

# Recovering Signals From Lowpass Data

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**Abstract**—The problem of recovering a signal from its low frequency components occurs often in practical applications due to the lowpass behavior of many physical systems. Here, we study in detail conditions under which a signal can be determined from its low-frequency content. We focus on signals in shift-invariant spaces generated by multiple generators. For these signals, we derive necessary conditions on the cutoff frequency of the lowpass filter as well as necessary and sufficient conditions on the generators such that signal recovery is possible. When the lowpass content is not sufficient to determine the signal, we propose appropriate pre-processing that can improve the reconstruction ability. In particular, we show that modulating the signal with one or more mixing functions prior to lowpass filtering, can ensure the recovery of the signal in many cases, and reduces the necessary bandwidth of the filter.

**Index Terms**—Lowpass signals, sampling, shift-invariant spaces.

## I. INTRODUCTION

**L**OWPASS filters are prevalent in biological, physical and engineering systems. In many scenarios, we do not have access to the entire frequency content of a signal we wish to process, but only to its low frequencies. For example, it is well known that parts of the visual system exhibit lowpass nature: the neurons of the outer retina have strong response to low frequency stimuli, due to the relatively slow response of the photoreceptors. Similar behavior is observed also in the cones and rods [1]. Another example is the lowpass nature of free space wave propagation [2]. This limits the resolution of optical image reconstruction to half the wave length. Many engineering systems introduce lowpass filtering as well. One reason is to allow subsequent sampling and digital signal processing at a low rate.

Clearly if we have no prior knowledge on the original signal, and we are given a lowpassed version of it, then we cannot recover the missing frequency content. However, if we have prior knowledge on the signal structure then it may be possible to interpolate it from the given data. As an example, consider a signal  $x$  that lies in a shift-invariant (SI) space generated by a function  $\phi$ , so that  $x(t) = \sum a_n \phi(t - nT)$  for some  $T$ . Even if  $x$  is not

bandlimited, it can be recovered from the output of a lowpass filter with cutoff frequency  $\pi/T$  as long as the Fourier transform of the generator is not zero for all  $\omega \in [-\pi/T, \pi/T]$  [3], [4].

The goal of this paper is to study in more detail under what conditions a signal  $x$  can be recovered from its low-frequency content. Our focus is on signals that lie in SI spaces, generated by multiple generators [5]–[7]. Following a detailed problem formulation in Section II, we begin in Section III by deriving a necessary condition on the cutoff frequency of the low pass filter (LPF) and sufficient conditions on the generators such that  $x$  can be recovered from its lowpassed version. As expected, there are scenarios in which recovery is not possible. For example, if the bandwidth of the LPF is too small, or if one of the generators is zero over a certain frequency interval and all of its shifts with period  $2\pi/T$ , then recovery cannot be obtained. For cases in which the recovery conditions are satisfied, we provide a concrete method to reconstruct  $x$  from its lowpass frequency content in Section IV.

The next question we address is whether we can improve our ability to determine the signal by appropriate preprocessing, in scenarios where the recovery conditions are not satisfied. In Section V we show that pre-processing with linear time-invariant (LTI) filters does not help, even if we allow for a bank of LTI filters. As an alternative, in Section VI we consider pre-processing by modulation. Specifically, the signal  $x$  is modulated by multiplying it with a periodic mixing function prior to lowpass filtering. We then derive conditions on the mixing function to ensure perfect recovery. As we show, a larger class of signals can be recovered this way. Moreover, by applying a bank of mixing functions, the necessary cutoff frequency in each channel may be reduced. In Section VII, we briefly discuss how our results apply to sampling sparse signals in SI spaces at rates lower than Nyquist. These ideas rely on the recently developed framework for analog compressed sensing [8]–[11]. In our setting, they translate to reducing the LPF bandwidth, or the number of modulators. Finally, Section VIII summarizes and points out some open problems.

Modulation architectures have been previously incorporated into different sampling schemes. In [12], modulation was utilized to obtain high-rate sigma-delta converters. More recently, modulation has been used in order to sample sparse high bandwidth signals at low rates [13], [14]. Our specific choice of periodic functions is rooted in [14] in which a similar bank of modulators was proposed for sampling multiband signals at sub-Nyquist rates. Here, our focus is on signals in general SI spaces and our goal is to develop a broad framework that enables pre-processing such as to ensure perfect reconstruction. We treat signals that lie in a predefined subspace, in contrast to the union of subspaces assumption used in the context of sparse signal models [15]. Our results may be used in practical systems

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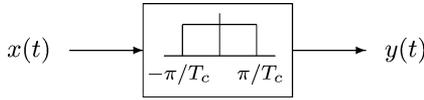


Fig. 1. Lowpass filtering of  $x(t)$ .

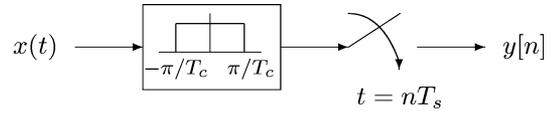


Fig. 2. Sampling of  $x(t)$  after lowpass filtering.

that involve lowpass filtering to preprocess the signal so that all its content can be recovered from the received low-frequency signal (without requiring a sparse signal model).

II. PROBLEM FORMULATION

A. Notations

We use the following notation throughout:  $\mathbb{C}^N$ ,  $L^2$ , and  $\ell^2$  denote the  $N$ -dimensional Euclidean space, the space of square integrable functions on the real line, and the space of square summable sequences, respectively. All these spaces are Hilbert spaces with the usual inner products. We write  $\hat{x}$  for the Fourier transform of a function  $x \in L^2$ :

$$\hat{x}(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt, \quad \omega \in \mathbb{R}.$$

The Paley–Wiener space of functions in  $L^2$  that are bandlimited to  $[-B, B]$  will be denoted by  $PW(B)$

$$PW(B) = \{x \in L^2 : \hat{x}(\omega) = 0 \text{ for all } \omega \notin [-B, B]\}$$

and  $P_B$  is the orthogonal projection  $L^2 \rightarrow PW(B)$  onto  $PW(B)$ . Clearly,  $P_B$  is a bounded linear operator on  $L^2$ . We will also need the Paley–Wiener space of functions whose inverse Fourier transform is supported on a compact interval, i.e.,

$$\widehat{PW}(B) = \{\hat{x} \in L^2 : x(t) = 0 \text{ for all } t \notin [-B, B]\}.$$

For any  $a \in \mathbb{R}$ , the shift (or translation) operator  $S_a : L^2 \rightarrow L^2$  is defined by  $(S_a x)(t) = x(t - a)$ .

