

# Nonlinear and Nonideal Sampling Revisited

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**Abstract**—We revisit the problem of recovering a continuous-time signal lying within a known shift-invariant subspace from nonlinear and nonideal samples. Recently, an iterative algorithm for perfect recovery of such signals was proposed. This method requires operations which are not linear time-invariant (LTI), rendering it impractical due to its polynomial dependency on the data length. We describe an alternative iterative algorithm for recovering the signal, which involves only LTI operations. In the revised method, each iteration is much faster and implementation is simpler. Furthermore, the overall running time of our approach depends linearly on the number of samples.

**Index Terms**—Generalized sampling, interpolation, nonlinear sampling, shift-invariant spaces.

## I. INTRODUCTION

WE consider the problem of recovering a continuous-time signal that was distorted by a memoryless nonlinear mapping, and sampled after passing through an anti-aliasing filter. Nonlinear distortions appear in various setups and applications of digital signal processing, such as CCD image sensors, power electronics and radiometric photography (see [1] and references therein).

Sampling problems in purely linear setups were studied extensively [2]–[4]. The theory in this area deals with arbitrary Hilbert space settings. In this letter we focus on shift-invariant (SI) settings in which both sampling and reconstruction are obtained by filtering operations and the sampling grid is uniform. SI spaces are commonly used in various applications; some important examples include bandlimited signals and splines [3], [4].

We consider a signal  $y(t)$  that is known to lie in a SI subspace  $\mathcal{A}$ , generated by the kernel  $a(t)$ :

$$y(t) = \sum_n d[n]a(t-n) \quad (1)$$

for some coefficients  $d[n]$ . We assume that  $y(t)$  is sampled at the integers  $t = n$  after passing through the memoryless nonlinear mapping  $m(y(t))$  and the filter  $s(-t)$ , as shown in Fig. 1. In the linear setup,  $m(y(t)) = y(t)$ . In this case it is well known that under a simple condition on  $\mathcal{A}$ , and on the SI subspace  $\mathcal{S}$  generated by  $s(t)$ ,  $y(t)$  can be perfectly reconstructed from the

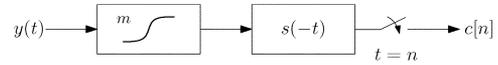


Fig. 1. Nonlinear and nonideal sampling.

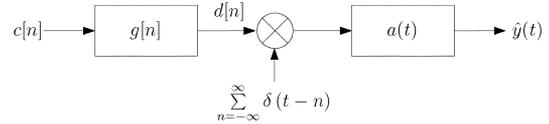


Fig. 2. Restoration with linear sampling.

samples [2]. The recovery is obtained by applying a digital correction filter  $g[n]$  on the samples before performing reconstruction with the kernel  $a(t)$ , as shown in Fig. 2.

In the more complicated situation, where  $m(y(t)) \neq y(t)$ , several approaches can be taken. Assuming  $m$  is invertible, a naive method would be to first apply  $m^{-1}$  on the samples, and then proceed with the same correction and reconstruction stages as in the linear case. However, as we show in Section V, this method is suboptimal in general.

The special case where the sampling filter matches the subspace prior, namely the subspaces  $\mathcal{S}$  and  $\mathcal{A}$  coincide, was considered in several previous works. In [5] uniqueness and existence theorems were developed for the recovery of nonlinearly distorted bandlimited signals. It was shown that perfect recovery is guaranteed if the nonlinearity is monotonic with a bounded derivative. This result was generalized in [6], [7] to the case of an arbitrary subspace prior.

A recent work [1] considers the more general case where the sampling filter does not match the subspace prior. There, the authors develop simple sufficient conditions on the nonlinear distortion and the spaces involved, which guarantee the existence of a unique solution. Moreover, this work proposes an iterative algorithm, which is proved to achieve perfect recovery. The method is based on linearization of the distortion in each iteration around the current signal guess. One main drawback of this approach is that it requires operations which are not linear time-invariant (LTI). This introduces inherent complexity to the scheme, since a matrix of the size of the data length has to be inverted.

