility problem in (9) has a unique solution. On the other hand, if  $|\mathcal{K}| < \min\{2L, M_x\}$  the problem in (9) is not uniquely identifiable.

Without assuming that  $x_1$  and  $x_2$  are sparse, the worst-case  $\|q\|_0$  becomes  $2M_x - 1$ , which results in the following corollary.

**Corollary 3 (Non-Sparse FIR Filters):** Assume the hypotheses of Theorem 1. Let  $L = M_x$ . Then the same identifiability result as in Theorem 1 holds iff  $|\mathcal{K}| \geq 2M_x - 1$ .

Compared with the results of [9], where  $3M_x$  time-samples are sufficient to identify the filters, in our frequency-domain approach  $2M_x - 1$  Fourier measurements are necessary and sufficient. Further, the Fourier measurements can be computed uniquely from  $2M_x - 1$  time-measurements by using a sampling kernel as shown in Section III. Hence, for a large  $M_x$ , we

gain significantly in terms of reducing the number of measurements compared with the approach in [9]. A detailed comparison of these methods is presented in Section V.

Next we show how the identifiability result in Theorem 1 for the two-channel case generalizes to the case of more than two channels. We obtain a particular sufficient condition by assuming that there exists a pair of coprime channels from which at least  $2L^2$  Fourier measurements are available.

**Theorem 2:** Suppose (A1), (A3), (A4), and (A5) hold for  $N \geq 2$ . Then compressive MBD is partially identifiable from the Fourier measurements if the following conditions are satisfied: i) There exist  $1 \leq n_1 < n_2 \leq N$  such that  $\mathcal{K}_{n_1} = \mathcal{K}_{n_2} = \mathcal{K}$  for a universal set  $\mathcal{K}$  with  $|\mathcal{K}| \geq \min\{2L^2, 2M_s - 1\}$ ; ii)  $\mathcal{K}_n$  is a universal set such that  $\mathcal{K}_n \subseteq \mathcal{K}$  and  $|\mathcal{K}_n| \geq 2L$  for all  $n \notin \{n_1, n_2\}$ .

*Proof:* By Theorem 1, the triplet  $(\{S(e^{jk\omega_0})\}_{k \in \mathcal{K}}, x_{n_1}, x_{n_2})$  is uniquely identified. Then for an appropriate choice of the sets  $\mathcal{K}_n$ , the recovery of the filters for the rest of the channels (those indexed by  $n \notin \{n_1, n_2\}$ ) reduces to a non-blind problem. Since  $Y_n(e^{jk\omega_0}) = S(e^{jk\omega_0})X_n(e^{jk\omega_0})$ , if the sets  $\mathcal{K}_n$  are chosen such that  $\mathcal{K}_n \subseteq \mathcal{K}_{n_1}$ , then we can compute  $\{X(e^{jk\omega_0})\}_{k \in \mathcal{K}_n}$  from  $\{Y_n(e^{jk\omega_0})\}_{k \in \mathcal{K}_n}$  as  $\{S(e^{jk\omega_0})\}_{k \in \mathcal{K}_{n_1}}$  is already identified. Next, from the measurements  $\{X(e^{jk\omega_0})\}_{k \in \mathcal{K}_n}$ , the filters are identifiable uniquely if  $|\mathcal{K}_n| \geq 2L$ . ■

Theorem 2 implies that a total  $4L^2 + (N - 2)2L$  Fourier measurements from  $N$  channels are sufficient for unique identification of the filters. Therefore on average it suffices to take  $(4L^2 + (N - 2)2L)/N$  measurements per channel. Particularly, with sufficiently many channels ( $N \gtrsim L$ ), the average number of measurements per channel is on the order of  $L$  which is the same as required by the necessary condition. Note that when the source is known, a minimum of  $2L$  Fourier measurements are necessary to uniquely identify the filters. Hence, for  $N \gtrsim L$  the requirement on the average number of measurements per channel matches that of the known source case in order.

### B. Recovering the Common Source Signal

As discussed in the introduction, in certain applications such as radar, sonar, and ultrasound, it suffices to identify only the filters, which describe the target. On the other hand, there exist applications where the recovery of the source signal is important. For example, in communications or imaging, the source signal carries information and the filters describe the channel impulse response or sensitivity functions. In this section, we present conditions under which the source and filters are simultaneously identified.

The results in Theorems 1 and 2 guarantee that the filters are fully identified but the source signal is partially identified up to its Fourier measurements at the selected frequencies. In general, the recovery of the source  $s$  from its partial Fourier measurements is ill-posed. However, recovery becomes feasible by introducing further restrictions on  $s$ .

For example, suppose that  $s$  is supported within  $[M_s]$  for a finite integer  $M_s > 0$ . To uniquely determine an arbitrary source signal  $s$  supported within  $[M_s]$ , the number of measurements needs to be at least  $M_s$ . On the other hand, since we choose  $\omega_0$  and  $\mathcal{K}$  such that the elements in  $\{e^{jk\omega_0}\}_{k=0}^{M_s-1}$  are distinct, the

linear system that generates the Fourier measurements at  $\mathcal{K}\omega_0$  corresponds to a Vandermonde matrix of full column rank and  $s$  is uniquely determined.

Combining the above argument with Theorem 1 provides the following result in the two-channel case: All  $s$ ,  $x_1$ , and  $x_2$  are identified from the sampling pattern given by  $\mathcal{K}_1 = \mathcal{K}_2 = \mathcal{K}$  if and only if  $|\mathcal{K}| \geq \max\{M_s, 2L^2\}$  and (A5) is satisfied.

Below we show that when there are more than two channels, with a carefully designed sampling pattern, one can significantly reduce the peak number of measurements per channel, where the gain is almost proportional to the number of channels. This is interesting, particularly when  $M_s$  dominates  $L$ , that is,  $M_s \gg 2L^2$ .

A naïve approach is to consider  $M_s$  Fourier measurements from any pair of channels and to apply Theorem 1 to identify the corresponding filters and the source  $s$ . As the source is identified, the problem is reduced to a non-blind one for the remaining channels, and to identify the filters in those channels,  $2L$  Fourier measurements are necessary and sufficient. However, this naïve approach may not be well suited for practical applications. For example, in radar and ultrasound applications, the Fourier measurements are computed as follows: The analog signal is first pre-filtered with a kernel followed by an analog-to-digital converter (ADC). Then, as discussed in Section III, the time-domain samples are linearly combined to give the Fourier measurements [6], [31]. Here the sampling rate is determined by the number of Fourier measurements. In practice, it has been shown that the bit resolution of ADCs is limited when the sampling rate is high [41]. Therefore, it is desirable to minimize the maximum number of Fourier measurements per channel.

To achieve recovery we consider a pairwise strategy. For example, assume there are four channels ( $N = 4$ ). Consider universal sets  $\{\mathcal{K}_n\}_{n=1}^4$  such that they satisfy the following conditions: (i)  $\mathcal{K}_1 = \mathcal{K}_2$  and  $\mathcal{K}_3 = \mathcal{K}_4$ ; and (ii)  $|\mathcal{K}_n| \geq 2L^2$  for  $n = 1, 2, 3, 4$ . By applying Theorem 1, independently to the measurements from the pair of channels (1, 2) and (3, 4), we identify the filters  $\{x_n\}_{n=1}^4$  as well as the Fourier measurements  $\{S(e^{jk\omega_0})\}_{k \in \mathcal{K}_1}$  and  $\{S(e^{jk\omega_0})\}_{k \in \mathcal{K}_3}$ . These two sets of partial Fourier measurements may be differently scaled due to shift and scaling ambiguities. Let us assume that there are no such inter-pair ambiguities. Then we have overall  $\{S(e^{jk\omega_0})\}_{k \in \mathcal{K}_1 \cup \mathcal{K}_3}$  Fourier measurements of the source. If  $|\mathcal{K}_1 \cup \mathcal{K}_3| \geq M_s$ , then we can uniquely recover the source. Here, the maximum number of Fourier measurements can be  $4L^2$ . In this particular case, if we consider more channels, it is necessary and sufficient to consider  $2L$  measurements from the additional channels to identify the corresponding filters as the source is identified from the first four channels. We generalize the example to any  $N$  channels and show how to choose the universal sets  $\{\mathcal{K}_n\}_{n=1}^N$  to eliminate the inter-pair ambiguity and uniquely identify the source and the filters.

To apply the pairwise strategy for any  $N \geq 2$  channels, we consider the Fourier-domain sampling grids given by

$$\mathcal{K}_{2r-1} = \mathcal{K}_{2r}, |\mathcal{K}_{2r-1}| = K, \mathcal{K}_{2r-1} \text{ is a universal set,} \\ \text{and } |\mathcal{K}_{2r-1} \cap \mathcal{K}_{2r+1}| = 2, \quad r = 1, 2, \dots, R, \quad (11)$$



where  $R \leq \lfloor N/2 \rfloor$ . The first three conditions with  $K \geq 2L^2$  are necessary and sufficient for the recovery of the filters and the partial Fourier measurements of the source. The last condition is that there should be overlap of two samples in successive pairs of sample sets. We show that this overlap aids in removing inter-pair ambiguity of shift and scaling.

