

# Compressive Shift Retrieval

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**Abstract**—The classical shift retrieval problem considers two signals in vector form that are related by a shift. This problem is of great importance in many applications and is typically solved by maximizing the cross-correlation between the two signals. Inspired by compressive sensing, in this paper, we seek to estimate the shift directly from compressed signals. We show that under certain conditions, the shift can be recovered using fewer samples and less computation compared to the classical setup. We also illustrate the concept of superresolution for shift retrieval. Of particular interest is shift estimation from Fourier coefficients. We show that under rather mild conditions only one Fourier coefficient suffices to recover the true shift.

**Index Terms**—Parameter estimation, compressed sensing, signal processing algorithms, signal sampling.

## I. INTRODUCTION

SHIFTS retrieval between two given signals is a fundamental problem in many signal processing applications. For example, to map the ocean floor, an active sonar can be used. The sonar transmits reference sound pulses through the water, and the time it takes to receive the echoes of the pulses indicates the depth of the ocean floor. In target tracking using two acoustic sensors, the time shift when a sound wave of a vehicle reaches the microphones indicates the direction to the vehicle. In global positioning system (GPS) receivers, the cor-

rect alignment between CDMA (code division multiple access) codes is sought [1] in order to calculate the synchronization delay necessary for determining its position. In the case of a time shift, the shift retrieval problem is often referred to as *time delay estimation* (TDE) [2]. In computer vision, the spatial shift relating two images is often sought and referred to as image registration or alignment [3]–[5].

Traditionally, the shift retrieval problem is solved by maximizing the cross-correlation between the two signals [6]. In this paper, we revisit this classical problem, and show how the basic premise of *compressive sensing* (CS) [7]–[10] can be used in the context of shift retrieval. This allows to recover the shift from compressed data leading to computational and storage savings.

Compressive sensing is a sampling scheme that makes it possible to sample at the *information rate* instead of the classical Nyquist rate predicted by the bandwidth of the signal [11]. The majority of the results in compressive sensing discuss conditions and methods for guaranteed reconstruction from an under-sampled version of the signal. Therefore, the information rate is typically referred to as the one that guarantees recovery of the sparse signal.

However, for many applications such as the aforementioned examples in shift retrieval, obtaining the signal may not be needed. The goal is to recover some properties or statistics of the unknown signal. Taking the active sonar for example, one may wonder if it is really necessary to sample at a rate which is twice that of the bandwidth of the transmitted signal so that the received signal can be exactly reconstructed? Clearly the answer is no. Since the signal itself is not of interest to the application, we might consider an alternative sampling scheme to directly estimate the shift without first reconstructing the signal. These ideas have in fact been recently explored in the context of radar and ultrasound [12]–[14] with continuous time signals and multiple shifts. Here we consider a related problem and ask: *What is the minimal information rate for shift retrieval when two related discrete-time signals are under-sampled?*

It turns out that under rather mild conditions, we only need fractions of the signals. In fact, we will show that only one Fourier coefficient from each of the signals suffices to recover the true shift. We refer to this method as *compressive shift retrieval* (CSR). It should be made clear that CSR does not assume that any of the involved signals are sparse.

As the main contribution of the paper, we will show that when the sensing matrix is taken to be a partial Fourier matrix, under suitable conditions, the true shift can be recovered from both noise-free and noisy measurements using CSR. In fact, our results show that in some cases sampling as few as one Fourier coefficient is enough to perfectly recover the true shift. Furthermore, CSR also reduces the computational load. This is of particular interest since recent developments in sampling [13], [15],

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[16] have shown that Fourier coefficients can be efficiently obtained from space (or time) measurements by the use of an appropriate filter and by subsampling the output. From a user perspective, for example, reduced computational complexity could elongate battery life of GPS enabled devices. Finally, we introduce the concept of shift retrieval in superresolution, whereby shifts can be recovered at finer granularities than those of the received signals, based on the proposed CSR framework.

### A. Prior Work

Compressive signal alignment problems have been addressed in only a few publications and, to the authors' best knowledge, not in the same setup studied in this paper. In [17], the authors considered alignment of images under random projection. The work was based on the Johnson-Lindenstrauss property of random projection and proposed an objective function that can be solved efficiently using *difference-of-two-convex programming* algorithms. In this paper, we instead focus on proving theoretical guarantees of exact shift recovery when the signal is subsampled by a partial Fourier basis. The theory developed in [17] does not apply to this setup.

The *smashed filter* [18] is another related technique. It is a general framework for maximum likelihood hypothesis testing and can be seen as a *matched filter* of reduced dimension. It can therefore be applied to the shift retrieval problem. The underlying idea of both the smashed filter and CSR are the same in that both approaches try to avoid reconstructing the signal and extract the sought descriptor, namely, the shift, from compressive measurements. However, the analysis and assumptions are very different. For CSR, we develop requirements for guaranteed recovery of the true shift using a given measurement matrix. For the smashed filter, the analysis focuses on random orthogonal projections and provides probabilities for correct recovery as a function of the number of projections.

Motivated by the GPS locking problem, [19], [20] studied computationally efficient algorithms for recovering shifts of a random code sequence, which may be corrupted by Gaussian noise. Their algorithms exploit the sparse nature of the signal matching problem, where the optimal signal alignment causes the cross-correlation between the source signal and the measurement to spike. The main limitation of the work is that their analysis assumes the source signals are sampled randomly with binary values in  $\{-1, 1\}$ . In this paper, our analysis of CSR and the conditions for guaranteed recovery is not restricted to signals with binary values. Motivated by the compressive sensing framework, the new algorithms are also sufficiently different from the ones in [19], [20].

### B. Notation

We use normal fonts to represent scalars and bold fonts for vectors and matrices. The notation  $|\cdot|$  represents the absolute value for scalars and it returns the cardinality of a set if the argument is a set. For both vectors and matrices,  $\|\cdot\|_0$  is the  $\ell_0$ -norm that counts the number of nonzero elements of its argument. Similarly,  $\|\cdot\|_p$  represents the  $\ell_p$ -norm: for a vector  $\mathbf{x}$ ,  $\|\mathbf{x}\|_p \triangleq (\sum_i |x_i|^p)^{1/p}$ , where  $x_i$  is the  $i$ th element of  $\mathbf{x}$ , and for a matrix  $\mathbf{X}$ ,  $\|\mathbf{X}\|_p \triangleq (\sum_{i,j} |X_{i,j}|^p)^{1/p}$ , where  $X_{i,j}$  is the  $(i, j)$ -th element of  $\mathbf{X}$ . Furthermore,  $\mathbf{X}^*$  denotes the complex conjugate transpose of  $\mathbf{X}$ . Let  $\mathbf{I}_{n \times n}$  denote an  $n \times n$  identity

matrix,  $\mathbf{0}_{m \times n}$  an  $m \times n$  matrix of zeros, and  $\mathbb{Z}$  be the set of integers.  $\Re\{\cdot\}$  returns the real part of its argument and  $\lfloor \cdot \rfloor$  denotes the floor function.

We say that two  $n$ -dimensional vectors  $\mathbf{y}$  and  $\mathbf{x}$  are related by an  $l$  cyclic-shift if  $\mathbf{y} = \mathbf{D}^l \mathbf{x}$ , where  $\mathbf{D}^l$  is defined as

$$\mathbf{D}^l = \begin{bmatrix} \mathbf{0}_{l \times (n-l)} & \mathbf{I}_{l \times l} \\ \mathbf{I}_{(n-l) \times (n-l)} & \mathbf{0}_{(n-l) \times l} \end{bmatrix}. \quad (1)$$

Throughout the paper, we assume that the (cyclic) shift is unique up to a multiple of  $n$ .

### C. Organization

In Sections II and III, we study the CSR problem under the assumption that the measurements are noise free. We extend the results to noisy measurements in Section IV. As we are particularly interested in Fourier measurements, we tailor the results to this choice of sensing matrix. We illustrate the concept of superresolution for shift retrieval in Section V, and conclude in Section VI.