In this letter we describe an alternative iterative algorithm that does not rely on linearization. In each iteration we use the same stages as in the previous method, except for the non-LTI operation, which we replace by an LTI filter. This filter is in fact the standard correction filter shown in Fig. 2 that is used for recovery in the purely linear sampling setup. We show that if there exists a unique solution, then our algorithm converges to it. Furthermore, we prove that when no unique solution exists, our algorithm still outputs a consistent recovery of the signal if a condition that is less strict than the uniqueness condition is satisfied. A consistent recovery means that its nonlinear and nonideal samples coincide with the samples of the original signal.

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The letter is organized as follows. Section II summarizes known uniqueness results for linear and nonlinear sampling setups. The previous recovery algorithm and its drawbacks are discussed in Section III. Our method is presented in Section IV. Finally, Section V shows simulation results and compares performance of the different recovery methods.

## II. CONDITIONS FOR UNIQUENESS

Throughout this letter the signal  $y(t)$  is assumed to lie in the SI space  $\mathcal{A} = \overline{\text{span}}\{a(t-n)\}_{n \in \mathbb{Z}}$  for some generator  $a(t) \in L_2$ . We denote by  $\mathcal{S} = \overline{\text{span}}\{s(t-n)\}_{n \in \mathbb{Z}}$  the SI space spanned by the filter  $s(t)$  in Fig. 1.

We begin by treating linear setups. In this case the nonideal samples of  $y(t)$  are

$$c_0[n] = \int_{-\infty}^{\infty} s(t-n)y(t)dt. \quad (2)$$

It is well known [3] that if  $L_2$  can be decomposed into a direct sum  $\mathcal{A} \oplus \mathcal{S}^\perp = L_2$ , then perfect recovery of  $y(t)$  from the samples (2) can be accomplished. This is obtained using the scheme shown in Fig. 2 with a digital correction filter  $g[n]$  whose discrete-time Fourier transform is

$$G(e^{j\omega}) = \frac{1}{\phi_{SA}(e^{j\omega})}. \quad (3)$$

Here

$$\phi_{SA}(e^{j\omega}) \triangleq \sum_{k \in \mathbb{Z}} S^*(\omega - 2\pi k)A(\omega - 2\pi k) \quad (4)$$

and  $S(\omega)$  and  $A(\omega)$  denote the continuous-time Fourier transforms of  $s(t)$  and  $a(t)$  respectively. The direct sum condition ensures that  $\phi_{SA}(e^{j\omega})$  is stably invertible. Indeed, it can be shown that it is equivalent to the existence of a scalar  $\alpha > 0$  such that  $|\phi_{SA}(e^{j\omega})| \geq \alpha$  [8].

For a general signal  $y$  (not necessarily in  $\mathcal{A}$ ), the resulting reconstruction  $\hat{y}$  equals  $P_{\mathcal{AS}^\perp}y$ , where  $P_{\mathcal{AS}^\perp}$  is the oblique projection onto  $\mathcal{A}$  along  $\mathcal{S}^\perp$ . This projection is the unique operator satisfying  $P_{\mathcal{AS}^\perp}y = y$  for all  $y \in \mathcal{A}$  and  $P_{\mathcal{AS}^\perp}y = 0$  for all  $y \in \mathcal{S}^\perp$  [3].

We now turn to treat the memoryless nonlinear setting, and summarize the uniqueness theorems developed in past works for this setup. The samples of  $y(t)$  in this case are given by

$$c[n] = \int_{-\infty}^{\infty} s(t-n)m(y(t))dt. \quad (5)$$

The following theorem provides a sufficient condition for uniqueness in this setting. This theorem is simply a merge of Theorems 2 and 5 of [1].