The identifiability result for both the source and filter for compressive MBD is stated in the following theorem.

**Theorem 3:** Suppose that (A1) to (A5) hold for  $N \geq 2$  and  $L < \sqrt{M_x}$ . Then compressive MBD is uniquely identifiable according to Definition 1 from the Fourier measurements if the following conditions are satisfied: i) For at least  $\max\{\frac{M_s-K}{K-2} + 1, 1\}$  pair of channels, the corresponding sampling sets satisfy the conditions in (11) with  $K \geq 2L^2$ ; and ii) At least  $2L$  Fourier measurements are available from the rest of the channels.

*Proof:* Let us assume that we can recover  $M_s$  Fourier measurements of  $s$  from the first  $2R$  channels together with the corresponding filters where  $2R \leq N$ . For these channels, the sampling pattern is chosen such that  $\mathcal{K}_{2r-1} = \mathcal{K}_{2r}$  for  $r = 1, \dots, R$  with  $|\mathcal{K}_{2r}| = K \geq 2L^2$ . For each pair of  $(2r-1)$ th and  $2r$ th channels for  $r = 1, \dots, R$ , the assumptions imply via Theorem 1 that  $x_{2r-1}$ ,  $x_{2r}$ , and  $S(e^{jk\omega_0})$  for  $k \in \mathcal{K}_{2r-1}$  are uniquely identified up to a scaling and shift ambiguity. In other words,  $S(e^{jk\omega_0})$  is identified up to multiplication by  $\alpha_r e^{jk\omega_0 p_r}$  for  $k \in \mathcal{K}_{2r-1}$  for unknown constants  $\alpha_r \neq 0$  and  $p_r \in \mathbb{Z}$ . Due to the overlaps  $|\mathcal{K}_{2r-1} \cap \mathcal{K}_{2r+1}| = 2$  for  $r = 1, \dots, R$  in the design of  $\{\mathcal{K}_{2r-1}\}_{r=1}^R$ . These inter-pair ambiguity constants can be removed up to a global constant in a sequential manner. For example, let us assume that  $\mathcal{K}_1 \cap \mathcal{K}_3 = \{k_1, k_2\}$ . In other words, for the channel pairs (1, 2) and (3, 4), Fourier measurements are taken at the overlapped frequencies  $k_1\omega_0$  and  $k_2\omega_0$ . By applying Theorem 1 to these pairs, we obtain the Fourier measurements of the source at the overlapped frequencies up to inter-pair ambiguities, which are  $\alpha_1 S(e^{jk_2\omega_0}) e^{jk_2\omega_0 p_1}$ ,  $\alpha_1 S(e^{jk_1\omega_0}) e^{jk_1\omega_0 p_1}$ ,  $\alpha_2 S(e^{jk_2\omega_0}) e^{jk_2\omega_0 p_2}$ , and  $\alpha_2 S(e^{jk_1\omega_0}) e^{jk_1\omega_0 p_2}$ . Then  $\alpha_1/\alpha_2$  and  $p_1 - p_2$  are computed from the ratios among these measurements and enable to obtain  $\{\alpha_1 S(e^{jk\omega_0}) e^{jk p_1 \omega_0}\}_{k \in \mathcal{K}_3}$  from  $\{\alpha_2 S(e^{jk\omega_0}) e^{jk p_2 \omega_0}\}_{k \in \mathcal{K}_3}$ , where the former is aligned to the first pair.

Applying this process successively, we identify the filters  $x_1, x_2, \dots, x_{2R}$ , up to a global scaling factor  $1/\alpha_1$  and a shift by  $-p_1$ , together with the Fourier measurements  $\alpha_1 S(e^{jk\omega_0}) e^{jk p_1 \omega_0}$  for  $k \in \bigcup_{n=1}^R \mathcal{K}_n$ .

To identify  $s$  from the above Fourier measurements, it is sufficient to satisfy

$$\bigcup_{r=1}^{N_1} \mathcal{K}_{2r-1} = (R-1)(K-2) + K \geq M_s. \quad (12)$$

In other words the source can be identified from a minimum  $2R \geq 2\frac{M_s-K}{K-2} + 2$  channels if  $K \geq 2L^2$ . From the remaining  $N - 2R$  channels it is sufficient to consider any  $2L$  Fourier measurements to identify the corresponding filters. ■

The maximum number of measurements per channel in Theorem 3 can be restricted to  $2L^2$ . The inequality in (12) implies that when  $K$  increases beyond  $2L^2$ , the number of channels for the identifiability can be reduced. In other words, one can

trade-off between the number of measurements per channel and the number of channels. For  $K = 2L^2$  the source and the filters are identifiable if  $N \geq \max\{\frac{M_s-2L^2}{L^2-1} + 2, 2\}$ .

By Corollary 3, we obtain an immediate extension of Theorem 3 to the non-sparse case.

**Corollary 4 (Identifiability Results for Source and Non-Sparse Filters):** In Theorem 3, let  $\text{supp}\{x_n\} = [M_x]$ , where  $M_x \leq M/2$ . Then compressive MBD is uniquely identifiable from  $2M_x - 1$  Fourier measurements from each of  $N \geq 2 \cdot \frac{M_s-2M_x+1}{2M_x-3} + 2$  channels.

Even in the non-sparse case, the measurement system can be compressive when the number of Fourier measurements  $2M_x - 1$  is smaller than the available time-domain measurements  $M = M_x + M_s - 1$ , that is,  $M_s > M_x$ .

### C. Extension to Sparse MBD With Circular Convolution

The MBD problem considered in the previous sections assumes that the measurements consist of a linear convolution of the source and filters, whereas, the recent results in the literature consider the MBD problem with circular convolutions. Here we extend our results to the case of circular convolution. In this setup, the  $N$ -channel MBD time-domain outputs are given as  $\{y_n = s \circledast x_n\}_{n=1}^N$ , where  $\circledast$  denotes circular convolution, and the supports of the filters and source are within the set  $[M]$ . In other words, we assume that  $\text{supp}\{s\} \subseteq M$  and  $\text{supp}\{x_n\} \subseteq M$ . Due to circular convolution, the measurements  $y_n = s \circledast x_n$  are  $M$ -periodic. We further assume that the filters are  $L$ -sparse and they are coprime. In this case, the goal is to derive identifiability conditions to uniquely recover the source  $s$  and the filters  $\{x_n\}_{n=1}^N$  from the discrete Fourier transform (DFT) measurements  $\{Y_n(e^{jk\omega_0})\}_{k \in \mathcal{K}_n}$ , where  $\omega_0 = \frac{2\pi}{M}$  and  $\mathcal{K}_n \subseteq [M]$ . With these settings, for  $N = 2$ , following the steps in the proof of Lemma 1, the sequence  $q$  in (10) is given as  $q = x_1 \circledast \hat{x}_2 - x_2 \circledast \hat{x}_1$ . To follow the remaining steps of the proof and prove the identifiability results, we have to ensure that  $q = x_1 \circledast \hat{x}_2 - x_2 \circledast \hat{x}_1 = x_1 * \hat{x}_2 - x_2 * \hat{x}_1$ . This is indeed true if we assume that the filters are supported within the set  $\lfloor M/2 \rfloor$ . With these assumptions, we state the extension of Theorem 3 for the circular convolution case.

**Theorem 4:** Let  $N \geq 2$  and  $M \in \mathbb{N}$ . Let  $(s, \{x_n\}_{n=1}^N)$  be arbitrary while satisfying (A1) to (A5) with  $M_x = \lfloor M/2 \rfloor$ ,  $M_s = M$ , and  $L < \sqrt{M_x}$ . Let  $y_n = s \circledast x_n$  for  $n = 1, 2, \dots, N$ . Then  $s$  and  $\{x_n\}_{n=1}^N$  are simultaneously identified from the DFT measurements  $\{Y_n(e^{j\omega_0 k})\}_{k \in \mathcal{K}_n}$  if the following conditions are satisfied: (i) For at least  $\max\{\frac{M-K}{K-2} + 1, 1\}$  pair of channels, the corresponding sampling sets  $\mathcal{K}_n$  satisfy the conditions in (11) where  $K \geq 2L^2$ ; and (ii) a minimum of  $2L$  DFT measurements are available from the rest of the channels.

The following result for the non-sparse case is obtained as an immediate corollary.

**Corollary 5 (Non-Sparse FIR With Circular Convolution):** Consider the assumptions of Theorem 4. Let  $\text{supp}\{x_n\} = [M_x]$  where  $M_x \leq M/2$ . Suppose that  $N \geq \frac{2M-4}{2M_x-3}$ . Then both the source and the filters are identifiable iff the number of Fourier measurements are greater than or equal to  $2M_x - 1$ .