If  $\{\phi_k\}_{k \in \mathcal{I}}$  is a set of functions in  $L^2$  with an arbitrary index set  $\mathcal{I}$ , then  $\text{span}\{\phi_k : k \in \mathcal{I}\}$  denotes the closed linear subspace of  $L^2$  spanned by  $\{\phi_k\}_{k \in \mathcal{I}}$ .

B. Problem Formulation

We consider the problem of recovering a signal  $x(t)$ ,  $t \in \mathbb{R}$  from its low-frequency content. Specifically, suppose that  $x$  is filtered by a LPF with cut off frequency  $\pi/T_c$ , as in Fig. 1. We would like to answer the following questions:

- What signals  $x$  can be recovered from the output  $y$  of the LPF?
- Can we perform preprocessing of  $x$  prior to filtering to ensure that  $x$  can be recovered from  $y$ ?

Filtering a signal  $x \in L^2$  with a LPF with cutoff frequency  $\pi/T_c$  corresponds to a projection of  $x$  onto the Paley–Wiener space  $PW(\pi/T_c)$ . Therefore, we can write  $y = P_{\pi/T_c} x$ .

Note, that we assume here that the output  $y(t)$ ,  $t \in \mathbb{R}$  is analog. Since  $y$  is a lowpass signal, an equivalent formulation

is to sample  $y$  with period  $T_s = 1/f_s$  lower than the Nyquist period  $T_c$  to obtain the sequence of samples  $\{y[n]\}_{n \in \mathbb{Z}}$ . The problem is then to recover  $x(t)$ ,  $t \in \mathbb{R}$  from the samples  $\{y[n]\}_{n \in \mathbb{Z}}$ , as in Fig. 2. Since  $\{y[n]\}_{n \in \mathbb{Z}}$  uniquely determines  $y$ , the two formulations are equivalent. For concreteness, we focus here on the problem in which we are given  $y(t)$ ,  $t \in \mathbb{R}$  directly. Thus, our emphasis is not on the sampling rate, but rather on the information content in the lowpass regime, regardless of the sampling rate to follow.

Clearly, if  $x$  is bandlimited to  $[-\pi/T_c, \pi/T_c]$ , then it can be recovered from  $y$ . However, we will assume here that  $x$  is a general SI signal, not necessarily bandlimited. These signals have the property that if  $x(t)$  lies in a given SI space, then so do all its shifts  $(S_{kT}x)(t) = x(t - kT)$  by integer multiples of some given  $T$ . Bandlimited signals are a special class of SI signals. Indeed, if  $x$  is bandlimited then so are all its shifts  $S_{kT}x$ ,  $k \in \mathbb{Z}$  for a given  $T$ . In fact, bandlimited signals have an even stronger property that all their shifts  $S_a x$  by any number  $a \in \mathbb{R}$  are bandlimited. Throughout, we assume that  $x$  lies in a generally complex SI space with multiple generators.

Let  $\phi = \{\phi_1, \dots, \phi_N\}$  be a given set of functions in  $L^2$  and let  $T \in \mathbb{R}$  be a given real number. Then the shift-invariant space defined by  $\phi$  is formally defined as [5]–[7]

$$\mathcal{S}_T(\phi) = \text{span}\{S_{kT}\phi_n : k \in \mathbb{Z}, 1 \leq n \leq N\}.$$

The functions  $\phi_n$  are referred to as the generators of  $\mathcal{S}_T(\phi)$ . Thus, every function  $x \in \mathcal{S}_T(\phi)$  can be written as

$$x(t) = \sum_{n=1}^N \sum_{k \in \mathbb{Z}} a_n[k] \phi_n(t - kT), \quad t \in \mathbb{R} \quad (1)$$

where for each  $1 \leq n \leq N$ ,  $\{a_n[k]\}_{k \in \mathbb{Z}}$  is an arbitrary sequence in  $\ell^2$ . Examples of such SI spaces include multiband signals [16] and spline functions [3], [17]. Expansions of the type (1) are also encountered in communication systems, when the analog signal is produced by pulse amplitude modulation.

In order to guarantee a unique and stable representation of any signal in  $\mathcal{S}_T(\phi)$  by coefficients  $\{a_n[k]\}$ , the generators  $\phi$  are typically chosen to form a Riesz basis for  $\mathcal{S}_T(\phi)$ . This means that there exist constants  $\alpha > 0$  and  $\beta < \infty$  such that

$$\alpha \|\mathbf{a}\|^2 \leq \left\| \sum_{n=1}^N \sum_{k \in \mathbb{Z}} a_n[k] \phi_n(t - kT) \right\|_{L^2}^2 \leq \beta \|\mathbf{a}\|^2 \quad (2)$$

where  $\|\mathbf{a}\|^2 = \sum_{n=1}^N \sum_{k \in \mathbb{Z}} |a_n[k]|^2$ . Condition (2) implies that any  $x \in \mathcal{S}_T(\phi)$  has a unique and stable representation in terms of the sequences  $\{a_n[k]\}_{k \in \mathbb{Z}}$ . In particular, it guarantees that these sequences can be recovered from  $x \in \mathcal{S}_T(\phi)$  by means of a linear bounded operator.