*Theorem 1:* Assume that  $\mathcal{A} \oplus \mathcal{S}^\perp = L_2$ . If  $m$  is invertible and its derivative  $m'$  satisfies

$$|1 - m'(z)| < \frac{1 - \sin(\mathcal{A}, \mathcal{S})}{1 + \sin(\mathcal{A}, \mathcal{S})} \quad (6)$$

for all  $z \in \mathbb{R}$ , then there is a unique  $\hat{y}(t) \in \mathcal{A}$  whose nonideal and nonlinear samples

$$\hat{c}[n] = \int_{-\infty}^{\infty} s(t-n)m(\hat{y}(t))dt \quad (7)$$

coincide with the measurements  $c[n]$  of (5).

The cosine between two SI spaces [2] is defined by

$$\cos(\mathcal{A}, \mathcal{S}) = \text{ess inf}_{\omega \in [-\pi, \pi]} \frac{\phi_{SA}(e^{j\omega})}{\sqrt{\phi_{SS}(e^{j\omega})\phi_{AA}(e^{j\omega})}} \quad (8)$$

where  $\phi_{SA}(e^{j\omega})$ ,  $\phi_{SS}(e^{j\omega})$ , and  $\phi_{AA}(e^{j\omega})$  are as in (4) with the appropriate substitution of filters  $S(\omega)$  and  $A(\omega)$ . The sine between  $\mathcal{S}$  and  $\mathcal{A}$  can be computed using the relation  $\sin^2(\mathcal{A}, \mathcal{S}) = 1 - \cos^2(\mathcal{A}, \mathcal{S})$ .

For the special case of  $\mathcal{S} = \mathcal{A}$  condition (6) becomes  $0 < |m'(z)| < 2$ . We note that a weaker condition for this special case was derived in earlier works [5]–[7]. Specifically, it was shown that  $m$  needs only to be monotonic to guarantee uniqueness.

Once uniqueness is guaranteed, perfect recovery is ensured for a signal  $\hat{y}(t)$  that satisfies the consistency requirement  $\hat{c}[n] = c[n]$ . In the next section we review the algorithm of [1] that seeks a consistent recovery.

## III. RESTORATION VIA LINEARIZATION

The recovery method developed in [1] is an iterative algorithm that is based on linearization of the distortion in each iteration around the current guess of the signal. It is designed to yield a recovery

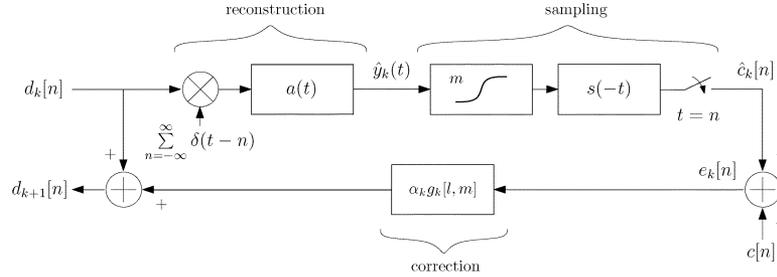
$$\hat{y}(t) = \sum_n \hat{d}[n]a(t-n) \quad (9)$$

that minimizes the cost function  $\varepsilon(\hat{d}) = \|c - \hat{c}\|_{\ell_2}^2$ , where  $c$  is the known sequence of samples (5) and  $\hat{c}$  is given by (7). By assumption, the minimal cost is  $\varepsilon(\hat{d}) = 0$ . However, since  $m$  is nonlinear, the cost function is in general nonconvex and therefore optimization algorithms for minimizing it might trap a stationary point. Nevertheless, it was shown in [1] that under the conditions of Theorem 1 there is only one stationary point, which is the global minimum. The method can be viewed as updating the sequence of coefficients  $d[n]$  in each iteration. A block diagram of the iterative recovery algorithm is shown in Fig. 3.