Note that the condition  $\text{supp}\{x_n\} = [M_x]$  implies that the filters are in a low-dimensional subspace of dimension  $M_x$ .

#### D. Recovery From Samples in the $z$ -Domain

A DTFT can be considered as a special case of the  $z$ -domain sample evaluated at a complex number of unit modulus. In this section we show that the results in the previous sections generalize to the case where the measurements of the output channels are given as samples in the  $z$ -domain

For example, as in (9), let us consider the filter identification problem for the two-channel case from samples  $\{Y_n(z_k) = S(z_k)X_n(z_k)\}_{k \in [K]}$  where  $\{z_k\}_{k \in [K]}$  denotes sampling grid in the  $z$ -domain. We assume that  $\{S(z_k)\}_{k \in [K]}$  is non vanishing. To show the identifiability, we can follow the lines of the proof of Theorem 1 by substituting  $e^{jk\omega_0}$  by  $z_k$ . With the  $z$ -domain measurements, all the steps of the proof of Theorem 1 remain valid except the spark properties of the resulting  $\mathbf{A}$  matrix (see (31)).

With  $z$ -domain sampling, the matrix  $\mathbf{A}$  is given by

$$\mathbf{A} = \begin{pmatrix} 1 & z_1 & z_1^2 & \dots & z_1^{2M_x-1} \\ 1 & z_2 & z_2^2 & \dots & z_2^{2M_x-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & z_K & z_K^2 & \dots & z_K^{2M_x-1} \end{pmatrix} \in \mathbb{C}^{K \times 2M_x}. \quad (13)$$

The matrix  $\mathbf{A}$  in (13) need not have full spark for any arbitrary choice of  $z_k$ . Here we show a particular choice of  $z_k$ s such that the matrix has full spark.

Let us assume that  $z_0 \in \mathbb{C}$  such that  $z_0 \neq 1$ . Let  $K \in [2M_x]$  be a distinct set of integers such that  $|\mathcal{K}| = K$ . Let  $z_k = z_0^{p_k}$  for  $p_k \in \mathcal{K}$ . Then the matrix  $\mathbf{A}$  in (13) has full spark if  $\mathcal{K}$  is a universal set. To show the full spark property, let us consider a submatrix of  $\mathbf{A}$  which consists of  $K$  distinct columns indexed by  $m_1, m_2, \dots, m_K$ . With  $z_k = z_0^{p_k}$ , the submatrix is given by

$$\begin{pmatrix} z_0^{p_1 m_1} & z_0^{p_1 m_2} & \dots & z_0^{p_1 m_K} \\ z_0^{p_2 m_1} & z_0^{p_2 m_2} & \dots & z_0^{p_2 m_K} \\ \vdots & \vdots & \ddots & \vdots \\ z_0^{p_K m_1} & z_0^{p_K m_2} & \dots & z_0^{p_K m_K} \end{pmatrix} \in \mathbb{C}^{K \times K}. \quad (14)$$

Since the choice of the columns is arbitrary, the matrix  $\mathbf{A}$  will have full spark if the submatrix is invertible. Since  $z_0 \neq 1$ , the submatrix is similar to matrix  $\mathbf{V}$  in Section II-B with seeds  $\{z_0^{m_k}\}_{k=1}^K$ . Since  $\mathcal{K}$  is a universal set and  $p_k \in \mathcal{K}$ , the submatrix has full spark and is invertible. Hence, the matrix  $\mathbf{A}$  with the particular choice of sampling grid in the  $z$ -domain has full spark which implies that the filters can be uniquely identifiable in the two-channel case if  $K \geq 2L^2$ . The problem is not identifiable if  $K < 2L$ . Similarly, we can extend the results of Theorem 2 and Theorem 3 to the case where the samples are measured in the  $z$ -domain.

A major difference between the identifiability results from the Fourier measurements and from measurements in the  $z$ -domain is that the sampling grid in the former case depends on the support of the source and the filters. For example, the results in Theorem 3 assumes that the sampling interval  $\omega_0$  is selected

such that the set  $\{e^{j\omega_0 k}\}_{k=0}^{\max\{2M_x-2, M_s-1\}}$  has distinct elements. However, with  $z$ -domain sampling, the sampling grid can be designed independent of the support of the source or the filters.

#### E. Relaxation of Coprime Condition

Under the assumptions (A1) and (A2), the condition (A3) is only a sufficient condition for the identifiability. Indeed, the assumption (A3) can be either dropped or relaxed into less stringent conditions as shown in the following examples.

*Example 1:* Let (A2') be defined by

- (A2') The support of  $s$  is included in  $[M_s]$ ,  $s[0] \neq 0$ , and  $s[M_s - 1] \neq 0$ .

Then (A1), (A2'), (A4), and (A5) provide another sufficient condition on the identifiability in Corollary 1. (The original Corollary 1 assumed that (A1)–(A5) hold.) the assumption in (A2') can be justified in the context of certain applications. For example, there are radar systems where the transmit pulse is exactly supported on a known finite interval. Therefore (A2') holds there.

*Proof:* Suppose that  $(\hat{s}, \hat{x}_1, \hat{x}_2)$  is feasible, that is, it satisfies (A1), (A2'), (A4), and (A5), and is consistent with the measurements. In Lemma 1, the proof of the sufficiency part up to (28) do not depend on support of the source. Hence, the sufficiency part of Lemma 1 also follows from these modified assumptions, and provides that if  $|\mathcal{K}| \geq 2L^2$  then

$$X_1(z)/X_2(z) = \hat{X}_1(z)/\hat{X}_2(z). \quad (15)$$

Hence, there exists a function  $G(z)$  such that  $X_n(z) = G(z)\hat{X}_n(z)$  for  $n = 1, 2$ . Alternatively, in the time-domain, we have  $x_n = g * \hat{x}_n$ , which leads to

$$y_n = s * x_n = s * g * \hat{x}_n = \hat{s} * \hat{x}_n,$$

where  $\hat{s} = s * g$ . If the length of  $g$  is strictly greater than one, then  $\hat{s}$  does not satisfy (A2'), which contradicts our assumption. Hence, the length of  $g$  should be one. This implies that  $(\hat{s}, \hat{x}_1, \hat{x}_2)$  is equivalent to  $(s, x_1, x_2)$  up to the fundamental ambiguities. ■

*Example 2:* Let (A3') be defined by

- (A3') **Sparse-coprime filters:** If  $x_n = h_0 * \bar{x}_n$  for all  $n$  and  $|\text{supp}\{h_0\}| > 1$ , then  $\|\bar{x}_n\|_0 > L$  for all  $n$ .

In words, the assumption in (A3') ensures that when the filters  $x_1, \dots, x_N$  share common zeros, the novel factors  $\bar{x}_1, \dots, \bar{x}_N$  are not  $L$ -sparse. Then substituting (A3) by (A3') in Corollary 1 provides the same identifiability result while making it applies to a broader set of signals.

*Proof:* Consider measurements  $y_1 = s * x_1$  and  $y_2 = s * x_2$ , where  $(s, x_1, x_2)$  satisfy assumptions (A1), (A3'), (A4), and (A5). Suppose that  $(\hat{s}, \hat{x}_1, \hat{x}_2)$  is also feasible. In Lemma 1, the proof of the sufficiency part up to (28) does not depend on (A3). Hence, it also follows from these modified assumptions, and provides that if  $|\mathcal{K}| \geq 2L^2$  then

$$X_1(z)/X_2(z) = \hat{X}_1(z)/\hat{X}_2(z). \quad (16)$$

Hence, there exists a sequence  $g$  such that  $x_n = g * \hat{x}_n$  for  $n = 1, 2$ . However, since  $x_1$  and  $x_2$  are sparse coprime, it implies that either  $\text{supp}\{g\} = 1$  or  $\|\hat{x}_n\|_0 > L$  for  $n = 1, 2$ . Since the former



is not true, we have that  $\text{supp}\{g\} = 1$  and hence,  $(s, \hat{x}_1, \hat{x}_2)$  belongs to fundamental ambiguity class of the ground-truth. ■