By taking Fourier transforms in (2) it can be shown that the generators  $\phi$  form a Riesz basis<sup>1</sup> if and only if [6]

$$\alpha \mathbf{I} \preceq \mathbf{M}_\phi(\omega) \preceq \beta \mathbf{I}, \quad \text{a.e. } \omega \in \left[-\frac{\pi}{T}, \frac{\pi}{T}\right]. \quad (3)$$

Here  $\mathbf{M}_\phi(\omega)$  is called the *Grammian* of the generators  $\phi = \{\phi_1, \dots, \phi_N\}$ , and is the  $N \times N$  matrix

$$\mathbf{M}_\phi(\omega) = \begin{bmatrix} R_{\phi_1\phi_1}(\omega) & \dots & R_{\phi_1\phi_N}(\omega) \\ \vdots & \ddots & \vdots \\ R_{\phi_N\phi_1}(\omega) & \dots & R_{\phi_N\phi_N}(\omega) \end{bmatrix} \quad (4)$$

where for any two generators  $\phi_i, \phi_j$  the function  $R_{\phi_i\phi_j}$  is given by

$$R_{\phi_i\phi_j}(\omega) = \sum_{k \in \mathbb{Z}} \overline{\hat{\phi}_i\left(\omega - 2k\frac{\pi}{T}\right)} \hat{\phi}_j\left(\omega - 2k\frac{\pi}{T}\right). \quad (5)$$

Note that the functions  $R_{\phi_i\phi_j}$  are  $2\pi/T$ -periodic. Therefore, condition (3) is equivalent to  $\alpha \mathbf{I} \preceq \mathbf{M}_\phi(\omega - a) \preceq \beta \mathbf{I}$  for every arbitrary real number  $a$ . We will need in particular the case  $a = \pi/T$ , for which the entries of the matrix  $\mathbf{M}_\phi(\omega - a)$  are

$$R_{\phi_i\phi_j}\left(\omega - \frac{\pi}{T}\right) = \sum_{k \in \mathbb{Z}} \overline{\hat{\phi}_i\left(\omega - [2k+1]\frac{\pi}{T}\right)} \hat{\phi}_j\left(\omega - [2k+1]\frac{\pi}{T}\right). \quad (6)$$

### III. RECOVERY CONDITIONS

The first question we address is whether we can recover  $x \in \mathcal{S}_T(\phi)$  of the form (1) from the output  $y = P_{\pi/T_c}x$  of a LPF with cutoff frequency  $\pi/T_c$ , assuming that the generators  $\phi$  satisfy (3). We further assume that the generators are not bandlimited to  $\pi/T_c$ , namely they have energy outside the frequency interval  $[-\pi/T_c, \pi/T_c]$ . We provide conditions on  $\phi$  and on the bandwidth of the LPF such that  $x$  can be recovered from  $y$ . As we show, even if the generators  $\phi$  are not bandlimited,  $x$  can often be determined from  $y$ .

First we note that in order to recover  $x \in \mathcal{S}_T(\phi)$  from the low-pass signal  $y = P_{\pi/T_c}x$  it is sufficient to recover the sequences  $\{a_n[k]\}_{k \in \mathbb{Z}}$ ,  $1 \leq n \leq N$  because the generators  $\phi$  are assumed to be known. The output of the LPF can be written as

$$y(t) = (P_{\pi/T_c}x)(t) = \sum_{n=1}^N \sum_{k \in \mathbb{Z}} a_n[k] \psi_n(t - kT)$$

where  $\psi_n := P_{\pi/T_c}\phi_n$  denotes the lowpass filtered generator  $\phi_n$ , and the sum on the right-hand side converges in  $L^2$  since  $P_{\pi/T_c}$  is bounded. Therefore, we immediately have the following observation: The sequences  $\{a_n[k]\}_{k \in \mathbb{Z}}$ ,  $1 \leq n \leq N$  can be recovered from  $y$  if  $\psi$  forms a Riesz basis for  $\mathcal{S}_T(\psi)$ . This is equivalent to the following statement.

<sup>1</sup>Here and in the sequel, when we say that a set of generators  $\phi$  form (or generate) a basis, we mean that the basis functions are  $\{\phi_n(t - kT), k \in \mathbb{Z}, 1 \leq n \leq N\}$ .

*Proposition 1:* Let  $\phi = \{\phi_1, \dots, \phi_N\}$  be a set generators, and let  $\psi_n = P_{\pi/T_c}\phi_n$ ,  $1 \leq n \leq N$  be the lowpass filtered generators where  $\pi/T_c$  is the bandwidth of the LPF. Then the signal  $x \in \mathcal{S}_T(\phi)$  can be recovered from the observations  $y = P_{\pi/T_c}x$  if the Grammian  $\mathbf{M}_\psi(\omega)$  satisfies (3) for some  $0 < \alpha \leq \beta < \infty$ .

*Example 1:* We consider the case of one generator ( $N = 1$ )

$$\phi_1(t) = \begin{cases} \frac{1}{2D}, & t \in [-D, D] \\ 0, & t \notin [-D, D] \end{cases} \quad (7)$$

for some  $D > 0$ . The Fourier transform of this generator is  $\hat{\phi}_1(\omega) = \sin(\omega D)/(\omega D)$  which becomes zero at  $\omega = k\pi/D$  for all  $k = \pm 1, \pm 2, \dots$ . We assume that  $D/T$  is not an integer. Then one can easily see that this choice satisfies (3), i.e., there exists  $\alpha, \beta$  such that

$$0 < \alpha \leq \sum_{k \in \mathbb{Z}} \left| \frac{\sin(\omega D - 2\pi k \frac{D}{T})}{\omega D - 2\pi k \frac{D}{T}} \right|^2 \leq \beta < \infty \quad (8)$$

for all  $\omega \in [-\pi/T, \pi/T]$ . The lower bound follows from the assumption that  $D/T$  is not an integer, so that all the functions in the above sum have no common zero in  $[-\pi/T, \pi/T]$ . The upper bound  $\beta$  follows from:

$$\begin{aligned} \sum_{k \in \mathbb{Z}} \left| \frac{\sin(\omega D - 2\pi k \frac{D}{T})}{\omega D - 2\pi k \frac{D}{T}} \right|^2 &\leq \sum_{k \in \mathbb{Z}} \frac{1}{|\omega D - 2\pi k \frac{D}{T}|^2} \\ &\leq \left(\frac{T}{\pi D}\right)^2 \left[ 1 + 2 \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \right] \leq \left(\frac{2T}{\pi D}\right)^2 \end{aligned}$$

using that  $|\omega D - 2\pi k D/T| \geq \pi D/T(2|k| - 1)$  for all  $k = \pm 1, \pm 2, \dots$  and all  $\omega \in [-\pi/T, \pi/T]$ .