In the figure, the operation  $g_k[l, m]$  is a linear system which is the inverse of

$$h_k[l, m] = \int_{-\infty}^{\infty} s(t-l)m'(\hat{y}_k(t))a(t-m)dt. \quad (10)$$

When the number  $N$  of available samples is finite,  $h_k[l, m]$  can be described by an  $N \times N$  matrix. Consequently, the algorithm of Fig. 3 requires the computation and inversion of an  $N \times N$  matrix at each iteration. For example, if we deal with images of megapixel size, a matrix of size  $10^6 \times 10^6$  must be computed and inverted, which is clearly infeasible. Furthermore, for an infinite number of samples, a closed form expression for  $g_k[l, m]$  is rarely available.

Fig. 3. The  $k$ th iteration of the recovery algorithm of [1].

As explained in [1], each iteration uses a varying step size  $\alpha_k$ . This step size is chosen by a line search method that uses a backtracking procedure satisfying the Armijo and curvature conditions. Convergence of this algorithm is guaranteed if the conditions of Theorem 1 hold and if the derivative of  $m$  is Lipschitz continuous. This iterative approach can be viewed in several ways, one being quasi-Newton iterations, which explains its fast convergence rate.

#### IV. RESTORATION VIA SUCCESSIVE APPROXIMATIONS

Due to the complex correction stage introduced by the linearization strategy, here we take a different approach. In fact, our method originates from the early works [5]–[7] which considered the special case in which the subspaces  $\mathcal{S}$  and  $\mathcal{A}$  coincide. In this section we suggest a recovery algorithm that is similar to the one proposed there and determine the conditions that guarantee consistent recovery for situations where the sampling filter does not necessarily match the subspace prior. Consequently, the recovery algorithm that appears in [5]–[7] can be obtained as a special case.

*Theorem 2:* Assume that  $\mathcal{A} \oplus \mathcal{S}^\perp = L_2$  and that  $m$  is differentiable almost everywhere. If  $m$  is continuous and its derivative (when it exists) is bounded below by  $q > 0$  and above by  $Q < \infty$ , where

$$\frac{q}{Q} > \frac{1 - \cos(\mathcal{A}, \mathcal{S})}{1 + \cos(\mathcal{A}, \mathcal{S})} \quad (11)$$

then there exists a signal  $\hat{y}(t) \in \mathcal{A}$  whose nonideal and nonlinear samples  $\hat{c}[n]$  coincide with  $c[n]$ . The expansion coefficients  $\hat{d}[n]$  of  $\hat{y}(t)$  can be obtained from the iterative scheme that appears in Fig. 3 by replacing the linear system  $g_k[l, m]$  by the LTI filter  $g[n]$  of (3), and setting a constant step size  $\alpha_k \equiv \alpha$  satisfying

$$\frac{1 - \cos(\mathcal{A}, \mathcal{S})}{q} < \alpha < \frac{1 + \cos(\mathcal{A}, \mathcal{S})}{Q}. \quad (12)$$

*Proof:* See Appendix. ■

Note that we use the same LTI filter for all iterations. The correction stage is now simply the standard correction filter used for recovery with linear sampling. Moreover, notice that for convergence of the algorithm to a consistent recovery we only need the nonlinearity to have a first derivative which is bounded below and above by finite and nonzero boundaries, and that those boundaries are not too far apart. An interesting observation arises from a comparison to the uniqueness condition (6) of Theorem 1 which not only requires separate limits:

$$q > \frac{2 \sin(\mathcal{A}, \mathcal{S})}{1 + \sin(\mathcal{A}, \mathcal{S})}, \quad Q < \frac{2}{1 + \sin(\mathcal{A}, \mathcal{S})} \quad (13)$$

but also imposes the joint condition:

$$\frac{q}{Q} > \sin(\mathcal{A}, \mathcal{S}). \quad (14)$$

It can be shown that for any pair of subspaces  $\mathcal{A}, \mathcal{S}$ , (13) and (14) impose a stricter condition than (11). For example, let  $\mathcal{A}$  and  $\mathcal{S}$  be spline spaces of orders 2 and 0 respectively. In this case  $\cos(\mathcal{A}, \mathcal{S}) = 0.873$  and  $\sin(\mathcal{A}, \mathcal{S}) = 0.488$ . The right-hand side of (11) becomes 0.068, whereas that of (14) becomes 0.488, which is clearly a stricter condition. Therefore, our algorithm is guaranteed to converge to a consistent solution even when there is no guarantee that there is a unique solution.