Next we present an example of filters that are sparse-coprime but not coprime. Suppose that  $s$  satisfies (A2) and (A4). Let  $N = 2$  and  $L = 4$ . We construct  $x_1$  and  $x_2$  as follows: Let

$$\bar{x}_n[m] = \begin{cases} a_n^m, & 0 \leq m \leq M_x - 3, \\ 0, & \text{else} \end{cases} \quad (17)$$

for  $n = 1, 2$ , where  $a_1 \neq a_2 \in \mathbb{C}$ . Then the  $z$ -transform of  $\bar{x}_1$  (resp.  $\bar{x}_2$ ) has zeros at  $\{e^{j2\pi m/(M_x-2)} a_1\}_{m=0}^{M_x-2}$  (resp.  $\{e^{j2\pi m/(M_x-2)} a_2\}_{m=0}^{M_x-2}$ ). Therefore,  $\bar{x}_1$  and  $\bar{x}_2$  are coprime. Furthermore,  $h_1$ , defined by  $h_1[m] = \delta[m] - a_1 \delta[m-1]$ , sparsifies  $x_1$  by

$$(\bar{x}_1 * h_1)[m] = \delta[m] - a_1^{M_x-2} \delta[m - M_x + 2], \quad (18)$$

where  $\delta$  denotes the Kronecker's delta. Similarly,

$$(\bar{x}_2 * h_2)[m] = \delta[m] - a_2^{M_x-2} \delta[m - M_x + 2], \quad (19)$$

where  $h_2[m] = \delta[m] - a_2 \delta[m-1]$ . Let  $x_1$  and  $x_2$  be given by

$$x_n = \bar{x}_n * h_0, \quad n = 1, 2, \quad (20)$$

where  $h_0 = h_1 * h_2$ . By plugging (18) and (19) into (20), we obtain  $\|x_1\|_0 = \|x_2\|_0 = L$ . Note that  $x_1$  and  $x_2$  are not coprime as they share non-trivial zeros arising from the common factor  $h_0 = h_1 * h_2$ . Furthermore, the coprimeness of  $\bar{x}_1$  and  $\bar{x}_2$  implies that there is no other nontrivial common factor than  $h_0$ . Since  $\|\bar{x}_1\|_0 > L$  and  $\|\bar{x}_2\|_0 > L$ ,  $x_1$  and  $x_2$  are sparse-coprime.

## V. COMPARISON WITH PRIOR ART

We first compare our results for non-sparse cases and then provide a comparison for sparse MBD.

### A. Comparison of Non-Sparse Case

Xu *et al.* [9], considered the problem of estimating filters only without sparsity assumption with linear-convolution. They made the assumptions that the filters are coprime and the source has a linear complexity<sup>1</sup> greater than or equal to  $2M_x$ , which implies that  $M_s \geq 4M_x$ . With these assumptions, the authors show that the filters are identifiable from  $N = 2$  channels if  $3M_x$  consecutive measurements of  $y_n$  are available. Note that with  $M_s \geq 4M_x$ , the length of  $y_n$  is given by  $M \geq 5M_x - 1$  out of which  $3M_x$  are sufficient to identify the filters.

In comparison, our results in Corollary 3, together with the time-domain results in Corollary 1, state that we can identify the filters with  $2M_x - 1$  time samples. As in [9], we too impose the coprimeness condition on the filters. However, we do not restrict the source to have a longer support compared with that of the filters. In our approach, the support of the source could be either larger or smaller compared with the support of the filters. Furthermore, our results are valid for any source signal whose DTFT samples do not vanish at a given frequency location. The source need not satisfy a linear complexity constraint.

<sup>1</sup>Mathematically, the linear complexity of a sequence  $s$  is defined as the smallest integer  $L_c$  such that there exists a set of complex-valued amplitudes  $\{c_\ell\}_{\ell=1}^{L_c}$  and complex-valued roots  $\{r_\ell\}_{\ell=1}^{L_c}$  such that  $s[m] = \sum_{\ell=1}^{L_c} c_\ell r_\ell^m$ .

For example, let us assume that the source  $s$  has linear-complexity of one, that is, the samples of the source sequence are given as  $s[m] = c_1 r_1^m$  for  $m \in [M_s]$ . In addition, let us assume that  $r_1 = e^{j\omega_1}$ . In this case, the DTFT of the source sequence is given as  $S(e^{j\omega}) = c_1 \frac{1 - e^{j(\omega - \omega_1)M_s}}{1 - e^{j(\omega - \omega_1)}}$ . The DTFT vanishes at  $\omega = p \frac{2\pi}{M_s} + \omega_1$  where  $p \in \mathbb{Z} \setminus \{0\}$ . If we chose our sampling set  $\mathcal{K}$  and  $\omega_0$  such that the set  $\mathcal{K}\omega_0$  does not have zeros of  $S(e^{j\omega})$  then our method identifies the filters. In particular, let the cardinality of the set  $\mathcal{K}$  be given, that is, the number of Fourier measurements to be taken is known. For a given  $\omega_1$  and  $M_s$ , if  $\omega_0$  and  $\mathcal{K}$  are chosen as  $\min\{\frac{4\pi}{M_s|\mathcal{K}|}, \frac{2\pi}{2M_x-1}\}$  and  $\left\lceil \frac{\omega_1 - 2\pi/M_s}{\omega_0} \right\rceil + [|\mathcal{K}|]$ , then  $\{S(e^{jk\omega_0})\}_{k \in \mathcal{K}}$  does not vanish. Hence, our approach identifies the filters for  $L_c = 1$ , whereas, the method by Xu *et al.* [9] cannot identify the filters uniquely.

Recently, Xia and Li [19] considered MBD with circular convolution in a deterministic setup where the source and filters are real and deterministic, but not sparse. Similar to our assumption in the case of non-sparse circular MBD, they assumed that  $\text{supp}\{x_n\} = [M_x]$  with  $M_x < M$ . They showed that by using all  $M$  time samples from each channel *almost all* the sources and the filters are uniquely identifiable if  $N \geq \frac{M-1}{M-M_x}$ . The authors used the conjugate symmetry property of the Fourier transform of real signals to restrict the feasible solution sets. Their results provided “generic” identifiability, that might exist a null set including instances for which unique identification fails. The major weakness is that this null set is not clearly identified and it may include signals of interest in applications. In contrast, our “deterministic” identifiability results hold for any source filter pairs when they satisfy the conditions (A1)–(A5) (cf. Corollary 5). We show that  $N \geq \frac{2M-4}{2M_x-3}$  channels are sufficient for unique identification of the source and filters iff  $2M_x - 1$  Fourier measurements per channel are available. For  $M_x = M/2$ , both results work for  $N = 2$  channels and require all the measurements. On the other hand, when  $M_x \ll M$ , our result requires more number of channels but with fewer measurements per channel than that by Xia and Li [19] with two channels.

### B. Comparison With Recent Results on Sparse MBD

All the results discussed in this section consider the MBD setup with circular convolution. Hence, we compare them with our results in Theorem 4. The results in [20]–[23] consider identifiability of MBD problems with the assumption that the filters are random and sparse. Balzano and Nowak [20] considered a BGPC problem with oversampled DFT matrix and showed that when  $L$ -sparse signals  $x_n s$  are generic and have common known support, it is necessary to have measurements from  $N \geq \lceil (M-1)/(M-L) \rceil$  channels for perfect recovery of unknown gains. Li *et al.* [23] studied the identifiability conditions for a general BGPC problem with subspace and sparsity constraints. The authors showed that under generic sparsity constraints on the filters, the problem is identifiable with high-probability up to acceptable ambiguities as long as  $N = \mathcal{O}(M \log M)$ . In [21], [22], the authors considered a sparse MBD problem with circular convolution by assuming that the

source is invertible and the filters are sparse with randomly chosen support and amplitudes. Specifically, Wang and Chi [22] assumed that the sparsity of the filters follows a Bernoulli-subgaussian model. With the assumption that the source  $s$  is approximately flat in the Fourier domain, the authors show that the problem could be efficiently solved through an  $\ell_1$ -minimization approach, as long as  $N = \mathcal{O}(M \log^4 M)$ . Cosse [21] assumed that the source is invertible and the location of the non-zero values of the sparse filters are chosen uniformly at random over  $[M]$ . The recovery is guaranteed with high-probability as soon as the number of channels  $N$  and the dimension of the filters  $M$ , satisfy  $N \lesssim M$  and  $N \gtrsim L^2$  where the filters are assumed to be  $L$ -sparse.<sup>2</sup>

Our results differ from previous results as follows:

- 1) The aforementioned recent results considered random sparsity or subspace models on the filters, whereas, we consider a deterministic sparsity model for the filters.
- 2) In [20]–[23], sparsity is introduced to derive the identifiability results but not with the goal of compressing the measurements. The results are derived by assuming that all  $M$  time samples  $y_n$  are available in all the channels. We show that sparsity also helps in identifiability by using compressive measurements in the frequency domain. Instead of using  $M$  time samples, we show identifiability using  $2L^2$  Fourier samples of  $y_n$ .
- 3) In [23] and [22], the number of channels required does not depend on the sparsity level of the filters. We show that the source and filters are identifiable from  $N \geq \max\{\frac{M-2L^2}{L^2-1} + 2, 2\}$  channels.
- 4) In [23] and [22], the total number of measurements is on the order of  $\mathcal{O}(M \log M)$  and  $\mathcal{O}(M \log^4 M)$ , respectively, whereas, in our setup, we need  $2L^2$  measurements from at least  $\max\{\frac{M-2L^2}{L^2-1} + 2, 2\}$  channels. This results overall in  $\frac{(M-2)2L^2}{L^2-1}$  measurements which is on the order of  $M$ .
- 5) Comparing the results in the case of known support, in [20], all  $M$  samples are considered per channel and overall  $\mathcal{O}(M)$  measurements are required. In our case, we need only  $L^2$  measurements per channel with overall measurements on the order  $M$ . We gain in terms of the number of measurements per channel but we require more channels compared with [20].