Assume now that the LPF has cutoff frequency  $\pi/T_c = \pi/T$ . Then the Fourier transform  $\hat{\psi}_1$  of the filtered generator  $\psi_1 = P_{\pi/T}\phi_1$  will satisfy a relation like (8) only if  $D \leq T$ , i.e., only if  $\hat{\phi}_1$  has no zero in  $[-\pi/T, \pi/T]$ . In cases where  $D > T$  the cutoff frequency has to be larger in order to allow a recovery of the original signal. One easily sees that the cutoff frequency of the LPF has to lie at least  $\pi/T - \pi/D$  above  $\pi/T$  in order that  $\hat{\psi}_1$  will satisfy a relation similar to (8). In this case, the shifts  $\hat{\psi}_1(\omega \pm 2\pi/T)$  compensate for the zero of  $\hat{\psi}_1(\omega)$  in the sum (8). Thus, for cutoff frequencies  $\pi/T_c \geq 2\pi/T - \pi/D$  a recovery of the signal  $x$  from the LPF signal  $y$  will be possible.

The previous example illustrates that the question whether  $\psi$  forms a Riesz basis for  $\mathcal{S}_T(\psi)$  depends on the given generators  $\phi$  and on the bandwidth  $\pi/T_c$  of the LPF. The next proposition derives a necessary condition on the required bandwidth  $\pi/T_c$  of the LPF such that  $\psi$  can be a Riesz basis for  $\mathcal{S}_T(\psi)$ .

*Proposition 2:* Let  $\phi = \{\phi_1, \dots, \phi_N\}$  be a Riesz basis for the space  $\mathcal{S}_T(\phi)$  and let  $\psi_n = P_{\pi/T_c}\phi_n$  with  $1 \leq n \leq N$ . Then a necessary condition for  $\psi = \{\psi_1, \dots, \psi_N\}$  to be a Riesz basis for  $\mathcal{S}_T(\psi)$  is that  $\pi/T_c \geq N\pi/T$ .

*Proof:* We consider the Grammian  $\mathbf{M}_\psi(\omega)$  whose entries are equal to

$$R_{\psi_i\psi_j}(\omega) = \sum_{|k| \leq 1/2(T/T_c+1)} \overline{\hat{\psi}_i\left(\omega - k\frac{2\pi}{T}\right)} \hat{\psi}_j\left(\omega - k\frac{2\pi}{T}\right).$$

All other terms in the generally infinite sum (cf. (5)) are identically zero since  $\hat{\psi}_n(\omega)$  is bandlimited to  $[-\pi/T_c, \pi/T_c]$ . This Grammian can be written as  $\mathbf{M}_\psi(\omega) = \mathbf{\Psi}^*(\omega)\mathbf{\Psi}(\omega)$  with

$$\mathbf{\Psi}(\omega) = \begin{bmatrix} \hat{\psi}_1(\omega + [L_0 + 1]\frac{2\pi}{T}) & \dots & \hat{\psi}_N(\omega + [L_0 + 1]\frac{2\pi}{T}) \\ \hat{\psi}_1(\omega + L_0\frac{2\pi}{T}) & \dots & \hat{\psi}_N(\omega + L_0\frac{2\pi}{T}) \\ \vdots & & \vdots \\ \hat{\psi}_1(\omega) & \dots & \hat{\psi}_N(\omega) \\ \vdots & & \vdots \\ \hat{\psi}_1(\omega - L_0\frac{2\pi}{T}) & \dots & \hat{\psi}_N(\omega - L_0\frac{2\pi}{T}) \\ \hat{\psi}_1(\omega - [L_0 + 1]\frac{2\pi}{T}) & \dots & \hat{\psi}_N(\omega - [L_0 + 1]\frac{2\pi}{T}) \end{bmatrix} \quad (9)$$

where  $L_0$  is the largest integer such that  $L_0 \leq (T/T_c - 1)/2$ . Since every  $\hat{\psi}_n(\omega)$  is banded to  $[-\pi/T_c, \pi/T_c]$ , the first and the last row of this matrix are identically zero for some  $\omega \in [-\pi/T, \pi/T]$ . At these  $\omega$ 's, the matrix  $\mathbf{\Psi}(\omega)$  has effectively  $L = 2L_0 + 1$  rows and  $N$  columns, and it holds that  $L \leq T/T_c$ . Since  $\mathbf{M}_\psi(\omega) = \mathbf{\Psi}^*(\omega)\mathbf{\Psi}(\omega)$ , the Grammian can have full rank for every  $\omega \in [-\pi/T, \pi/T]$  only if  $L \geq N$ , i.e., only if  $\pi/T_c \geq N\pi/T$ . ■

The necessary condition on the bandwidth of the LPF given in the previous proposition is not generally sufficient. However, given a bandwidth  $\pi/T_c$  which satisfies the necessary condition of Proposition 2, sufficient conditions on the generators  $\phi$  can be derived such that the lowpass filtered generators  $\psi$  form a Riesz basis for  $\mathcal{S}_T(\psi)$ , i.e., such that  $x$  may be recovered from  $y$ .