## II. NOISE-FREE COMPRESSIVE SHIFT RETRIEVAL

Consider two vectors  $\mathbf{x} \in \mathbb{C}^n$  and  $\mathbf{y} \in \mathbb{C}^n$  that are related by a cyclic shift, i.e.,  $\mathbf{y} = \mathbf{D}^l \mathbf{x}$ . The shift retrieval problem is a multi-hypothesis testing problem: Define the  $s$ th hypothesis  $\mathcal{H}_s$ ,  $s = 0, \dots, n-1$ , as

$$\mathcal{H}_s : \mathbf{x} \text{ is related to } \mathbf{y} \text{ via a } s\text{-cyclic-shift,}$$

and accept  $\mathcal{H}_s$  if  $\mathbf{y} = \mathbf{D}^s \mathbf{x}$  and otherwise reject. Since the true shift is assumed unique, only one hypothesis will be accepted.

To determine  $s$  we minimize the error  $\|\mathbf{y} - \mathbf{D}^s \mathbf{x}\|_2$ . Now,

$$\begin{aligned} \|\mathbf{y} - \mathbf{D}^s \mathbf{x}\|_2^2 &= \|\mathbf{y}\|_2^2 + \|\mathbf{D}^s \mathbf{x}\|_2^2 - \mathbf{y}^* \mathbf{D}^s \mathbf{x} - \mathbf{x}^* \mathbf{D}^s \mathbf{y} \\ &= \|\mathbf{y}\|_2^2 + \|\mathbf{x}\|_2^2 - 2\Re\{\langle \mathbf{y}, \mathbf{D}^s \mathbf{x} \rangle\} \end{aligned} \quad (2)$$

where we used the fact that  $\|\mathbf{D}^s \mathbf{x}\|_2^2 = \|\mathbf{x}\|_2^2$ . Therefore, minimizing  $\|\mathbf{y} - \mathbf{D}^s \mathbf{x}\|_2^2$  is equivalent to maximizing the real part of the cross-correlation with respect to  $s$ :

$$\max_{s \in \{0, \dots, n-1\}} \Re\{\langle \mathbf{y}, \mathbf{D}^s \mathbf{x} \rangle\}. \quad (3)$$

The goal of CSR is to recover the shift  $l$  relating  $\mathbf{x}$  and  $\mathbf{y}$  from compressed measurements  $\mathbf{z}$  and  $\mathbf{v}$ . The compressed measurement signals are assumed related to the ground-truth signals  $\mathbf{x} \in \mathbb{C}^n$  and its shifted version  $\mathbf{y} = \mathbf{D}^l \mathbf{x} \in \mathbb{C}^n$  via a sensing matrix  $\mathbf{A} \in \mathbb{C}^{m \times n}$ , with  $m \leq n$ :

$$\mathbf{z} = \mathbf{A}\mathbf{y} \in \mathbb{C}^m \quad \text{and} \quad \mathbf{v} = \mathbf{A}\mathbf{x} \in \mathbb{C}^m. \quad (4)$$

Since only the compressed measurements  $\mathbf{z}$  and  $\mathbf{v}$  are assumed available, we cannot evaluate  $\mathbf{y} = \mathbf{D}^s \mathbf{x}$  or maximize  $\Re\{\langle \mathbf{y}, \mathbf{D}^s \mathbf{x} \rangle\}$  for each hypothesis  $s = 0, \dots, n-1$ . However, if  $\mathbf{A}^* \mathbf{A}$  and  $\mathbf{D}^s$  commute for all  $s = 0, \dots, n-1$ , then

$$\mathbf{y} = \mathbf{D}^s \mathbf{x} \quad \Rightarrow \quad \mathbf{A}^* \mathbf{A}\mathbf{y} = \mathbf{A}^* \mathbf{A} \mathbf{D}^s \mathbf{x} = \mathbf{D}^s \mathbf{A}^* \mathbf{A}\mathbf{x}. \quad (5)$$

Therefore,

$$\mathbf{A}^* \mathbf{z} = \mathbf{D}^s \mathbf{A}^* \mathbf{v}. \quad (6)$$

Hence, in this case, we can consider the test:

$$\text{Accept } \mathcal{H}_s \text{ if } \mathbf{A}^* \mathbf{z} = \mathbf{D}^s \mathbf{A}^* \mathbf{v} \text{ and otherwise reject.} \quad (7)$$

It is clear that if  $s$  is such that  $\mathbf{y} = \mathbf{D}^s \mathbf{x}$ , then  $\mathbf{A}^* \mathbf{z} = \mathbf{D}^s \mathbf{A}^* \mathbf{v}$  will also hold. However, the other way around might not be true. Therefore, we might erroneously accept a wrong hypotheses using (7). Theorem 1 below lists conditions under which testing (7) is guaranteed to accept the correct hypothesis.

Before stating the theorem, note that testing the condition  $\mathbf{A}^* \mathbf{z} = \mathbf{D}^s \mathbf{A}^* \mathbf{v}$  is equivalent to minimizing  $\|\mathbf{A}^* \mathbf{z} - \mathbf{D}^s \mathbf{A}^* \mathbf{v}\|_2^2$  with respect to  $s$ . Now,

$$\begin{aligned} & \|\mathbf{A}^* \mathbf{z} - \mathbf{D}^s \mathbf{A}^* \mathbf{v}\|_2^2 \\ &= \|\mathbf{A}^* \mathbf{z}\|_2^2 + \|\mathbf{D}^s \mathbf{A}^* \mathbf{v}\|_2^2 - \mathbf{z}^* \mathbf{A} \mathbf{D}^s \mathbf{A}^* \mathbf{v} - \mathbf{v}^* \mathbf{A} \mathbf{D}^s \mathbf{A}^* \mathbf{z} \\ &= \|\mathbf{A}^* \mathbf{z}\|_2^2 + \|\mathbf{A}^* \mathbf{v}\|_2^2 - 2\Re\{\langle \mathbf{z}, \mathbf{A} \mathbf{D}^s \mathbf{A}^* \mathbf{v} \rangle\}. \end{aligned} \quad (8)$$

Since the only term depending on  $s$  is  $2\Re\{\langle \mathbf{z}, \mathbf{A} \mathbf{D}^s \mathbf{A}^* \mathbf{v} \rangle\}$ , seeking  $s$  satisfying  $\mathbf{A}^* \mathbf{z} = \mathbf{D}^s \mathbf{A}^* \mathbf{v}$  is equivalent to maximizing  $\Re\{\langle \mathbf{z}, \mathbf{A} \mathbf{D}^s \mathbf{A}^* \mathbf{v} \rangle\}$ . Note that if  $\mathbf{A}^* \mathbf{A} = \mathbf{I}_{n \times n}$ , then the implication in (5) holds in both directions and maximizing  $\Re\{\langle \mathbf{z}, \mathbf{A} \mathbf{D}^s \mathbf{A}^* \mathbf{v} \rangle\}$  reduces to the classical test using uncompressed signals given in (3).

*Theorem 1 (Shift Recovery from Low-Rate Data):* Let  $\mathbf{X}$  be an  $n \times n$  matrix with  $i$ th column equal to  $\mathbf{D}^i \mathbf{x}$ ,  $i = 1, \dots, n$ , and define  $\bar{\mathbf{D}}^s = \mathbf{A} \mathbf{D}^s \mathbf{A}^*$ . If the sensing matrix  $\mathbf{A}$  satisfies the following conditions:

- 1)  $\mathbf{A}^* \mathbf{D}^s = \mathbf{D}^s \mathbf{A}^* \mathbf{A}$ ,
  - 2)  $\exists \alpha \in \mathbb{R}, \alpha \mathbf{A} \mathbf{A}^* = \mathbf{I}$  and
  - 3) all columns of  $\mathbf{A} \mathbf{X}$  are different,
- then

$$\max_{s \in \{0, \dots, n-1\}} \Re\{\langle \mathbf{z}, \bar{\mathbf{D}}^s \mathbf{v} \rangle\} \quad (9)$$

or equivalently the test (7) recovers the true shift.