### C. Relation to Blind CS

The proposed compressive MBD problem can be viewed as a special case of blind compressive sensing (BCS) [42]. In BCS, a set of signals that are sparse in an unknown bases are uniquely identified from their compressed measurements. Similarly, in compressive MBD, the output sequences  $\{y_n = s * x_n\}_{n=1}^N$  are sparse in the unknown dictionary that is given by the convolution matrix corresponding to the sequence  $s$ , the filters denote the sparse vectors, and the objective is to identify them from compressive measurements in the Fourier domain. However, the

existing dictionary models in [42] do not include convolutional dictionaries.

## VI. NUMERICAL RESULTS

The main goal of this section is to compare the identifiability results of Theorem 1 to empirical observations. While Theorem 1 provides a set of necessary and sufficient conditions for the identifiability in the worst case, in practice, it is not feasible to test in the worst case scenario. Therefore, instead, we run a set of Monte Carlo simulations and observe conditions under which most cases are successful (we count the frequency of empirical successes). On the other hand, the identifiability result of Theorem 1 implies the existence of a method that uniquely determines the solution of compressive MBD, which means one has to consider the optimal algorithm regardless of its computational cost. This is also infeasible in practice. For a set of small sized problems, we performed enumeration over all possible supports, which provides an optimal reconstruction algorithm. For larger scaled problems, we consider a few selected heuristics described below. In the non-sparse case with noise-free measurements, an optimal algorithm can be obtained by a standard eigenvalue decomposition. Thus, we use it for the study of the identifiability.

### A. Practical Algorithms for Compressive MBD

To describe the algorithms used in the experiments, we rewrite the measurements succinctly in a compact matrix form. Let  $y_n \in \mathbb{C}^{[K]}$  denote the compressible measurements from the  $n$ th channel. Let  $s \in \mathbb{C}^{[K]}$  denote the column vector whose entries are  $\{S(e^{jk\omega_0})\}_{k \in K}$ . The measurements in (4) can then be compactly written as

$$\mathbf{Y} = [\mathbf{y}_1 \ \mathbf{y}_2] = \text{diag}(\mathbf{s})\bar{\mathbf{A}}\mathbf{X}, \quad (21)$$

where  $\mathbf{X} = [\mathbf{x}_1 \ \mathbf{x}_2]$ . Estimating  $s$  and  $\mathbf{X}$  from  $\mathbf{Y}$  is a BGPC or BDC problem with partial Fourier matrix. Practical algorithms to solve the sparse MBD or BGPC/BDC problem were proposed in [22], [26], [30]. The algorithm proposed in [22] is based on  $\ell_1$ -norm minimization and requires the matrix  $\bar{\mathbf{A}}$  to be a full DFT matrix. Hence, the algorithm is not applicable to compressive MBD. In [30] and [26], the authors proposed algorithms based on BDC and truncated-power iteration (TPI), respectively. We will use modifications of these two approaches in the simulation.

The BDC problem is similar to a sparse dictionary learning (DL) problem [43]; however, there is a major difference. In the DL problem, the composite matrix  $\text{diag}(\mathbf{s})\bar{\mathbf{A}}$  is unknown and needs to be estimated along with the sparse vectors  $\mathbf{X}$ . In contrast, BDC requires to estimate only  $s$  with known  $\bar{\mathbf{A}}$ . In [30], the authors proposed a convex optimization based solution to the BDC problem. However, we experimentally observed that for  $N = 2$  solving the BDC problem in (21) by using an alternate minimization approach provides desirable identifiability results. Hence, we consider the alternate minimization-based BDC approach as one of the practical algorithms for assessing the results. In alternating minimization, first, we estimate the sparse vectors by assuming that  $s$  is known and then, in the dictionary update step, we estimate  $s$  by using the estimated  $\mathbf{X}$ .

<sup>2</sup> $M \gtrsim N \Leftrightarrow \exists c \in \mathbb{R}$  s.t.  $M \geq cN$ .

**Algorithm 1:** BDC for Solving (21).**Output:**  $\mathbf{s}$  and  $\mathbf{X}$ **Input:**  $\mathbf{Y}$ ,  $\bar{\mathbf{A}}$ ,  $L$ , and the initial estimate  $\mathbf{s}^{(0)}$ 

- 1: Let  $i \leftarrow 1$
- 2: **repeat**
- 3: Estimate columnwise  $L$ -sparse matrix  $\mathbf{X}^{(i)}$  by OMP to minimize  $\|\text{diag}(\mathbf{s}^{(i-1)})^{-1} \mathbf{Y} - \bar{\mathbf{A}} \mathbf{X}^{(i-1)}\|_2^2$ .
- 4:  $\mathbf{s}^{(i)} \leftarrow \text{argmin}_{\mathbf{s}} \|\mathbf{Y} - \text{diag}(\mathbf{s}) \bar{\mathbf{A}} \mathbf{X}^{(i)}\|_2^2$
- 5:  $i \leftarrow i + 1$
- 6: **until** convergence criterion is reached

We apply orthogonal matching pursuit (OMP) [44] to estimate the sparse vectors. Then  $\mathbf{s}$  is estimated as a minimizer of the error  $\|\mathbf{Y} - \text{diag}(\mathbf{s}) \bar{\mathbf{A}} \mathbf{X}\|_2^2$ . We present the proposed approach in Algorithm 1. The solution of the optimization problem in Step 4 is given as  $\mathbf{s}^{(i)} = \mathcal{D}\{\bar{\mathbf{A}} \mathbf{X}^{(i)} \mathbf{Y}^H\} / \mathcal{D}\{(\bar{\mathbf{A}} \mathbf{X}^{(i)})(\bar{\mathbf{A}} \mathbf{X}^{(i)})^H\}$ . Here the operator  $\mathcal{D}$  acts on a square matrix to output a vector consisting of the diagonal elements of the matrix, and the symbol  $/$  denotes element-wise division. In our simulations, the algorithm stops at the  $i$ th iteration if  $\|\mathbf{X}^{(i)} - \mathbf{X}^{(i-1)}\|_2 \leq 10^{-3}$ .

Next, we discuss an alternative method to identify the filters from the measurements  $\mathbf{Y}$ . By applying cross-correlation, as in the proof of our main results (cf. Section VII), (21) can be rewritten as

$$\underbrace{\begin{bmatrix} \text{diag}(\mathbf{y}_2) \bar{\mathbf{A}} & -\text{diag}(\mathbf{y}_1) \bar{\mathbf{A}} \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}}_{\boldsymbol{\gamma}} = \mathbf{0}. \quad (22)$$

In the sparse MBD framework, identifying the filters from the matrix  $\mathbf{B}$  is equivalent to identifying a  $2L$ -sparse null vector of  $\mathbf{B}$ . The solution to the problem in (22) can be computed as the solution to the following non-convex optimization problem:

$$\begin{aligned} & \underset{\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 \in \mathbb{C}^{M_x}}{\text{minimize}} \begin{bmatrix} \boldsymbol{\gamma}_1^H & \boldsymbol{\gamma}_2^H \end{bmatrix} \mathbf{B}^H \mathbf{B} \begin{bmatrix} \boldsymbol{\gamma}_1 \\ \boldsymbol{\gamma}_2 \end{bmatrix} \\ & \text{subject to } \|\boldsymbol{\gamma}_1\|_0 \leq L, \|\boldsymbol{\gamma}_2\|_0 \leq L, \left\| \begin{bmatrix} \boldsymbol{\gamma}_1 \\ \boldsymbol{\gamma}_2 \end{bmatrix} \right\|_2 = 1. \end{aligned} \quad (23)$$

Problem (23) can be solved by adapting the truncated power iteration (TPI) algorithm proposed in [45]. In its original version, TPI is developed to compute the largest sparse eigenvector of a positive semidefinite matrix. In [26], Li *et al.* adopt the TPI algorithm in [45] to solve the BGPC problem by assuming that measurement matrix is random Gaussian and the filters are jointly sparse. We present TPI to solve the optimization problem in (23).