*Proposition 3:* Let  $\phi = \{\phi_1, \dots, \phi_N\}$  be a Riesz basis for  $\mathcal{S}_T(\phi)$  and let  $\psi_n = P_{\pi/T_c}\phi_n$  for  $1 \leq n \leq N$  with  $\pi/T_c \geq N\pi/T$ . Denote by  $L$  the largest integer such that  $L \leq T/T_c$ . If  $L = 2L_0 + 1$  is an odd number, then we define the  $L \times N$  matrix  $\mathbf{\Phi}_L(\omega)$  by

$$\mathbf{\Phi}_L(\omega) = \begin{bmatrix} \hat{\phi}_1(\omega + 2L_0\frac{\pi}{T}) & \dots & \hat{\phi}_N(\omega + 2L_0\frac{\pi}{T}) \\ \vdots & & \vdots \\ \hat{\phi}_1(\omega + 2\frac{\pi}{T}) & \dots & \hat{\phi}_N(\omega + 2\frac{\pi}{T}) \\ \hat{\phi}_1(\omega) & \dots & \hat{\phi}_N(\omega) \\ \hat{\phi}_1(\omega - 2\frac{\pi}{T}) & \dots & \hat{\phi}_N(\omega - 2\frac{\pi}{T}) \\ \vdots & & \vdots \\ \hat{\phi}_1(\omega - 2L_0\frac{\pi}{T}) & \dots & \hat{\phi}_N(\omega - 2L_0\frac{\pi}{T}) \end{bmatrix}. \quad (10)$$

For  $L = 2L_0$  even, we define

$$\mathbf{\Phi}_L(\omega) = \begin{bmatrix} \hat{\phi}_1(\omega + [2L_0 - 1]\frac{\pi}{T}) & \dots & \hat{\phi}_N(\omega + [2L_0 - 1]\frac{\pi}{T}) \\ \vdots & & \vdots \\ \hat{\phi}_1(\omega + \frac{\pi}{T}) & \dots & \hat{\phi}_N(\omega + \frac{\pi}{T}) \\ \hat{\phi}_1(\omega - \frac{\pi}{T}) & \dots & \hat{\phi}_N(\omega - \frac{\pi}{T}) \\ \vdots & & \vdots \\ \hat{\phi}_1(\omega - [2L_0 - 1]\frac{\pi}{T}) & \dots & \hat{\phi}_N(\omega - [2L_0 - 1]\frac{\pi}{T}) \end{bmatrix}. \quad (11)$$

If there exists a constant  $\alpha > 0$  such that

$$\mathbf{M}_L(\omega) := \mathbf{\Phi}_L^*(\omega)\mathbf{\Phi}_L(\omega) \succeq \alpha\mathbf{I} \quad \text{a.e. } \omega \in \left[-\frac{\pi}{T}, \frac{\pi}{T}\right], \quad (12)$$

then  $\psi = \{\psi_1, \dots, \psi_N\}$  forms a Riesz basis for  $\mathcal{S}_T(\psi)$ .

Moreover, if  $T/T_c$  is an integer, then condition (12) is also necessary for  $\psi$  to be a Riesz basis for  $\mathcal{S}_T(\psi)$ .

When  $\pi/T_c \rightarrow \infty$ , i.e.,  $L \rightarrow \infty$ , the matrix  $\mathbf{M}_L(\omega)$  reduces to  $\mathbf{M}_\phi(\omega)$  of (4), which by definition satisfies (3). However, since for the calculation of the entries of  $\mathbf{M}_L(\omega)$ , we are only summing over a partial set of the integers, we are no longer guaranteed that  $\mathbf{M}_L(\omega)$  satisfies the lower bound of (3).

The requirements of Proposition 3 imply that  $L \geq N$ . Consequently, the matrix  $\mathbf{M}_L(\omega) = \mathbf{\Phi}_L^*(\omega)\mathbf{\Phi}_L(\omega)$  is positive definite for almost all  $\omega \in [-\pi/T, \pi/T]$  if and only if  $\mathbf{\Phi}_L(\omega)$  has full column rank for almost all  $\omega \in [-\pi/T, \pi/T]$ .

Note that Example 1 shows that (12) is not necessary, in general: With  $T < D < 2T$  and a cutoff frequency of  $\pi/T_c > 2\pi/T - \pi/D$ , the corresponding  $\psi$  form a Riesz basis for  $\mathcal{S}_T(\psi)$ . However, it can easily be verified that (12) is not satisfied.

*Proof:* We consider the case of  $L$  being odd. It has to be shown that the Grammian  $\mathbf{M}_\psi(\omega)$  satisfies (3). Since  $NT_c \leq T$ , the Grammian can be written as  $\mathbf{M}_\psi(\omega) = \mathbf{\Psi}^*(\omega)\mathbf{\Psi}(\omega)$  with  $\mathbf{\Psi}(\omega)$  defined by (9). Next  $\mathbf{\Psi}(\omega)$  is written as  $\mathbf{\Psi}(\omega) = \mathbf{\Psi}_L(\omega) + \mathbf{\Psi}_\perp(\omega)$  where  $\mathbf{\Psi}_\perp(\omega)$  is the  $(2L_0 + 1) \times N$  matrix whose first and last row coincide with those of  $\mathbf{\Psi}(\omega)$  and whose other rows are identically zero. Similarly  $\mathbf{\Psi}_L(\omega)$  denotes the matrix whose first and last row is identically zero and whose remaining rows coincide with those of  $\mathbf{\Psi}(\omega)$ . Since  $\psi_n(\omega) = \phi_n(\omega)$  for all  $\omega \in [-\pi/T_c, \pi/T_c]$  and for every  $1 \leq n \leq N$ , we have that  $\mathbf{\Psi}_L^*(\omega)\mathbf{\Psi}_L(\omega) = \mathbf{\Phi}_L^*(\omega)\mathbf{\Phi}_L(\omega)$ . Therefore,

$$\begin{aligned} \mathbf{M}_\psi(\omega) &= \mathbf{\Psi}_L^*(\omega)\mathbf{\Psi}_L(\omega) + \mathbf{\Psi}_\perp^*(\omega)\mathbf{\Psi}_\perp(\omega) \\ &\quad + \mathbf{\Psi}_L^*(\omega)\mathbf{\Psi}_\perp(\omega) + \mathbf{\Psi}_\perp^*(\omega)\mathbf{\Psi}_L(\omega) \\ &= \mathbf{\Phi}_L^*(\omega)\mathbf{\Phi}_L(\omega) + \mathbf{\Psi}_\perp^*(\omega)\mathbf{\Psi}_\perp(\omega) \end{aligned} \quad (13)$$

since by the definition of  $\mathbf{\Psi}_L(\omega)$  and  $\mathbf{\Psi}_\perp(\omega)$ , we obviously have that  $\mathbf{\Psi}_L^*(\omega)\mathbf{\Psi}_\perp(\omega) \equiv 0$  and  $\mathbf{\Psi}_\perp^*(\omega)\mathbf{\Psi}_L(\omega) \equiv 0$ . Now it follows from (13) that for every  $\mathbf{x} \in \mathbb{C}^N$ ,