*Proof:* See Appendix A.  $\blacksquare$

The conditions of Theorem 1 may seem restrictive. However, as we will show in Lemma 3, if  $\mathbf{A}$  is chosen as a partial Fourier matrix, then the first two conditions are trivially satisfied. The last condition is the only one that needs to be checked and will lead to a requirement on the sampled Fourier coefficients.

The conditions of Theorem 1 can be checked prior to estimating the shift. However, knowing the estimate of the shift, it is easy to see from the proof (see the proof of Lemma 9) that it is enough to check if the column of  $\mathbf{A} \mathbf{X}$  associated with the estimate of the shift is different than all the other columns of  $\mathbf{A} \mathbf{X}$ . Hence, we do not need to check if all columns of  $\mathbf{A} \mathbf{X}$  are different. This conclusion is formulated in the following corollary, which is less conservative than Theorem 1.

*Corollary 2 (Test for True Shift):* Let  $\mathbf{X}$  be an  $n \times n$  matrix with the  $i$ th column equal to  $\mathbf{D}^i \mathbf{x}$ ,  $i = 1, \dots, n$ , and define  $\bar{\mathbf{D}}^s = \mathbf{A} \mathbf{D}^s \mathbf{A}^*$ . If the sensing matrix  $\mathbf{A}$  satisfies the following conditions:

- 1)  $\mathbf{A}^* \mathbf{A} \mathbf{D}^s = \mathbf{D}^s \mathbf{A}^* \mathbf{A}$ , and
  - 2)  $\exists \alpha \in \mathbb{R}, \alpha \mathbf{A} \mathbf{A}^* = \mathbf{I}$ ,
- then

$$s^* = \arg \max_{s \in \{0, \dots, n-1\}} \Re\{\langle \mathbf{z}, \bar{\mathbf{D}}^s \mathbf{v} \rangle\} \quad (10)$$

is the true shift if the  $s^*$ th column of  $\mathbf{A} \mathbf{X}$  is different than all the other columns of  $\mathbf{A} \mathbf{X}$ .

*Proof:* See Appendix A.  $\blacksquare$

### III. COMPRESSIVE SHIFT RETRIEVAL USING FOURIER COEFFICIENTS

Of particular interest is the case in which  $\mathbf{A}$  is made up of a partial Fourier basis. That is,  $\mathbf{A} \in \mathbb{C}^{m \times n}$  takes the form

$$\mathbf{A} = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & e^{-\frac{2j\pi k_1}{n}} & e^{-\frac{4j\pi k_1}{n}} & \cdots & e^{-\frac{2(n-1)j\pi k_1}{n}} \\ 1 & e^{-\frac{2j\pi k_2}{n}} & \cdot & \cdot & e^{-\frac{2(n-1)j\pi k_2}{n}} \\ \vdots & \vdots & \cdot & \cdot & \cdot \\ 1 & e^{-\frac{2j\pi k_m}{n}} & e^{-\frac{4j\pi k_m}{n}} & \cdots & e^{-\frac{2(n-1)j\pi k_m}{n}} \end{bmatrix}$$

where  $k_1, \dots, k_m \in \{0, 1, 2, \dots, n-1\}$ ,  $m \leq n$ . For this specific choice,

$$\mathbf{A} \mathbf{X} = \frac{1}{\sqrt{n}} \begin{bmatrix} X_{k_1} & X_{k_1} e^{\frac{2j\pi k_1}{n}} & \cdots & X_{k_1} e^{\frac{2(n-1)j\pi k_1}{n}} \\ X_{k_2} & \cdot & \cdot & X_{k_2} e^{\frac{2(n-1)j\pi k_2}{n}} \\ \vdots & \cdot & \cdot & \cdot \\ X_{k_m} & X_{k_m} e^{\frac{2j\pi k_m}{n}} & \cdots & X_{k_m} e^{\frac{2(n-1)j\pi k_m}{n}} \end{bmatrix}$$

where  $X_r$  denotes the  $r$ th Fourier coefficient of the Fourier transform of  $\mathbf{x}$ .

For a sensing matrix made up by a partial Fourier basis, we have the following useful result:

*Lemma 3:* Let  $\mathbf{A}$  be a partial Fourier matrix. Then  $\mathbf{D}^s \mathbf{A}^* \mathbf{A} = \mathbf{A}^* \mathbf{A} \mathbf{D}^s$  for all  $s = 0, \dots, n-1$ .

*Proof:* See Appendix A.  $\blacksquare$

Applying this result to Theorem 1 gives the following corollary:

*Corollary 4 (Shift Recovery from Low Rate Fourier Data):* Suppose  $\mathbf{A}$  is chosen as a partial Fourier matrix with  $k_1, \dots, k_m \in \{0, 1, 2, \dots, n-1\}$ ,  $m \leq n$ . Let  $z_i$  and  $v_i$  be the  $i$ th elements of  $\mathbf{z} = \mathbf{A} \mathbf{y}$  and  $\mathbf{v} = \mathbf{A} \mathbf{x}$ . Then (9) is simplified as

$$\max_{s \in \{0, \dots, n-1\}} \Re\left\{ \sum_{i=1}^m z_i^* v_i e^{-\frac{2j\pi k_i s}{n}} \right\}, \quad (11)$$

and it recovers the true shift if there exists  $p \in \{1, \dots, m\}$  such that  $X_{k_p} \neq 0$  and  $\{1, \dots, n-1\} \frac{k_p}{n}$  contains no integers. In particular, measuring only the first Fourier coefficients ( $k_1 = 1$ ) of  $\mathbf{x}$  and  $\mathbf{y}$  would, as long as the coefficients are nonzero, suffice to recover the true shift.

*Proof:* See Appendix A.  $\blacksquare$

Remarkably, in the extreme case when  $m = 1$ , the corollary states that all we need is two scalar measurements,  $z$  and  $v$ , to perfectly recover the true shift. The scalar measurements can be any nonzero Fourier coefficient of  $\mathbf{x}$  and  $\mathbf{y}$  as long as  $\{1, \dots, n-1\} \frac{k_1}{n}$  contains no integers. As noted in the corollary, the first Fourier coefficients ( $k_1 = 1$ ) of  $\mathbf{x}$  and  $\mathbf{y}$  would suffice. Also note that only  $2mn$  multiplications are required to evaluate the test. This should be compared with  $\mathcal{O}(n \log n)$  multiplications to evaluate the cross-correlation for the full uncompressed signals  $\mathbf{x}$  and  $\mathbf{y}$  [21].

*Example 1 (Noise Free Compressive Shift Retrieval):* To validate the results, we carry out the following example. In each trial we let the sample dimension  $m$  and the shift  $l$  be random integers between 1 and 9 and generate  $\mathbf{x}$  by sampling from an  $n$ -dimensional uniform distribution. We let  $n = 10$  and generate a partial Fourier matrix by picking  $k_1, \dots, k_m$  from  $\{0, 1, \dots, 9\}$  at random without replacement. The coefficients  $k_1, \dots, k_m$  are

regenerated if the assumptions of Corollary 4 are not met. The true shift is successfully recovered in each trial by the simplified test (11), namely, with 100% success rate. This is quite remarkable since when  $m = 1$ , we recover the true shift using only two scalar measurements  $\mathbf{z}$  and  $\mathbf{v}$  and one fifth of the multiplications that (3) would require.