A potential increase in convergence rate may be obtained by using a varying step size. However, our experiments indicate that there is no gain in the total number of basic loops of the type shown in Fig. 3, because each iteration requires several basic loops in order to determine the appropriate step size. Therefore, in the next section we only report the results of our method with a constant step size. As we show, our algorithm requires less running time than the previous one even though it necessitates more iterations. In addition, each iteration requires only LTI filtering.

#### V. SIMULATIONS

We now compare the following three recovery methods in terms of SNR  $\triangleq 10 \log_{10}(\|y\|_{L_2}^2 / \|y - \hat{y}\|_{L_2}^2)$ , convergence rate and running time:

- 1) Naive approach, namely applying  $m^{-1}$  on the samples and then proceeding with the correction and reconstruction stages of Fig. 2.
- 2) The algorithm of [1], outlined in Section III.
- 3) Our algorithm, as outlined in Section IV.

All algorithms were evaluated in the task of recovering a second degree spline, so that  $a(t)$  is a B-spline function of degree 2. The nonlinear distortion was taken to be  $\arctan(\gamma z)$ , where  $\gamma$  is a constant chosen so that condition (11) is satisfied for the possible range of values of  $y(t)$ . The sampling filter is a B-spline of order 0.

The SNRs achieved by the three methods are plotted in Fig. 4 as a function of the number of iterations. We averaged our results over 50 signals, each generated from 11 random expansion coefficients  $d[n]$ . We can see that both iterative algorithms outperformed the naive method after several iterations and achieved very high values of SNR. It is worth mentioning that if we wait until the algorithms converge, both of them achieve perfect recovery, even though the sufficient conditions of Theorem 1 do not hold. In terms of convergence rate, the results fit our expectations. The algorithm of [1] converges very quickly, whereas

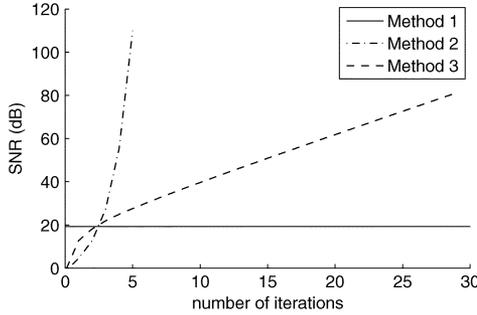


Fig. 4. The SNRs of the different recovery methods as a function of the number of iterations.

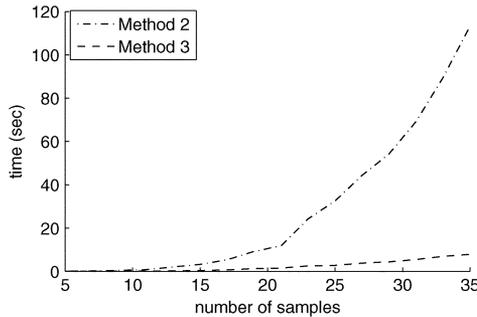


Fig. 5. The running times in Matlab of the different recovery methods as a function of the number of samples.

the algorithm proposed in Theorem 2 requires more iterations for good recovery results.

The running times in Matlab for both iterative methods are plotted in Fig. 5 as a function of the number of available samples. All the times were evaluated for an SNR of 60 dB. We can see that our method requires less running time than the previous one. Moreover, the running time of our algorithm increases linearly with the number of samples, whereas the increase for the previous approach is polynomial.