The TPI algorithm in [45] is developed for a single sparse vector. In (23) the vector to be estimated is a concatenation of two sparse vectors. By adopting the original TPI algorithm, the sparsity constraints on  $\boldsymbol{\gamma}$  in (23) are imposed by the composite sparse projector  $\mathcal{P}_L: \mathbb{C}^{2M_x} \rightarrow \mathbb{C}^{2M_x}$ . For any vector  $\boldsymbol{\gamma} \in \mathbb{C}^{2M_x}$ , the output of the sparse projector,  $\mathcal{P}_L\{\boldsymbol{\gamma}\}$ , is an  $2L$  sparse vector computed by independently retaining the  $L$  largest entries over the sets  $[M_x]$  and  $M_x + [M_x]$  from the support of

**Algorithm 2:** Truncated Power Iteration to Solve (23).**Output:**  $\boldsymbol{\gamma}$ ; **Parameter:**  $\beta$ **Input:**  $\mathbf{B}$ ,  $M_x$ ,  $L$ , and the initial estimate  $\boldsymbol{\gamma}^{(0)}$ 

- 1: Set  $\mathbf{G} \leftarrow \beta \mathbf{I}_{2M_x} - \mathbf{B}^H \mathbf{B}$
- 2: Let  $i \leftarrow 1$
- 3: **repeat**
- 4:  $\boldsymbol{\gamma}^{(i)} \leftarrow \mathbf{G} \boldsymbol{\gamma}^{(i-1)} / \|\mathbf{G} \boldsymbol{\gamma}^{(i-1)}\|_2$
- 5:  $\boldsymbol{\gamma}^{(i)} \leftarrow \mathcal{P}_L\{\boldsymbol{\gamma}^{(i)}\} / \|\mathcal{P}_L\{\boldsymbol{\gamma}^{(i)}\}\|_2$
- 6:  $i \leftarrow i + 1$
- 7: **until** convergence criterion is reached

$\boldsymbol{\gamma}$  and setting the rest of the entries to zero. The truncated power iteration method is presented in Algorithm 2.

In Step 2, we denote by  $\mathbf{I}_{2M_x}$  the identity matrix of size  $2M_x \times 2M_x$ . As was suggested in [26], a safe choice of the parameter  $\beta$  is  $\|\mathbf{B}\|$ . We initialize  $\boldsymbol{\gamma}$  as the concatenation of the outputs of the OMP algorithm to the inputs  $\mathbf{y}_1$  and  $\mathbf{y}_2$ . The algorithm stops when the update in  $\boldsymbol{\gamma}$  is not significant in successive iterations. Specifically, we stop at the  $i$ th iteration if  $\|\boldsymbol{\gamma}^{(i)} - \boldsymbol{\gamma}^{(i-1)}\|_2 \leq 10^{-3}$ .

**B. Comparison of Sparse MBD**

1) *Exhaustive Search for Compressive MBD:* The exhaustive search can be applied to either the measurement  $\mathbf{Y}$  in (21) or to  $\mathbf{B}$  in (22) to identify the unknowns. In the former case, one needs to search over three unknowns,  $\mathbf{s}$ ,  $\mathbf{x}_1$ , and  $\mathbf{x}_2$ , whereas, the search reduces to two unknowns  $\mathbf{x}_1$  and  $\mathbf{x}_2$  in the latter case. Hence, we apply the exhaustive search to identify the sparse filters from  $\mathbf{B}$  and compare the results with Theorem 1.

In this experiment, the measurements are directly generated by using (21) upon setting  $M_x = 2L^2$ ,  $L = 4$ , and  $M = 2M_x$ . We choose  $\mathcal{K}$  as  $\{1, 2, \dots, K\}$ . We assume that the full Fourier measurements of the source consist of a sum of two Gaussian pulses with their amplitudes, means, and variances given by the triplets:  $(4, M/2, 0.001)$  and  $(1, 2M/3, 0.01)$ . We choose the sum of Gaussian pulses to make sure that the source spectrum is not flat and non-vanishing. The experiment can also be performed for any other choices of the source. The support and the coefficients of the filters are generated randomly. Specifically, for each filter  $L$  non-zero values are chosen uniformly at random over the set  $[M_x]$  and their amplitudes are chosen uniformly at random between 1 and 2. Out of the available  $M$  Fourier measurements, we use  $K \leq M$  to identify the filters. If the filters are uniquely identifiable up to scaling and shift of the original filters, we consider the experiment to be successful or else we assume that the experiment has failed for that particular  $K$ .

Fig. 2 shows the success rate averaged over 200 independent experiments. Interestingly, we observe that  $2L$  Fourier samples are necessary and sufficient to identify the filters.

2) *TPI and BDC to Solve Compressive MBD:* In this section, we compare the performance of the TPI and BDC approaches to solve compressive MBD. We compare the performance of these two algorithms to that by OMP in the non-blind case where



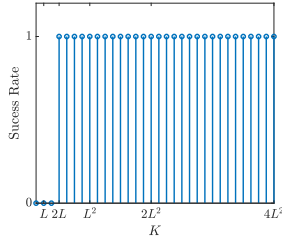


Fig. 2. Phase transition for the identification of  $L$ -sparse filters by exhaustive search ( $L = 4$ ):  $K \geq 2L$  is necessary and sufficient.

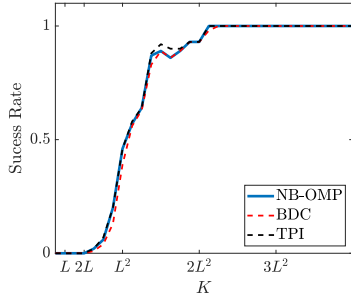


Fig. 3. Phase transition for the identification of  $L$ -sparse filters by NB-OMP, BDC, and TPI with known source ( $L = 4$ ,  $M_x = 2L^2$ );  $K \geq 2L$  is necessary and  $K > 2L^2$  is sufficient.

the source  $s$  is known. We call this method non-blind OMP (NB-OMP).

We use the same simulation settings as in the exhaustive search experiment. Let  $\tilde{\mathbf{X}}$  is an estimate of  $\mathbf{X}$ , then the filters are assumed to be identified if the normalized mean-squared error (MSE) is less than  $-50$  dB, that is, if  $20 \log(\|\mathbf{X} - \tilde{\mathbf{X}}\|_2 / \|\mathbf{X}\|_2) \leq -50$ . Fig. 3 shows the average success rate computed over 200 independent realizations of the sparse filters for  $L = 4$ . Performances of both TPI and BDC methods follow closely that of NB-OMP. We observe that the success rate gradually increases for  $K \geq 2L$  up to  $K = 2L^2$ . For  $K > 2L^2$ , both BDC and TPI algorithms always identify the filters uniquely. The curve shows that  $2L$  measurements are necessary and  $2L^2$  compressive MBD measurements are sufficient to uniquely identify the sparse filters.

Fig. 4 shows the success rate for different values of sparsity levels. We note that for  $L = 2$ , for some realizations of the filters, the algorithms are able to identify the filters for  $K = 2L$ . Except for the BDC method in the case of  $L > 8$ , we observe that  $K > 2L^2$  measurements are sufficient. The results also show that  $2L$  measurements are necessary.

3) *Experiments With Other Source Models:* The first two experiments are performed by assuming the source spectrum consists of a sum of two Gaussian pulses. In this experiment, we conducted another set of Monte Carlo simulations for other source and filter models. Specifically, we generated  $s$  so that its elements are i.i.d. following uniform distributions with mean fixed to 100. We observed how the performance varies as a function of the source variance. Indeed, a smaller variance results in a flatter source spectrum and the resulting reconstruction

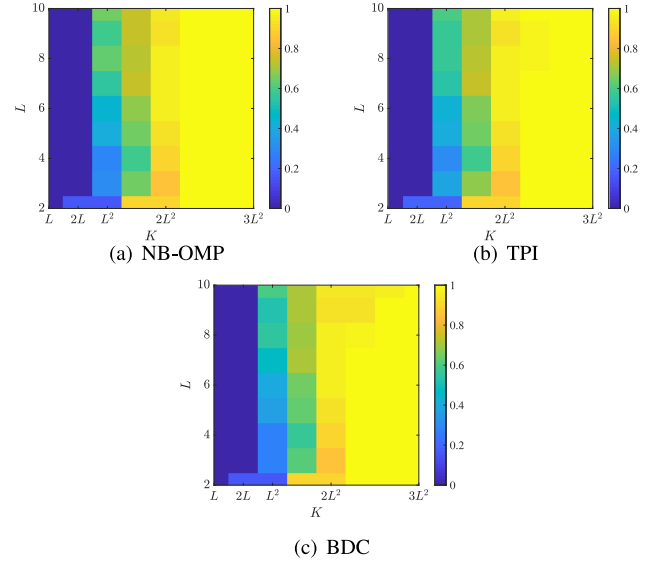


Fig. 4. Phase transition for the identification of  $L$  sparse filters in the two-channel case:  $K > 2L$  is necessary and  $K > 2L^2$  is sufficient.

problem gets closer to the non-blind case. The filters are assumed to be identified if the normalized MSE is less than  $-20$  dB. The success rates (averaged over 500 realizations) are shown in Fig. 6. We observe that the performance of the methods do not alter much with a random selection of the source. Moreover, the performance of the algorithms does not depend critically on the source variance.