$$\begin{aligned} \mathbf{x}^*\mathbf{M}_\psi(\omega)\mathbf{x} &= \|\mathbf{\Phi}_L(\omega)\mathbf{x}\|_{\mathbb{C}^N}^2 + \|\mathbf{\Psi}_\perp(\omega)\mathbf{x}\|_{\mathbb{C}^N}^2 \\ &\geq \|\mathbf{\Phi}_L(\omega)\mathbf{x}\|_{\mathbb{C}^N}^2 = \mathbf{x}^*\mathbf{\Phi}_L^*(\omega)\mathbf{\Phi}_L(\omega)\mathbf{x} \geq \alpha, \end{aligned}$$

where the last inequality follows from (12). This shows that the Grammian  $\mathbf{M}_\psi(\omega)$  is lower bounded as in (3). The existence of an upper bound for  $\mathbf{M}_\psi(\omega)$  is trivial since  $\mathbf{M}_\psi(\omega)$  has finite dimensions.

Assume now that  $T/T_c$  is an (odd) integer. In this case  $L_0 = (T/T_c - 1)/2$  and it can easily be verified that the matrix  $\mathbf{\Psi}_\perp(\omega)$  is identically zero. From (13),  $\mathbf{M}_\psi(\omega) = \mathbf{\Phi}_L^*(\omega)\mathbf{\Phi}_L(\omega) = \mathbf{M}_L(\omega)$  which shows that if the Grammian  $\mathbf{M}_\psi(\omega)$  satisfies (3) then  $\mathbf{\Phi}_L(\omega)$  satisfies (12). This proves that (12) is also necessary for  $\psi$  to be a Riesz basis for  $\mathcal{S}_T(\psi)$ .

The case of  $L$  even follows from the same arguments but starting with expression (6) for the entries of the Grammian instead of (5). Therefore, the details are omitted. ■

*Example 2:* We consider an example with two generators ( $N = 2$ ) which both have the form as in Example 1, with different values for  $D$ , i.e.,

$$\phi_i(t) = \begin{cases} \frac{1}{2D_i}, & t \in [-D_i, D_i] \\ 0, & t \notin [-D_i, D_i] \end{cases} \quad i = 1, 2$$

with Fourier transforms  $\hat{\phi}_i(\omega) = \sin(\omega D_i)/(\omega D_i)$ . As in Example 1 we assume that  $D_i/T$  are not integers and that  $D_1 \neq D_2$ . Under these conditions, the Grammian  $\mathbf{M}_\phi(\omega)$  of  $\phi = \{\phi_1, \phi_2\}$  satisfies (3). To see this, we consider the determinant of  $\mathbf{M}_\phi(\omega)$  for some arbitrary but fixed  $\omega \in [-\pi/T, \pi/T]$ :

$$\det[\mathbf{M}_\phi(\omega)] = \sum_{k \in \mathbb{Z}} \left| \hat{\phi}_1 \left( \omega - k \frac{2\pi}{T} \right) \right|^2 \sum_{k \in \mathbb{Z}} \left| \hat{\phi}_2 \left( \omega - k \frac{2\pi}{T} \right) \right|^2 - \left( \sum_{k \in \mathbb{Z}} \hat{\phi}_1 \left( \omega - k \frac{2\pi}{T} \right) \hat{\phi}_2 \left( \omega - k \frac{2\pi}{T} \right) \right)^2. \quad (14)$$

We know from Example 1, that the first term on the right-hand side is lower bounded by some constant  $\alpha_1 \alpha_2 > 0$ . Moreover, the Cauchy–Schwarz inequality shows that the second term on the right-hand side is always smaller or equal than the first term with equality only if the two sequences

$$\left\{ \hat{\phi}_i \left( \omega - k \frac{2\pi}{T} \right) \right\}_{k \in \mathbb{Z}}, \quad i = 1, 2$$

are linearly dependent. However, since  $D_1 \neq D_2$ , it is not hard to verify that these two sequences are linearly independent. Consequently  $\det[\mathbf{M}_\phi(\omega)] > 0$  for all  $\omega \in [-\pi/T, \pi/T]$  which shows that  $\mathbf{M}_\phi(\omega)$  satisfies the lower bound of (3). That  $\mathbf{M}_\phi(\omega)$  satisfies also the upper bound in (3) follows from a similar calculation as in Example 1 using that  $|\hat{\phi}_i(\omega)|$  decreases proportional to  $1/|\omega|$  as  $|\omega| \rightarrow \infty$ .

Assume now that the bandwidth of the LPF satisfies  $2\pi/T \leq \pi/T_c < 3\pi/T$ . In this case the matrix  $\Phi_L(\omega)$  of Proposition 3 is given by

$$\Phi_L(\omega) = \begin{bmatrix} \hat{\phi}_1 \left( \omega + \frac{\pi}{T} \right) & \hat{\phi}_2 \left( \omega + \frac{\pi}{T} \right) \\ \hat{\phi}_1 \left( \omega - \frac{\pi}{T} \right) & \hat{\phi}_2 \left( \omega - \frac{\pi}{T} \right) \end{bmatrix}$$

and the determinant of  $\mathbf{M}_L(\omega) := \Phi_L^*(\omega) \Phi_L(\omega)$  becomes

$$\det[\mathbf{M}_L(\omega)] = \sum_{k=\pm 1} \left| \hat{\phi}_1 \left( \omega - k \frac{\pi}{T} \right) \right|^2 \sum_{k=\pm 1} \left| \hat{\phi}_2 \left( \omega - k \frac{\pi}{T} \right) \right|^2 - \left( \sum_{k=\pm 1} \hat{\phi}_1 \left( \omega - k \frac{\pi}{T} \right) \hat{\phi}_2 \left( \omega - k \frac{\pi}{T} \right) \right)^2. \quad (15)$$

This expression is similar to (14) and the same arguments show that  $\det[\mathbf{M}_L(\omega)] > 0$  for all  $\omega \in [-\pi/T, \pi/T]$ . Namely, since  $D_i/T$  are not integers, the functions  $\hat{\phi}_i(\omega + \pi/T)$  and  $\hat{\phi}_i(\omega - \pi/T)$  have no common zero such that the first term on the right-hand side of (15) is lower bounded by some  $\alpha_1 \alpha_2 > 0$ . The Cauchy–Schwarz inequality implies that the second term is always smaller than the first one.