## VI. CONCLUSION

In this letter we addressed the problem of recovering a signal within a subspace from its nonlinear and nonideal samples. We presented a recovery algorithm that uses a much simpler, faster and practical correction stage than the recently proposed linearization-based method. The price paid for this improvement is a slower convergence rate. Nevertheless, the total running time is decreased in our algorithm and depends linearly on the number of samples. This is a significant improvement in comparison with the polynomial complexity of the previous approach.

### APPENDIX PROOF OF THEOREM 2

First, we define a mapping  $T$  from  $\mathcal{A}$  to itself:

$$T(x) = x + \alpha P_{\mathcal{AS}^\perp} (M(y) - M(x)) \quad (15)$$

where  $P_{\mathcal{AS}^\perp}$  is an oblique projection defined in Section II and the operator  $M : L_2 \rightarrow L_2$  is defined by  $\{(M(y))(t) =$

$m(y(t))$ . Using this mapping, the iterative scheme shown in Fig. 3 can be written in terms of  $\hat{y}_k, \hat{y}_{k+1}$  as  $\hat{y}_{k+1} = T(\hat{y}_k)$ .

Next, we show that for an appropriate choice of a step size  $\alpha$ , the mapping  $T$  is a contraction. Indeed

$$\begin{aligned} & \|T(x) - T(z)\|_{L_2}^2 \\ &= \|P_{\mathcal{AS}^\perp}(x - z) - \alpha P_{\mathcal{AS}^\perp}(M(x) - M(z))\|_{L_2}^2 \\ &\leq \frac{1}{\cos^2(\mathcal{A}, \mathcal{S})} \|(x - z) - \alpha(M(x) - M(z))\|_{L_2}^2 \\ &= \frac{1}{\cos^2(\mathcal{A}, \mathcal{S})} \int_{-\infty}^{\infty} \left(1 - \alpha \frac{m(x(t)) - m(z(t))}{x(t) - z(t)}\right)^2 \\ &\quad \times (x(t) - z(t))^2 dt \\ &\leq \frac{1}{\cos^2(\mathcal{A}, \mathcal{S})} \max_{z_1, z_2} \left|1 - \alpha \frac{m(z_1) - m(z_2)}{z_1 - z_2}\right|^2 \\ &\quad \times \|x - z\|_{L_2}^2 \end{aligned} \quad (16)$$

for any  $x, z \in \mathcal{A}$ , where we used the following inequality [9]:

$$\|P_{\mathcal{AS}^\perp} x\|_{L_2} \leq \frac{1}{\cos(\mathcal{A}, \mathcal{S})} \|x\|_{L_2}. \quad (17)$$

We conclude that we need to choose a step size  $\alpha$  such that

$$\max_{z_1, z_2} \left|1 - \alpha \frac{m(z_1) - m(z_2)}{z_1 - z_2}\right| < \cos(\mathcal{A}, \mathcal{S}) \quad (18)$$

leading to (12). Condition (11) ensures that such an  $\alpha$  exists.

We showed that  $T$  is a contraction mapping and thus, by the fixed point theorem [10], the sequence  $\hat{y}_k$  converges to the unique fixed point of  $T$ , which remains to be found. To this end we substitute  $T(\hat{y}) = \hat{y}$  into (15), leading to

$$\hat{y} = \hat{y} + \alpha P_{\mathcal{AS}^\perp} (M(y) - M(\hat{y})). \quad (19)$$

This implies that  $P_{\mathcal{AS}^\perp}(M(y) - M(\hat{y})) = 0$ , meaning that  $M(y) - M(\hat{y}) \in \mathcal{S}^\perp$ . Consequently,  $\langle s(t - n), M(y(t)) - M(\hat{y}(t)) \rangle_{L_2} = 0$  for every  $n \in \mathbb{Z}$ , which is equivalent to the consistency requirement  $\hat{c}[n] = c[n]$ .

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