#### IV. NOISY COMPRESSIVE SHIFT RETRIEVAL

We now consider the noisy version of CSR, where the measurements  $\mathbf{z}$  and  $\mathbf{v}$  are perturbed by noise:

$$\tilde{\mathbf{z}} = \mathbf{z} + \mathbf{e}_z, \quad \tilde{\mathbf{v}} = \mathbf{v} + \mathbf{e}_v. \quad (12)$$

Similar to the noise-free case, we can guarantee recovery of the true shift. In particular, if the columns of the noisy version of  $\mathbf{A}\mathbf{X}$  are far enough apart with respect to the noise, then it can be shown that the columns of the noise free version of  $\mathbf{A}\mathbf{X}$  are distinct and the true shift is recovered. We note that a more natural scenario in some applications might be to assume that  $\mathbf{v}$  is a known reference signal and noise free. Our derivations below also handle this case by setting  $\mathbf{e}_v = 0$ .

Our main result is given in the following theorem:

*Theorem 5 (Noisy Shift Recovery from Low-Rate Data):* Let  $\tilde{\mathbf{x}}$  be such that  $\tilde{\mathbf{v}} = \mathbf{A}\tilde{\mathbf{x}}$  and let the  $i$ th column of  $\tilde{\mathbf{X}}$  be  $\tilde{\mathbf{x}}$  shifted by  $i$ , i.e.,  $\mathbf{D}^i \tilde{\mathbf{x}}$ . Assume that  $\mathbf{A}$  is a partial Fourier matrix and that the noisy measurements are used in (11) to estimate the shift. If the  $\ell_2$ -norm difference between any two columns of  $\mathbf{A}\tilde{\mathbf{X}}$  is greater than

$$\Delta_{zv} \triangleq \|\mathbf{e}_z\|_2 + \|\mathbf{e}_v\|_2 + \sqrt{\|\tilde{\mathbf{v}}\|_2^2 + \|\tilde{\mathbf{z}}\|_2^2 - 2 \max_{s \in \{0, \dots, n-1\}} \Re \left\{ \langle \tilde{\mathbf{z}}, \mathbf{D}^s \tilde{\mathbf{v}} \rangle \right\}}, \quad (13)$$

then the estimate of the shift is not affected by the noise.

*Proof 4:* See Appendix B. ■

The result of Theorem 5 is that, by requiring the difference between columns of  $\mathbf{A}\tilde{\mathbf{X}}$  to be greater than  $\Delta_{zv}$ , we assure that the noise does not affect the outcome of the test (11).

Note that  $\|\mathbf{e}_z\|_2$  and  $\|\mathbf{e}_v\|_2$  might not be available in practice but could be replaced by an upper bound on the  $\ell_2$ -norm of the noise if the noise is known to be bounded. Also note that the theorem only states that the noise does not affect the estimate of the shift. It does not state that the shift will be the true shift.

*Example 2 (Recovery of a Shift from Noisy Data):* We illustrate the results by running a Monte Carlo simulation consisting of 10,000 trials for each sample dimension  $m = 1, \dots, 10$ , and for two different SNR levels. In Fig. 1, 10 histograms are shown (corresponding to  $m = 1, \dots, 10$ ) for SNR =  $\|\mathbf{z}\|_2^2 / \|\mathbf{e}_z\|_2^2$  being 2 (low SNR) and in Fig. 2, SNR = 10 (high SNR). The errors  $\mathbf{e}_z$  and  $\mathbf{e}_v$  are both generated by sampling from  $\mathcal{N}(0, \sigma^2) + j\mathcal{N}(0, \sigma^2)$ . We further use  $n = 10$ ,  $l = 5$  and sample  $\mathbf{x}$  from a uniform (0,1)-distribution. The conclusion from the simulation is that the smaller the  $m$ , the more the estimate of the shift is sensitive to noise. Notice that when  $m = 10$ , the test (11) reduces to the classical test of maximizing the cross-correlation.

We can use Theorem 5 to check if the noise affects the estimate of the shift or not in each of the trials. For  $m = 2$  and high SNR, 40% of the trials satisfy the conditions of Theorem 5 and the noise therefore does not affect the shift estimates in

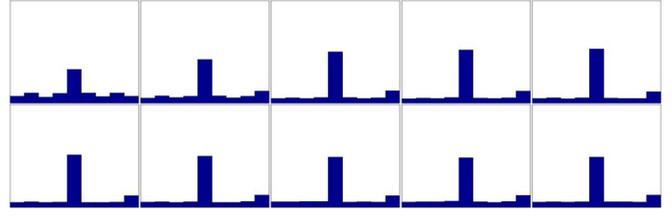


Fig. 1. Histogram plots for the estimated shift and low SNR. From left to right, top to bottom,  $m = 1, \dots, 10$ . The true shift is set to 5 in all trials.

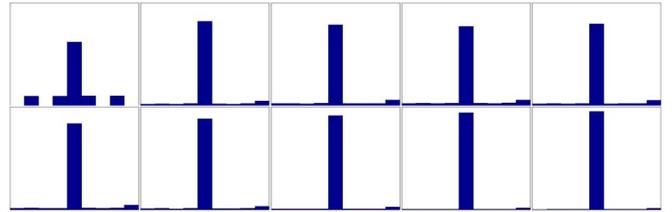


Fig. 2. Histogram plots for the estimated shift and high SNR. From left to right, top to bottom,  $m = 1, \dots, 10$ . The true shift is set to 5 in all trials.

those cases. Of the trials that satisfy the conditions, all successfully predict the true shift and none of them are false shifts. Note however that Theorem 5 only states that if the conditions are satisfied, then the estimated shift is the same as if we would have used the noise free compressed measurements in the test (11). It does not state that the estimate will be the true shift.

*Example 3 (Varying SNR):* To further illustrate the ability of CSR to handle different SNR levels, in this example, we vary SNR and study the recovery rate predicted by the theory and obtained in simulations. Let  $n = 100$ ,  $m = 10$  and generate  $\mathbf{x}$  by sampling from a uniform distribution between 0 and 1 (we have also sampled from a standard Gaussian distribution but the results were essentially the same and therefore not shown here). We generate the true shift by sampling an integer randomly between 0 and 99, add Gaussian complex noise to  $\mathbf{z}$  and  $\mathbf{v}$  and repeat the experiment 1,000 times for each SNR level. The results are shown in Fig. 3. The solid curve shows the rate of recovery seen in simulations. The dashed curve shows rate at which the condition (13) of Theorem 5 holds. The dashed-dotted line shows the rate at which the difference between columns of  $\mathbf{A}\tilde{\mathbf{X}}$  is greater than  $2\|\mathbf{e}_v\|$ . This test, as shown in Corollary 6, is relevant for guaranteeing that the estimated shift is the true one.

Theorem 5 provides conditions under which the noise does not affect the estimate of the shift. A better result would be to guarantee the recovery of the true shift. We saw in the first part of this paper that if the columns of  $\mathbf{A}\mathbf{X}$  are distinct, then the true shift is recovered. To guarantee the recovery of the true shift from noisy measurements we need:

- 1) that (11) gives the same shift estimate for the noisy measurements as for the noise free measurement (Theorem 5), and in addition,
- 2) that the columns of  $\mathbf{A}\tilde{\mathbf{X}}$  are far enough apart so that if the noise would be removed, the columns would still be distinct. The details are given by the following corollary.

*Corollary 6 (Recovery of the True Shift from Noisy Low-Rate Data):* If the  $\ell_2$ -norm difference between any two columns of  $\mathbf{A}\tilde{\mathbf{X}}$  is greater than  $2\|\mathbf{e}_v\|_2$  and the conditions of Theorem 5 are fulfilled, then (11) recovers the true shift.