We conclude that  $\Phi_L(\omega)$  satisfies the condition of Proposition 3, so that the signal  $x$  can be recovered from its low frequency components  $y = P_{\pi/T_c} x$ .

If for a certain bandwidth  $\pi/T_c$  of the LPF the generators  $\phi$  satisfy the conditions of Proposition 3, then the signal  $x$  can be recovered from  $y = P_{\pi/T_c} x$ . However, if the generators  $\phi$  do not satisfy these conditions, then there exists in principle two ways to enable recovery of  $x$ :

- increasing the bandwidth of the LPF;
- preprocess  $x$  before lowpass filtering, i.e., modify the generators  $\phi$ .

It is clear that for a given set  $\phi = \{\phi_1, \dots, \phi_N\}$  of generators an increase of the LPF can only increase the “likelihood” that the matrix  $\Phi_L(\omega)$  of Proposition 3 will have full column rank. This is because enlarging  $\pi/T_c$  increases the number  $L$  i.e., it adds additional rows to the matrix which can only enlarge the column rank of  $\Phi_L(\omega)$ . Preprocessing of  $x$  will be discussed in detail in Sections V and VI.

#### IV. RECOVERY ALGORITHM

We now describe a simple method to reconstruct the desired signal  $x$  from its low frequency components. This method is used in later sections to show how preprocessing of the signal  $x$  may facilitate its recovery. Throughout this section, we assume that the bandwidth  $\pi/T_c$  of the LPF satisfies the necessary condition of Proposition 2, and that the generators satisfy the sufficient condition of Proposition 3.

Taking the Fourier transform of (1), we see that every  $x \in \mathcal{S}_T(\phi)$  can be expressed in the Fourier domain as

$$\hat{x}(\omega) = \sum_{n=1}^N \hat{a}_n(e^{j\omega T}) \hat{\phi}_n(\omega), \quad \omega \in \mathbb{R} \quad (16)$$

where

$$\hat{a}_n(e^{j\omega T}) = \sum_{k \in \mathbb{Z}} a_n[k] e^{-j\omega k T}$$

is the  $2\pi/T$ -periodic discrete time Fourier transform of the sequence  $\{a_n[k]\}_{k \in \mathbb{Z}}$  at frequency  $\omega T$ . Denoting by  $\hat{\mathbf{a}}(e^{j\omega T})$  the vector whose  $n$ th element is equal to  $\hat{a}_n(e^{j\omega T})$  and by  $\hat{\phi}(\omega)$  the vector whose  $n$ th element is equal to  $\hat{\phi}_n(\omega)$  we can write (16) in vector form as

$$\hat{x}(\omega) = \hat{\phi}^T(\omega) \hat{\mathbf{a}}(e^{j\omega T}).$$

The Fourier transform of the LPF output  $y = P_{\pi/T_c} x$  is bandlimited to  $\pi/T_c$ , and for all  $\omega \in [-\pi/T_c, \pi/T_c]$  we have  $\hat{y}(\omega) = \hat{x}(\omega)$ . Therefore,

$$\hat{y}(\omega) = \hat{\phi}^T(\omega) \hat{\mathbf{a}}(e^{j\omega T}), \quad \omega \in \left[ -\frac{\pi}{T_c}, \frac{\pi}{T_c} \right]. \quad (17)$$

For every  $\omega \in [-\pi/T_c, \pi/T_c]$ , (17) describes an equation for the  $N$  unknowns  $\hat{a}_n(e^{j\omega T})$ . Clearly, one equation is not sufficient to recover the length- $N$  vector  $\hat{\mathbf{a}}(e^{j\omega T})$ ; we need at least  $N$  equations. However, since according to Proposition 2 the bandwidth of the LPF has to be at least  $\pi/T_c \geq N\pi/T$ , we can form more equations from the given data by noting that  $\hat{\mathbf{a}}$  is periodic with period  $2\pi/T$ , while  $\hat{\phi}$ , and consequently  $\hat{y}$ , are

generally not. Specifically, let  $\omega_0 \in [-\pi/T, \pi/T]$  be an arbitrary frequency. For any  $\omega_k = \omega_0 + 2\pi k/T$  with  $k$  an integer we have that  $\hat{\mathbf{a}}(e^{j\omega_k T}) = \hat{\mathbf{a}}(e^{j\omega_0 T})$ . Therefore, by evaluating  $\hat{\mathbf{y}}$  and  $\hat{\boldsymbol{\phi}}$  at frequencies  $-\pi/T_c \leq \omega_k \leq \pi/T_c$ , we can use (17) to generate more equations. To this end, let  $L$  be the largest integer for which  $L \leq T/T_c$ . Assume first that  $L = 2L_0 + 1$  for some integer  $L_0$ , so that  $L$  is odd. We then generate the equations

$$\begin{aligned} \hat{y}_k(\omega) &:= \hat{y}\left(\omega - k\frac{2\pi}{T}\right) \\ &= \sum_{n=1}^N \hat{\phi}_n\left(\omega - k\frac{2\pi}{T}\right) \hat{a}_n\left(\omega - k\frac{2\pi}{T}\right), \end{aligned}$$

for  $-L_0 \leq k \leq L_0$  and  $\omega \in [-\pi/T, \pi/T]$ . Since by our assumption  $\pi/T_c \geq L\pi/T$ , all the observations  $\hat{y}_k(\omega) = \hat{y}(\omega - 2k\pi/T)$  are in the passband regime of the LPF. The above set of  $L$  equations may be written as

$$\hat{\mathbf{y}}(\omega) = \mathbf{\Phi}_L(\omega) \hat{\mathbf{a}}(e^{j\omega T}), \quad \omega \in \left[-\frac{\pi}{T}, \frac{\pi}{T}\right] \quad (18)$$