4) *Comparison of Algorithms for Varying Compression Ratios:* Corollary 3 states that the filters are identifiable if  $\|\mathcal{K}\| \geq \min\{2L^2, 2M_x - 1\}$ . In particular, when  $L \geq \sqrt{M_x}$ , it is sufficient to have  $2M_x - 1$  measurements. Since the bounds are based on the worst-case supports of the sequence  $q$  (cf. (10)), practical algorithms are able to identify the filters uniquely from fewer than  $2M_x - 1$  measurements for  $L > \sqrt{M_x}$ . To validate, we consider identifying the filters when the condition  $L \leq \sqrt{M_x}$  does not hold. In this simulation, we set  $M_x = 32$  and  $M = 2M_x$ . We generated the filters and source signal as in the first experiment. In this experiment, the filters are assumed to be identified if the normalized MSE is less than  $-20$  dB. The success rates that are averaged over 500 independent realizations of filters are shown in Fig. 7. We observe that for a sufficient number of Fourier measurements, the algorithms are able to identify the filters even for  $L \geq \sqrt{M_x} = 5.657$ .

5) *Comparison of TPI and BDC in the Presence of Measurement Noise:* In this experiment we compared the performance of the TPI and BDC methods together with NB-OMP in the presence of additive white Gaussian noise. We apply the settings of the first experiment (cf. Section VI-B1). The filters are assumed to be identified if the normalized MSE is less than  $-10$  dB. The success rates averaged over 500 independent realizations of filters are shown in Fig. 8. We observe that the algorithms are able to uniquely identify the filters for  $\text{SNR} > 5$  dB provided that  $K \geq 2L^2$ .

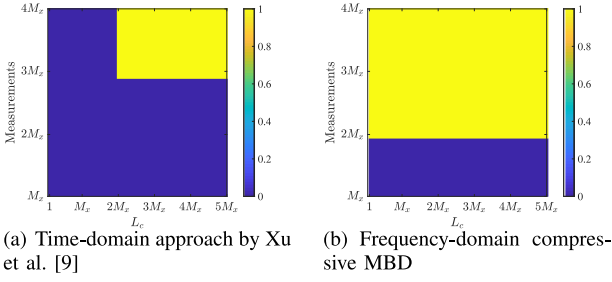


Fig. 5. Phase transition in the non-sparse FIR case: The success rate is plotted as a function of the linear complexity  $L_c$  of the source and the number of measurements  $M_s - M_x$ . Compressive MBD identifies the filter from at least  $2M_x - 1$  measurements and is independent of  $L_c$ . The approach by Xu *et al.* [9] requires at least  $3M_x$  measurements and  $L_c \geq 2M_x - 1$ .

### C. Comparison of Non-Sparse FIR MBD

Next, we show simulation results for the non-sparse case. As the present frequency-domain approach to derive the identifiability results and the time-domain approach taken by Xu *et al.* [9] are based on cross-correlation we compare these two techniques. In both approaches, the solution is computed by solving a homogeneous equation. In our frequency-domain settings, the solution is obtained by solving the homogeneous equation (22) via the optimization problem (23) without imposing the sparsity assumption. The solution is given as the eigenvector corresponding to the minimum eigenvalue of the matrix  $\mathbf{B}^H \mathbf{B}$ . A similar technique is applied in Xu *et al.* by replacing the matrix  $\mathbf{B}$  by a data matrix that consists of time-domain samples (cf. equations (7) and (20) in [9]). The identifiability results in the time-domain approach strongly depend on the linear complexity  $L_c$  of the source and on the length  $M_x$  of the source compared with the length of the filters  $M_x$ . We claim that our results do not depend on these parameters, but depend only on the number of frequency-measurements and non-vanishing property of the source at those frequencies. To justify our claim, we compare the MSE in estimating the filters by both approaches as a function of  $L_c$  and the number of measurements. To have a fair comparison, we use the same number of measurements for each experiment in both the methods. In [9], the number of measurements is given by  $M_s - M_x$  by assuming that  $M_s > M_x$ . In the proposed frequency-domain approach, we choose  $K$  as set of  $K = M_s - M_x$  consecutive integers. In the simulations, the source sequence  $s$  of length  $M_s$  with a linear complexity  $L_c$  is generated using the following model:

$$s[m] = \sum_{\ell=1}^{L_c} c_\ell r_\ell^m \quad \text{for } m = 0, 1, \dots, M_s - 1. \quad (24)$$

In the simulations, both  $c_\ell$  and  $r_\ell$  as well as the filters are generated randomly.

In both approaches, we assume that the filters are uniquely identifiable if the normalized MSE is less than  $-50$  dB. Fig. 5 shows the success rate over 200 independent realizations of the source and filters for Xu's [9] and the proposed methods. We observe that Xu's method is successful if  $L_c \geq 2M_x - 1$  and  $M_s \geq 4M_x$ , whereas our approach identifies the filters as long

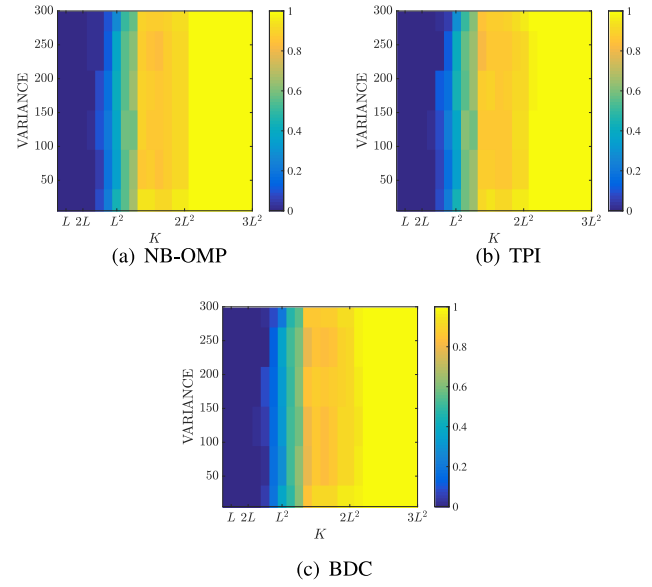


Fig. 6. Phase transition for the identification of  $L$  sparse filters in the two-channel case with varying source variance: The partial spectrum of the source is generated uniformly at random with mean value 100 and different variances. The SNR is 30 dB. The performance of all methods does not change considerably as variance of the source changes.

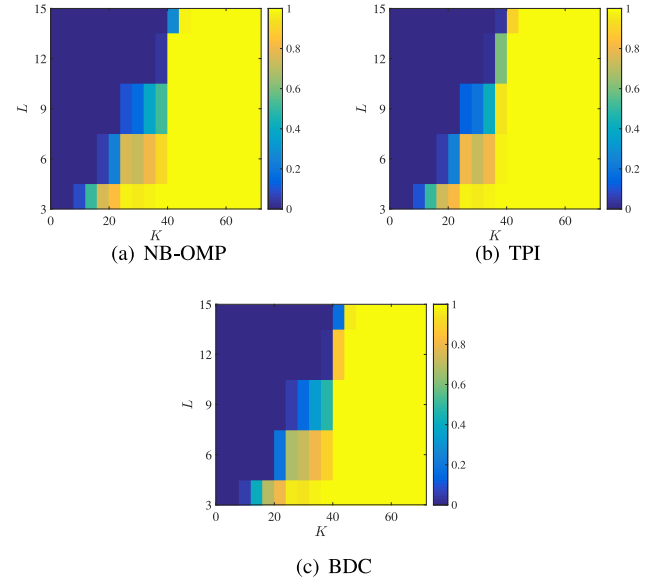


Fig. 7. Phase transition diagrams for success rate of different algorithms for a fixed  $M_x = 32$  and  $M = 2M_x$ . We observe that the algorithms identifies the filters even for  $L > \sqrt{M_x} = 5.657$  as opposed to the assumption in Theorem 1. In this case Corollary 3 for general sparsity case holds.

as  $K = M_s - M_x > 2M_x - 1$  and the results are independent of the linear complexity of the source.

## VII. PROOF OF LEMMA 1

Theorem 1 states that  $(\{S(e^{j\omega_0 k})\}_{k \in K}, x_1, x_2)$  is the unique solution to (9) up to the fundamental ambiguity in (2). Let  $(\{\hat{S}(e^{j\omega_0 k})\}_{k \in K}, \hat{x}_1, \hat{x}_2)$  be another solution to (9). Then there

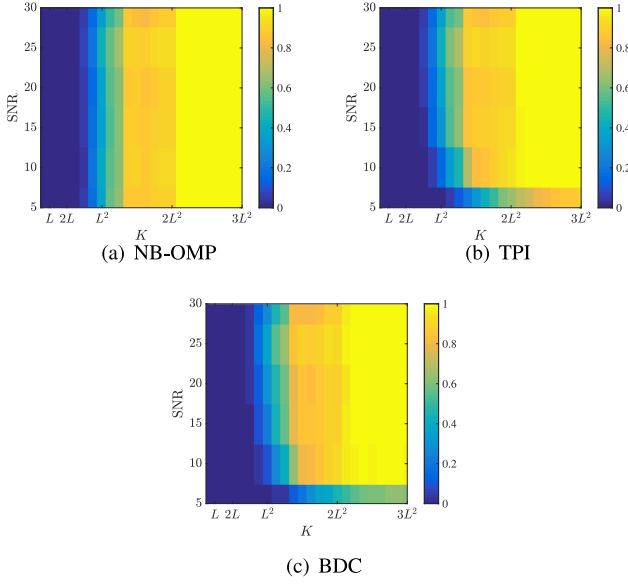


Fig. 8. Phase transition diagrams for success rate of different algorithms in the presence of noise. For SNR > 5 dB, the algorithms are able to uniquely identify the filters for  $K \geq 2L^2$ .

exist  $\alpha \neq 0$  and  $m_0 \in \mathbb{Z}$  such that

$$S(e^{j\omega_0 k}) = \alpha e^{j\omega_0 k m_0} \hat{S}(e^{j\omega_0 k}), \quad \forall k \in \mathcal{K}, \quad (25)$$

and

$$X_n(z) = \alpha^{-1} z^{-m_0} \hat{X}_n(z), \quad n = 1, 2, \quad (26)$$

if  $|\mathcal{K}| \geq 2L^2$ .