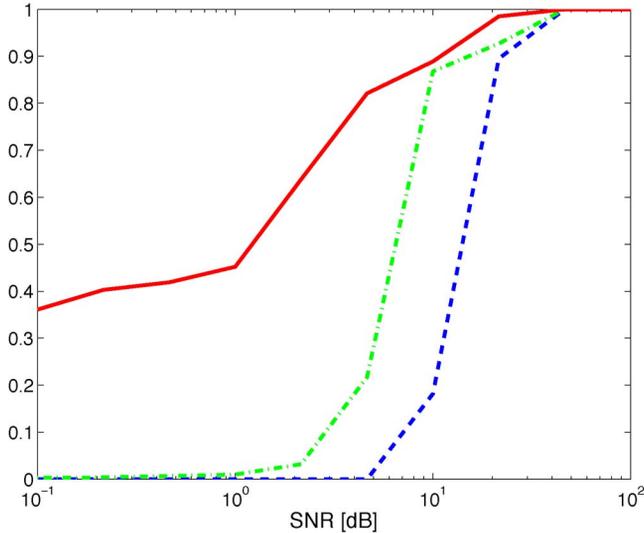


Fig. 3. The recovery rate (solid line), the rate at which the condition of Theorem 5 holds (dashed curve), the rate at which the difference between columns of  $\mathbf{A}\tilde{\mathbf{X}}$  are greater than  $2\|\mathbf{e}_v\|$  (dashed-dotted line) for SNRs between 0.1 and 100 dB.

*Proof:* See Appendix B. ■

Note that the result is not independent of the noise  $\mathbf{e}_z$  since the conditions of Theorem 5 depend on it.

If the estimate of the shift has been computed, a less conservative test can be used to check if the computed estimate has been affected by noise and if it is the true one. We summarize our conclusion in the following corollary.

*Corollary 7 (Test for True Shift in the Presence of Noise):* Assume that (11) gives  $s^*$  as an estimate of the shift. If the  $\ell_2$  difference between any column and the  $s^*$ -column of  $\mathbf{A}\tilde{\mathbf{X}}$  is greater than  $2\|\mathbf{e}_v\|_2$  and  $\Delta_{z\mathbf{v}}$ , then  $s^*$  is the true shift.

*Proof:* See Appendix B. ■

Note that  $\|\mathbf{e}_z\|_2$  and  $\|\mathbf{e}_v\|_2$  might not be available in practice but can be replaced by an upper bound if the noise is known to be bounded. This holds for both Corollaries 6 and 7.

## V. SHIFT RETRIEVAL IN SUPERRESOLUTION

The resolution of any classical electromagnetic sensing system is limited by the wavelength of the measured electromagnetic wave, and details finer than a wavelength cannot be observed [22]. This limitation can be improved to some extent by imposing some structural information about the image to enhance its resolution, also known as *superresolution* [23], [24]. In this section, we study an analogue of superresolution for shift retrieval. The goal is to recover shifts in higher resolution/precision than those by maximizing the cross-correlation in the source signal resolution.

First, we observe that under the conditions of Theorem 1, CSR recovers the shift with a resolution defined by  $n$ , the dimension of  $\mathbf{x}$  and  $\mathbf{y}$ . Hence, the resolution is independent of the number of measurements acquired. It implies that in the noise free case, the correct shift can be recovered up to any accuracy without increasing the number of measurements, as long as the conditions of Theorem 1 are satisfied.

Also note that the above observation would not be practical if we first had to sample  $\mathbf{x}$  and  $\mathbf{y}$  to compute their partial Fourier

transforms  $\mathbf{z}$  and  $\mathbf{v}$ . The concept of superresolution is more meaningful if the signals  $\mathbf{z}$  and  $\mathbf{v}$  are measured directly. The details are given in the following corollary:

*Corollary 8 (Shift Retrieval in Superresolution):* Let  $x(t), t \in [0, T]$  be a continuous time signal,  $y(t) = x(t-l), t \in [l, T]$  and  $y(t) = x(t-l+T), t \in [0, l]$ . Assume that  $l$  is a multiple of  $T/n$  for some  $n$ . Let the  $i$ th element of  $\mathbf{x} \in \mathbb{R}^n$  be  $x((i-1)T/n)$ ,  $i = 1, \dots, n$ , and define  $\mathbf{y}$  similarly by sampling  $y(t)$ . Let  $\mathbf{X}$  be an  $n \times n$  matrix with  $i$ th column equal to  $\mathbf{D}^i \mathbf{x}$ ,  $i = 1, \dots, n$ , and let  $\mathbf{A} \in \mathbb{C}^{m \times n}$  be a partial Fourier matrix. Suppose that we are given  $\mathbf{v} = \mathbf{x}$  and  $\mathbf{z} = \mathbf{A}\mathbf{y}$ . Then the true shift  $l$  can be recovered from  $\mathbf{v}$  and  $\mathbf{z}$  as  $l = Tl'/n$  with  $l'$  solving (11), as long as  $\mathbf{A}\mathbf{X}$  has distinct columns.

*Proof:* From the construction of  $\mathbf{x}, \mathbf{y}, \mathbf{v}, \mathbf{z}$ , we have that  $\mathbf{y} = \mathbf{D}^{l'} \mathbf{x}$  and  $\mathbf{z} = \mathbf{D}^{l'} \mathbf{v}$ , for some  $l'$ . Using Theorem 1 it follows that  $Tl'/n = l$  as long as  $\mathbf{A}\mathbf{X}$  has distinct columns. ■

Note that  $l$  has to be a multiple of  $T/n$  for Corollary 8 to hold. If this is not the case, and the grid does not include the true shift, we may ask under what conditions (11) recovers the shift estimate  $l' \in \{0, \dots, n-1\}$  that minimizes the error  $|l-l'T/n|$ . To answer this question, consider the following setup: Let  $x(t), t \in [0, T]$  be a continuous time signal,  $y(t) = x(t-l), t \in [l, T]$  and  $y(t) = x(t-l+T), t \in [0, l]$ . Introduce  $\tilde{\mathbf{x}} \in \mathbb{R}^n$  by stacking the (possibly noisy) samples  $x(iT/n), i = 0, \dots, n-1$ . Define  $\tilde{\mathbf{y}} \in \mathbb{R}^n$  accordingly. Let  $\tau \in (-T/(2n), T/(2n))$  be the smallest offset such that  $l-\tau$  is a multiple of  $T/n$  and introduce  $\mathbf{x} \in \mathbb{R}^n$  by stacking the samples  $x(iT/n+\tau), i = 0, \dots, n-1$ ; define  $\mathbf{y} \in \mathbb{R}^n$  accordingly. Let  $\mathbf{A} \in \mathbb{C}^{m \times n}$  be a partial Fourier matrix and generate  $\mathbf{v} = \mathbf{A}\mathbf{x}$ ,  $\tilde{\mathbf{v}} = \mathbf{A}\tilde{\mathbf{x}}$ ,  $\mathbf{z} = \mathbf{A}\mathbf{y}$  and  $\tilde{\mathbf{z}} = \mathbf{A}\tilde{\mathbf{y}}$ . By identifying  $\mathbf{e}_z = \tilde{\mathbf{z}} - \mathbf{z}$  and  $\mathbf{e}_v = \tilde{\mathbf{v}} - \mathbf{v}$ , we can view the misalignment in the grid as noise and use the theory developed for noisy compressive shift retrieval to give guarantees for recovery. We demonstrate this through an example.

*Example 4 (Superresolution):* The aim of this example is to illustrate superresolution for both the noise free and noisy cases. Let the continuous time signal  $x(t)$  be a realization of a fractional Brownian motion (a continuous-time Gaussian process) on the time interval  $[0, 1600)$ ,  $y(t) = x(t-l), t \in [l, 1600)$ ,  $y(t) = x(t-l+1600), t \in [0, l)$  and sample the delay  $l$  randomly from  $\{1, 3, 5, \dots, 1599\}$ . Assume that we are given 10 noise-free Fourier transform measurements of  $x$  and  $y$  at frequencies randomly chosen from  $\{0, 1/1664, \dots, 12/1664\}$ . We stack these 10 measurements in  $\tilde{\mathbf{z}}$  and  $\tilde{\mathbf{v}}$ , respectively.

To recover the shift we grid the time interval  $[0, 1600)$ . We start by a rather coarse grid:

$$\left\{ 0, 2^k, 2 \times 2^k, \dots, \left\lfloor \frac{1600}{2^k} \right\rfloor 2^k \right\}, \quad (14)$$

with  $k = 7$ . The grid is successively refined for  $k = 6, 5, 4, 3, 2, 1, 0$ , each time using the same 10 Fourier measurements. Note that by construction, the true shift does not match any grid points for  $k = 7, 6, 5, 4, 3, 2, 1$ . One example of the estimated shifts in the different resolutions is illustrated in Fig. 4.

For  $k = 7$ , the two grid points closest to the true shift are shown with blue circles and the true shift by the red vertical line. The filled circle shows the estimate of the shift given by (11) using the 10 Fourier samples. As seen, the grid point closest to the true time delay is correctly found. This was also verified by checking that the column difference of  $\mathbf{A}\tilde{\mathbf{X}}$  exceeded  $\Delta_{z\mathbf{v}}$

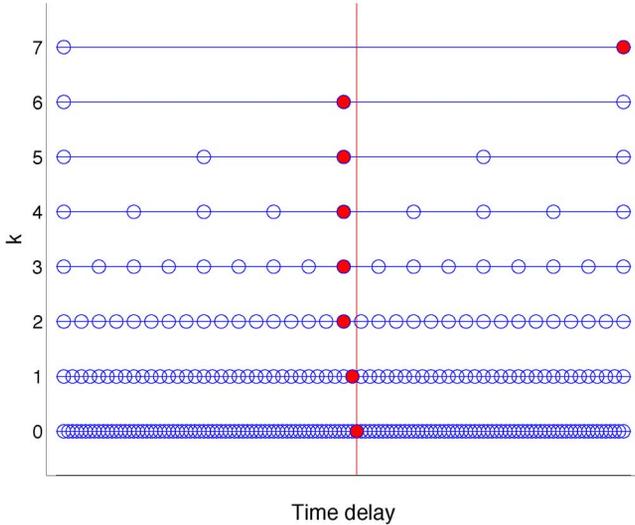


Fig. 4. An illustration of the grid for different  $k$ 's. The true shift is illustrated by a red vertical line and the shift predicted by CSR marked by a red bullet.

TABLE I

SUCCESS RATES OF SHIFT RETRIEVAL IN SUPERRESOLUTION IN A MONTE CARLO SIMULATION. FOR BOTH THE NOISE-FREE CASE AND THE NOISY CASE, THE PERCENTAGE OF SUCCESSFUL TRIALS IN WHICH THE OPTIMAL SHIFT ESTIMATES (CLOSEST TO THE TRUE SHIFT) IN DIFFERENT RESOLUTIONS  $k = 7, 6, \dots, 0$  ARE RECOVERED BY CSR AND THE PERCENTAGE PREDICTED BY THE THEORY ARE SHOWN

	$k = 7$	6	5	4	3	2	1	0
Noise-free (numerical)	95	62	53	59	63	60	100	100
Noise-free (theoretical)	86	51	45	51	55	55	0	100
Noisy (numerical)	79	43	26	30	10	4	4	0
Noisy (theoretical)	28	7	2	0	0	0	0	0

and  $2\|\mathbf{e}_v\|$ . Since the conditions of Corollary 6 are satisfied, the true shift must be within  $\pm 2^6$  of the estimate.

For each of  $k = 6, 5, 4, 3$ , the closest grid point to the true shift is found. For this particular example, since all column differences exceeded  $\Delta_{zv}$  and  $2\|\mathbf{e}_v\|$ , the true shift is guaranteed to be within  $\pm 2^{k-1}$  of the estimates by Corollary 6. For  $k = 2$ , CSR does not return the grid point closest to the true shift and  $\Delta_{zv}$  exceeds the smallest distances between two columns. For  $k = 1$ , CSR returns the grid point closest to the true shift but  $\Delta_{zv}$  exceeds the smallest distances between two columns. When  $k = 0$ , one grid point is aligned with the true shift and this grid point is correctly identified by CSR. It can be also verified that  $\mathbf{A}\mathbf{X}$  has distinct columns.

The above experiment is further repeated 100 times and the results summarized in Table I. We also run the above Monte Carlo simulation with noise added to  $\mathbf{z}$  and  $\mathbf{v}$  (SNR = 10 dB). The results are also reported in Table I.

## VI. CONCLUSION

To recover the cyclic shift relating two signals, the cross-correlation is usually evaluated for all possible shifts. Recent advances in hardware, signal acquisition and signal processing have made it possible to sample or compute Fourier coefficients of a signal efficiently. It is therefore of particular interest to see under what conditions the true shift can be recovered from the

Fourier coefficients. We have proposed a criterion that is computationally more efficient than using the time samples, and we have shown that the true shift can be recovered using as few as one Fourier coefficient. We have also derived bounds for perfect recovery for both noise free and noisy measurements and introduced the concept of superresolution for shift retrieval.

## APPENDIX A

### PROOFS: NOISE-FREE COMPRESSIVE SHIFT RETRIEVAL

Before proving Theorem 1, we state two lemmas.

*Lemma 9 (Recovery of a Shift using Projections):* Let  $\mathbf{X}$  be the  $n \times n$ -matrix made up of cyclically shifted versions of  $\mathbf{x}$  as columns. If the columns of  $\mathbf{A}\mathbf{X}$  are distinct, then the true shift can be recovered by

$$\min_{\mathbf{q} \in \{0,1\}^n} \|\mathbf{A}\mathbf{y} - \mathbf{A}\mathbf{X}\mathbf{q}\|_2^2 \quad \text{s.t.} \quad \|\mathbf{q}\|_0 = 1. \quad (15)$$

*Proof of Lemma 9:* Since the shift relating  $\mathbf{x}$  and  $\mathbf{y}$  is assumed unique, it is clear that the true shift is recovered by

$$\min_{\mathbf{q} \in \{0,1\}^n} \|\mathbf{y} - \mathbf{X}\mathbf{q}\|_2^2 \quad \text{s.t.} \quad \|\mathbf{q}\|_0 = 1. \quad (16)$$

Assume that the solution of (15) is not equivalent to that of (16). Namely, assume that (16) gives  $\mathbf{q}$ , (15) gives  $\tilde{\mathbf{q}}$  and  $\mathbf{q} \neq \tilde{\mathbf{q}}$ . Since  $\mathbf{q}$  will give a zero objective value in (15), so must  $\tilde{\mathbf{q}}$ . We therefore have that  $\mathbf{A}\mathbf{y} = \mathbf{A}\mathbf{X}\tilde{\mathbf{q}} = \mathbf{A}\mathbf{X}\mathbf{q}$  and hence

$$\mathbf{A}\mathbf{X}\tilde{\mathbf{q}} - \mathbf{A}\mathbf{X}\mathbf{q} = \mathbf{A}\mathbf{X}(\tilde{\mathbf{q}} - \mathbf{q}) = 0. \quad (17)$$

Since  $\mathbf{q}, \tilde{\mathbf{q}} \in \{0,1\}^n$ ,  $\|\tilde{\mathbf{q}}\|_0 = \|\mathbf{q}\|_0 = 1$ , and  $\mathbf{q} \neq \tilde{\mathbf{q}}$ ,  $\mathbf{A}\mathbf{X}(\tilde{\mathbf{q}} - \mathbf{q}) = 0$  implies that two columns of  $\mathbf{A}\mathbf{X}$  are identical. This is a contradiction and we therefore conclude that both (15) and (16) recover the true shift. ■

*Lemma 10 (From (15) to (9)):* Under conditions 1) and 2) of Theorem 1, the shifts recovered by (15) and (9) are the same.