*Sufficiency part:* Since both  $(\{S(e^{j\omega_0 k})\}_{k \in \mathcal{K}}, x_1, x_2)$  and  $(\{\hat{S}(e^{j\omega_0 k})\}_{k \in \mathcal{K}}, \hat{x}_1, \hat{x}_2)$  are solutions to (9), we have

$$Y_n(e^{j\omega_0 k}) = S(e^{j\omega_0 k})X_n(e^{j\omega_0 k}) = \hat{S}(e^{j\omega_0 k})\hat{X}_n(e^{j\omega_0 k}), \quad (27)$$

for  $n = 1, 2$ , and  $k \in \mathcal{K}$ . We have assumed that  $\{S(e^{j\omega_0 k})\}_{k \in \mathcal{K}}$  and  $\{\hat{S}(e^{j\omega_0 k})\}_{k \in \mathcal{K}}$  are nonzero. Therefore it follows from (27) that

$$X_1(e^{j\omega_0 k})\hat{X}_2(e^{j\omega_0 k}) = \hat{X}_1(e^{j\omega_0 k})X_2(e^{j\omega_0 k}), \quad k \in \mathcal{K}. \quad (28)$$

Let  $Q(e^{j\omega_0 k})$  denote the sample of DTFT of

$$q = x_1 * \hat{x}_2 - x_2 * \hat{x}_1 \quad (29)$$

at frequency  $k\omega_0$ . Then by the convolution theorem and the linearity of DTFT, the identity in (28) is equivalently rewritten as

$$Q(e^{j\omega_0 k}) = 0, \quad k \in \mathcal{K}. \quad (30)$$

Let  $\mathbf{A} \in \mathbb{C}^{|\mathcal{K}| \times 2M_x - 1}$  denote a Vandermonde matrix with its  $(k, m)$ th entry given as  $e^{j(k-1)(m-1)\omega_0}$ . Then we can rewrite (30) as

$$\mathbf{A}\mathbf{q} = \mathbf{0}, \quad (31)$$

where  $\mathbf{q} = [q[0], q[1], \dots, q[2M_x - 2]]^T \in \mathbb{C}^{2M_x - 1}$ . Since  $\mathbf{A}$  is a Vandermonde matrix constructed by  $2M_x - 1$  distinct generators  $\{e^{jm\omega_0}\}_{m \in [2M_x]}$  and  $\mathcal{K}$  is a universal set,  $\mathbf{A}$  has full

spark. Therefore, if  $|\mathcal{K}| \geq |\text{supp}\{q\}|$ , then (31) implies  $\mathbf{q} = \mathbf{0}$ . Furthermore, since  $\text{supp}\{q\} \subset [2M_x]$ , it follows that  $q = 0$ , that is,

$$x_1 * \hat{x}_2 = x_2 * \hat{x}_1, \quad (32)$$

which implies via the  $z$ -transform that

$$X_1(z)/X_2(z) = \hat{X}_1(z)/\hat{X}_2(z). \quad (33)$$

For brevity, we introduce the following notation. For an FIR sequence  $x$ , let  $\mathcal{Z}_x$  denote the set of the zeros of its  $z$ -transform  $X(z)$ .

Since  $X_1(z)$  and  $X_2(z)$  do not share any common zeros, we obtain

$$\mathcal{Z}_{x_1} \subseteq \mathcal{Z}_{\hat{x}_1}. \quad (34)$$

Hence, there is a polynomial  $H(z)$  that satisfies

$$\hat{X}_1(z) = X_1(z)H(z).$$

Then it follows from (33) that

$$\hat{X}_2(z) = X_2(z)H(z).$$

Since  $\hat{X}_1(z)$  and  $\hat{X}_2(z)$  do not share any common zeros except at  $z = 0$ , we conclude that  $H(z) = \alpha z^{m_0}$  for some  $m_0 \in \mathbb{N}$  and  $\alpha \in \mathbb{C} \setminus \{0\}$ . It remains to show that  $\|q\|_0 = |\text{supp}\{q\}| \leq 2L^2$ . Without any assumption on the support structure of the filters the support of sequence  $q$  depends only on  $L$ . Since  $L < \sqrt{M_x}$ , we have  $\|x_1 * \hat{x}_2\|_0 = \|x_2 * \hat{x}_1\|_0 \leq L^2$ . Hence

$$\|q\|_0 = \|x_1 * \hat{x}_2 - x_2 * \hat{x}_1\|_0 \leq 2L^2. \quad (35)$$

*Necessity part:* We show that if  $|\mathcal{K}| < 2L$ , then there exist distinct solutions  $(\{S(e^{j\omega_0 k})\}_{k \in \mathcal{K}}, x_1, x_2)$  and  $(\{\hat{S}(e^{j\omega_0 k})\}_{k \in \mathcal{K}}, \hat{x}_1, \hat{x}_2)$  to (9) such that (25) and (26) are not satisfied for any  $\alpha$  and  $n_0$ . It suffices to show that

$$q = x_1 * \hat{x}_2 - x_2 * \hat{x}_1 \neq 0, \quad \text{but } Q(e^{j\omega_0 k}) = 0, \quad \forall k \in \mathcal{K}.$$

Before proceeding further, we define the following notations to prove the necessary part. Let  $\bar{\mathbf{A}}$  denotes the  $|\mathcal{K}| \times M_x$  matrix that consists of first  $M_x$  columns of matrix  $\mathbf{A}$ . Next, let  $\mathbf{x}_1, \mathbf{x}_2, \hat{\mathbf{x}}_1$ , and  $\hat{\mathbf{x}}_2$  denote  $L$ -sparse vectors in  $\mathbb{C}^{M_x}$  that are constructed by considering the first  $M_x$  values of the sequences  $x_1, x_2, \hat{x}_1, \hat{x}_2$ , respectively.

Since the matrix  $\bar{\mathbf{A}}$  has full spark, there exist distinct pairs of  $L$ -sparse vectors  $(\mathbf{x}_1, \hat{\mathbf{x}}_1)$  and  $(\mathbf{x}_2, \hat{\mathbf{x}}_2)$  such that  $\bar{\mathbf{A}}\mathbf{x}_1 = \bar{\mathbf{A}}\hat{\mathbf{x}}_1$  and  $\bar{\mathbf{A}}\mathbf{x}_2 = \bar{\mathbf{A}}\hat{\mathbf{x}}_2$  as long as  $|\mathcal{K}| < 2L$ . Hence, we have

$$\bar{\mathbf{A}}\mathbf{x}_1 \odot \bar{\mathbf{A}}\hat{\mathbf{x}}_2 = \bar{\mathbf{A}}\hat{\mathbf{x}}_1 \odot \bar{\mathbf{A}}\hat{\mathbf{x}}_2, \quad (36)$$

which is equivalent to (28) or  $Q(e^{j\omega_0 k}) = 0$  for all  $k \in \mathcal{K}$ . Since  $\mathbf{x}_1 \neq \hat{\mathbf{x}}_1$  and  $\mathbf{x}_2 \neq \hat{\mathbf{x}}_2$ , we also have that corresponding sequence  $q \neq 0$ . Hence, for  $|\mathcal{K}| < 2L$  the problem in (9) does not have unique solution for all the filter pairs  $(x_1, x_2)$ .

## VIII. CONCLUSION

In this paper, we derived identifiability conditions for multichannel blind deconvolution when the filters follow a fully deterministic sparsity model and the source has finite support.



We showed that when there exist at least a pair of two mutually coprime filters, it is sufficient to take  $2L^2$  Fourier measurements from those channels for the unique identification of the filters. To identify the source uniquely, we derive conditions on the number of measurements and number of channels in terms of the support of the source and sparsity of the filters. The results improve upon existing MBD results both in terms of the number of measurements and the number of channels required for unique identifiability and also apply to the non-sparse settings.

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