*Proof of Lemma 10:* Consider the objective of (15):

$$\|\mathbf{A}\mathbf{y} - \mathbf{A}\mathbf{X}\mathbf{q}\|_2^2 = (\mathbf{A}\mathbf{y})^* \mathbf{A}\mathbf{y} + (\mathbf{A}\mathbf{X}\mathbf{q})^* \mathbf{A}\mathbf{X}\mathbf{q} - (\mathbf{A}\mathbf{y})^* \mathbf{A}\mathbf{X}\mathbf{q} - (\mathbf{A}\mathbf{X}\mathbf{q})^* \mathbf{A}\mathbf{y}. \quad (18)$$

Notice that we can write

$$\mathbf{X}\mathbf{q} = \mathbf{D}^s \mathbf{x}, \quad (19)$$

for some  $s$ . This follows from the construction of  $\mathbf{X}$  as a matrix with delayed versions of  $\mathbf{x}$  as its columns and from the fact that  $\mathbf{q}$  selects exactly one of these columns. Problem (15) is then equal to

$$\max_{s \in \{0, \dots, n-1\}} 2\Re \{ (\mathbf{A}\mathbf{y})^* \mathbf{A}\mathbf{D}^s \mathbf{x} \} - (\mathbf{A}\mathbf{D}^s \mathbf{x})^* \mathbf{A}\mathbf{D}^s \mathbf{x}. \quad (20)$$

Using the assumption  $\mathbf{A}^* \mathbf{A}\mathbf{D}^s = \mathbf{D}^s \mathbf{A}^* \mathbf{A}$  and that  $(\mathbf{D}^s)^* \mathbf{D}^s = \mathbf{I}$  for a shift matrix, we have

$$(\mathbf{A}\mathbf{D}^s \mathbf{x})^* \mathbf{A}\mathbf{D}^s \mathbf{x} = \mathbf{x}^* (\mathbf{D}^s)^* \mathbf{A}^* \mathbf{A}\mathbf{D}^s \mathbf{x} = \|\mathbf{A}\mathbf{x}\|_2^2, \quad (21)$$

which is independent of  $s$ . Therefore, the shift recovered by (20) is the same as that of

$$\max_{s \in \{0, \dots, n-1\}} \Re \{ (\mathbf{A}\mathbf{y})^* \mathbf{A}\mathbf{D}^s \mathbf{x} \}. \quad (22)$$

Lastly, if we again use that  $\mathbf{A}^* \mathbf{A} \mathbf{D}^s = \mathbf{D}^s \mathbf{A}^* \mathbf{A}$  and  $\alpha \mathbf{A} \mathbf{A}^* = \mathbf{I}$ , then (9) follows from

$$\begin{aligned} \Re \{ (\mathbf{A} \mathbf{y})^* \mathbf{A} \mathbf{D}^s \mathbf{x} \} &= \Re \{ \mathbf{y}^* \mathbf{A}^* \mathbf{A} \mathbf{D}^s \mathbf{x} \} \\ &= \alpha \Re \{ \mathbf{y}^* \mathbf{A}^* \mathbf{A} \mathbf{A}^* \mathbf{A} \mathbf{D}^s \mathbf{x} \} \\ &= \alpha \Re \{ \mathbf{y}^* \mathbf{A}^* \mathbf{A} \mathbf{D}^s \mathbf{A}^* \mathbf{A} \mathbf{x} \} \\ &= \alpha \Re \{ \langle \mathbf{z}, \bar{\mathbf{D}}^s \mathbf{v} \rangle \} \end{aligned} \quad (23)$$

where  $\mathbf{z} = \mathbf{A} \mathbf{y}$  and  $\mathbf{v} = \mathbf{A} \mathbf{x}$ .  $\blacksquare$

We are now ready to prove Theorem 1.

*Proof of Theorem 1:* The assumptions of Theorem 1 imply that the requirements of both Lemmas 9 and 10 are satisfied. The theorem therefore follows trivially.  $\blacksquare$

We next prove Corollary 2.

*Proof of Corollary 2:* In the proof of Lemma 9,  $\mathbf{A} \mathbf{X}(\tilde{\mathbf{q}} - \mathbf{q}) = 0$  leads to  $\tilde{\mathbf{q}} - \mathbf{q} = 0$  if the columns of  $\mathbf{A} \mathbf{X}$  were all distinct. Now, if

$$s^* = \arg \max_{s \in \{0, \dots, n-1\}} \Re \{ \langle \mathbf{z}, \bar{\mathbf{D}}^s \mathbf{v} \rangle \}, \quad (24)$$

then the  $s^*$ -th element of  $\tilde{\mathbf{q}}$  is one and all other elements zero. Hence, Lemma 9 can be made less conservative if  $s^*$  is known by requiring that only the  $s^*$ -th column of  $\mathbf{A} \mathbf{X}$  is different than all other columns.  $\blacksquare$

*Proof of Lemma 3:* Let  $\mathbf{M} = \mathbf{A} \mathbf{D}^s$  and  $\mathbf{Q} = \mathbf{A}(\mathbf{D}^s)^*$ . By the definition of  $\mathbf{D}^s$ ,  $\mathbf{M}$  is a column permutation of  $\mathbf{A}$  where the columns are shifted  $s$  times to the right. Thus, the  $r$ -th column of  $\mathbf{M}$  is equal to the  $t$ -th column of  $\mathbf{A}$  where  $t = (r - s) \bmod n$ . It is also easy to see that  $(\mathbf{D}^s)^*$  permutes the columns of  $\mathbf{A}$  by  $s$  to the left so that the  $r$ -th column of  $\mathbf{Q}$  is equal to the  $q$ -th column of  $\mathbf{A}$  where  $q = (r + s) \bmod n$ . Now, the  $pr$ -th element of  $\mathbf{A}^* \mathbf{M} = \mathbf{A}^* \mathbf{A} \mathbf{D}^s$  is given by

$$(\mathbf{A}_{:,p})^* \mathbf{M}_{:,r} = (\mathbf{A}_{:,p})^* \mathbf{A}_{:,r-s} = \frac{1}{n} \sum_{i=1}^m e^{2j\pi k_i(p-r+s)}, \quad (25)$$

where  $\mathbf{A}_{:,p}$  is used to denote the  $p$ -th column of  $\mathbf{A}$  and  $\mathbf{M}_{:,r}$  the  $r$ -th column of  $\mathbf{M}$ . On the other hand, the  $(p, r)$ -th element of  $\mathbf{Q}^* \mathbf{A} = \mathbf{D}^s \mathbf{A}^* \mathbf{A}$  is given by

$$(\mathbf{Q}_{:,p})^* \mathbf{A}_{:,r} = (\mathbf{A}_{:,p+s})^* \mathbf{A}_{:,r} = \frac{1}{n} \sum_{i=1}^m e^{2j\pi k_i(p+s-r)}. \quad (26)$$

Clearly, the two are equivalent.  $\blacksquare$

We are now ready to prove Corollary 4.

*Proof of Corollary 4:* Lemma 3 gives that Condition 1) of Theorem 1 is satisfied. Since a full Fourier matrix is orthonormal, a matrix made up of a selection of rows of a Fourier matrix satisfies Condition 2). The last condition of Theorem 1 requires columns of  $\mathbf{A} \mathbf{X}$  to be distinct. A sufficient condition is that there exists a row with all distinct elements. As shown previously, the  $(p, r)$ -th element of  $\mathbf{A} \mathbf{X}$  is  $X_{k_p} e^{\frac{2j\pi k_p(r-1)}{n}}$ . If  $X_{k_p}$  is assumed nonzero, then a sufficient condition for  $\mathbf{A} \mathbf{X}$  to have distinct columns is that  $e^{\frac{2j\pi k_p r_1}{n}} \neq e^{\frac{2j\pi k_p r_2}{n}}$ ,  $r_1, r_2 \in \{0, \dots, n-1\}$ ,  $r_1 \neq r_2$ . This condition can be simplified to  $\frac{k_p r_1}{n} \neq \frac{k_p r_2}{n} + \gamma$ ,  $\gamma \in \mathbb{Z}$ . By realizing that  $r_1 - r_2$  takes values in  $\{-n+1, \dots, -1, 1, \dots, n-1\}$  we get that the condition is equivalent to requiring that there are no integers in  $\{-n+1, \dots, -1, 1, \dots, n-1\} \frac{k_p}{n}$ . Due

to symmetry, a sufficient condition for distinct columns is that there exists a  $p \in \{1, \dots, m\}$  such that  $X_{k_p} \neq 0$  and  $\{1, \dots, n-1\} \frac{k_p}{n}$  contains no integers.

Lastly, writing out  $\mathbf{A} \mathbf{D}^s \mathbf{A}^*$  we get that the  $pr$ -th element is equal to  $\delta_{p,r} e^{-\frac{2j\pi k_p s}{n}} / n$ , leading to the simplified test proposed in (11).  $\blacksquare$

## APPENDIX B

### PROOFS: NOISY COMPRESSIVE SHIFT RETRIEVAL

*Proof of Theorem 5:* From Lemma 10 we can see that seeking  $s$  that maximizes  $\Re \{ \langle \tilde{\mathbf{z}}, \bar{\mathbf{D}}^s \tilde{\mathbf{v}} \rangle \}$  is equivalent to seeking  $\mathbf{q}$  that solves

$$\min_{\mathbf{q} \in \{0,1\}^n} \|\tilde{\mathbf{z}} - \mathbf{A} \tilde{\mathbf{X}} \mathbf{q}\|_2^2 \quad \text{s.t.} \quad \|\mathbf{q}\|_0 = 1, \quad (27)$$

where the first column of  $\mathbf{A} \tilde{\mathbf{X}}$  is equal to  $\tilde{\mathbf{v}}$  (which defines the first column of  $\tilde{\mathbf{X}}$ ) and the  $i$ -th column of  $\tilde{\mathbf{X}}$  is a circular shift of the first column of  $\tilde{\mathbf{X}}$   $i-1$  steps.

Assume that  $\hat{\mathbf{q}}$  solves (27). Since our measurements are noisy, we cannot expect a zero loss. The loss can be shown to be given by

$$\|\tilde{\mathbf{z}} - \mathbf{A} \tilde{\mathbf{X}} \hat{\mathbf{q}}\|_2^2 = \|\tilde{\mathbf{v}}\|_2^2 + \|\tilde{\mathbf{z}}\|_2^2 - \max_{s \in \{0, \dots, n-1\}} 2\Re \{ \tilde{\mathbf{z}}^* \bar{\mathbf{D}}^s \tilde{\mathbf{v}} \}. \quad (28)$$

Now, consider  $\|\tilde{\mathbf{z}} - \mathbf{A} \tilde{\mathbf{X}} \hat{\mathbf{q}}\|_2$ . Assume that  $\mathbf{q}_0$  solves the noise-free version of (27) and let  $\tilde{\mathbf{X}} = \tilde{\mathbf{X}} + \mathbf{H}$ . We have the following inequality:

$$\begin{aligned} \|\tilde{\mathbf{z}} - \mathbf{A} \tilde{\mathbf{X}} \hat{\mathbf{q}}\|_2 &= \|\mathbf{z} + \mathbf{e}_z - \mathbf{z} + \mathbf{A} \mathbf{X} \mathbf{q}_0 - \mathbf{A} \tilde{\mathbf{X}} \hat{\mathbf{q}}\|_2 \\ &= \|\mathbf{e}_z + \mathbf{A} \mathbf{X} \mathbf{q}_0 - \mathbf{A} \tilde{\mathbf{X}} \hat{\mathbf{q}}\|_2 \\ &= \|\mathbf{e}_z + \mathbf{A}(\tilde{\mathbf{X}} - \mathbf{H}) \mathbf{q}_0 - \mathbf{A} \tilde{\mathbf{X}} \hat{\mathbf{q}}\|_2 \\ &\geq \|\mathbf{A} \tilde{\mathbf{X}} \mathbf{q}_0 - \mathbf{A} \tilde{\mathbf{X}} \hat{\mathbf{q}}\|_2 - \|\mathbf{e}_z\|_2 - \|\mathbf{e}_v\|_2, \end{aligned}$$

where we used the fact that  $\mathbf{A} \mathbf{H} \mathbf{q}_0 = \mathbf{e}_v$  and the reverse triangle inequality. Therefore

$$\|\mathbf{A} \tilde{\mathbf{X}} \mathbf{q}_0 - \mathbf{A} \tilde{\mathbf{X}} \hat{\mathbf{q}}\|_2 \leq \Delta_{z\mathbf{v}}. \quad (29)$$

Since  $\|\hat{\mathbf{q}}\|_0 = \|\mathbf{q}_0\|_0 = 1$ ,  $\|\mathbf{A} \tilde{\mathbf{X}} \mathbf{q}_0 - \mathbf{A} \tilde{\mathbf{X}} \hat{\mathbf{q}}\|_2$  equals the  $\ell_2$  difference between two columns of  $\mathbf{A} \tilde{\mathbf{X}}$ . It is hence sufficient to require that the  $\ell_2$  difference between any two columns of  $\mathbf{A} \tilde{\mathbf{X}}$  is greater than  $\Delta_{z\mathbf{v}}$  for  $\mathbf{q}_0 = \hat{\mathbf{q}}$ .  $\blacksquare$

*Proof of Corollary 6:* Let  $\tilde{\mathbf{q}}$  and  $\hat{\mathbf{q}}$  be any vectors such that  $\|\hat{\mathbf{q}}\|_0 = \|\tilde{\mathbf{q}}\|_0 = 1$ ,  $\hat{\mathbf{q}} \neq \tilde{\mathbf{q}}$  and  $\hat{\mathbf{q}}, \tilde{\mathbf{q}} \in \{0, 1\}^n$ . Using the triangle inequality and the fact that  $\mathbf{H} \hat{\mathbf{q}}$  and  $\mathbf{H} \tilde{\mathbf{q}}$  are shifted versions of the same vector, we have that

$$\|\mathbf{A} \tilde{\mathbf{X}} \hat{\mathbf{q}} - \mathbf{A} \tilde{\mathbf{X}} \tilde{\mathbf{q}}\|_2 = \|\mathbf{A}(\tilde{\mathbf{X}} + \mathbf{H})(\hat{\mathbf{q}} - \tilde{\mathbf{q}})\|_2 \quad (30)$$

$$\leq \|\mathbf{A} \tilde{\mathbf{X}}(\hat{\mathbf{q}} - \tilde{\mathbf{q}})\|_2 + \|\mathbf{A} \mathbf{H}(\hat{\mathbf{q}} - \tilde{\mathbf{q}})\|_2 \quad (31)$$

$$\leq \|\mathbf{A} \tilde{\mathbf{X}}(\hat{\mathbf{q}} - \tilde{\mathbf{q}})\|_2 + 2\|\mathbf{e}_v\|_2. \quad (32)$$

Hence, if  $\|\mathbf{A} \tilde{\mathbf{X}} \hat{\mathbf{q}} - \mathbf{A} \tilde{\mathbf{X}} \tilde{\mathbf{q}}\|_2 - 2\|\mathbf{e}_v\|_2 > 0$ , then  $\|\mathbf{A} \tilde{\mathbf{X}}(\hat{\mathbf{q}} - \tilde{\mathbf{q}})\|_2$  is greater than zero. Now since Theorem 5 gives that (11) recovers the same shift as if the measurements would have been noise-free, and since Theorem 1 gives that the noise-free estimate is equal to the true shift if  $\|\mathbf{A} \tilde{\mathbf{X}}(\hat{\mathbf{q}} - \tilde{\mathbf{q}})\|_2$  is greater than zero (or equivalently that all columns of  $\mathbf{A} \tilde{\mathbf{X}}$  are distinct), we can guarantee the recovery of the true shift also in the noisy case.  $\blacksquare$

*Proof of Corollary 7:* The corollary follows trivially by setting the  $s^*$ th element of  $\hat{q}$  to one and all other elements to zero in the proofs of Theorem 5 and Corollary 6. ■

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