where  $\hat{\mathbf{y}}(\omega) = [\hat{y}_{-L_0}(\omega), \dots, 0, \dots, \hat{y}_{L_0}(\omega)]^T$  is a length  $L$  vector containing all the different observations  $\hat{y}_k$  of the output  $\hat{y}$ , and  $\mathbf{\Phi}_L(\omega)$  is the  $L \times N$  matrix given by (10). When  $L = 2L_0$  is an even number,<sup>2</sup> we generate additional equations by

$$\hat{y}_k(\omega) := \sum_{n=1}^N \hat{\phi}_n\left(\omega - [2k+1]\frac{\pi}{T}\right) \hat{a}_n\left(\omega - [2k+1]\frac{\pi}{T}\right), \quad (19)$$

for  $-L_0 \leq k \leq L_0 - 1$ . Here again all the observations in (19) are in the passband regime of the LPF. Therefore, (19) can be written as in (18), where  $\mathbf{\Phi}_L(\omega)$  is now given by (11), and the definition of  $\hat{\mathbf{a}}$  is changed accordingly.

If the matrix  $\mathbf{\Phi}_L(\omega)$  satisfies the sufficient conditions of Proposition 3, then the unknown vector  $\hat{\mathbf{a}}(e^{j\omega T})$  can be recovered from (18) by solving the linear set of equations for all  $\omega \in [-\pi/T, \pi/T]$ . In particular, there exists a left inverse  $\mathbf{G}(\omega)$  of  $\mathbf{\Phi}_L(\omega)$  such that  $\hat{\mathbf{a}}(e^{j\omega T}) = \mathbf{G}(\omega) \hat{\mathbf{y}}(\omega)$ . Finally, the desired sequences  $\{a_n[k]\}_{k \in \mathbb{Z}}$  are the Fourier coefficients of the  $2\pi/T$  periodic functions  $\hat{a}_n$ .

### V. PREPROCESSING WITH FILTERS

When  $\mathbf{\Phi}_L(\omega)$  does not have full column rank for all  $\omega \in [-\pi/T, \pi/T]$  and if the bandwidth of the LPF can not be increased, an interesting question is whether we can preprocess  $x$  before lowpass filtering in order to ensure that it can be recovered from the LPF output. In this and in the next section we consider two types of preprocessing: using a bank of filters, and using a bank of mixers (modulators), respectively.

Suppose we allow preprocessing of  $x$  with a set of  $N$  filters, as in Fig. 3. The question is whether we can choose the filters  $g_n$  in the figure so that  $x$  can be recovered from the outputs  $y_n$  of each of the branches under more mild conditions than those developed in Section III.

<sup>2</sup>In subsequent sections, we will only discuss the case where  $L$  is odd. The necessary changes for the case of  $L$  being even are obvious.

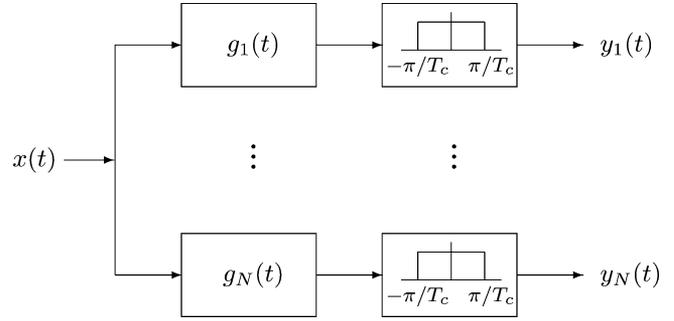


Fig. 3. Preprocessing of  $x(t)$  by a bank of  $N$  LTI filters.

Let  $\hat{\mathbf{y}}, \hat{\mathbf{g}}$  be the length- $N$  vectors with  $n$ th elements given by  $\hat{y}_n, \hat{g}_n$ . Then we can immediately verify that

$$\hat{\mathbf{y}}(\omega) = \hat{\mathbf{g}}(\omega) \hat{\boldsymbol{\phi}}^T(\omega) \hat{\mathbf{a}}(e^{j\omega T}), \quad \omega \in \left[-\frac{\pi}{T_c}, \frac{\pi}{T_c}\right]. \quad (20)$$

Clearly,  $\hat{\mathbf{a}}$  cannot be recovered from this set of equations as all the equations are linearly dependent (they are all multiples of each other). Thus, although we have  $N$  equations, only one of them provides independent information on  $\hat{\mathbf{a}}$ . We can, as before, use the periodicity of  $\hat{\mathbf{a}}$  if  $T_c$  is small enough. Following the same reasoning as in Section IV, assuming that  $\pi/T_c \geq L\pi/T$ , we can create  $L - 1$  new measurements using the same unknowns  $\hat{\mathbf{a}}$  by considering  $\hat{\mathbf{y}}(\omega)$  for different frequencies  $\omega + k2\pi/T$ . In this case though it is obvious that the prefiltering does not help, since only one equation can be used from the set of  $N$  (20) for each frequency. In other words, all the branches in Fig. 3 provide the same information. Following the same reasoning as in the previous section, the resulting equation is the same up to multiplication by  $\hat{g}_n$  for one index  $1 \leq n \leq N$ . Therefore, the recovery conditions reduce to the same ones as before, and having  $N$  branches does not improve our ability to recover  $x$ .

### VI. PREPROCESSING WITH MIXERS

We now consider a different approach, which as we will see leads to greater benefit. In this strategy, instead of using filters in each branch, we use periodic mixing functions  $p_n$ . Each sequence is assumed to be periodic with period equal to<sup>3</sup>  $T_p = T$ . By choosing the mixing functions appropriately, we can increase the class of functions that can be recovered from the low-pass filtered outputs.

#### A. Single Channel

Let us begin with the case of a single mixing function, as in Fig. 4. Since  $p$  is assumed to be periodic with period  $T$ , it can be written as a Fourier series

$$p(t) = \sum_{k \in \mathbb{Z}} b_k e^{j2\pi kt/T} \quad (21)$$

<sup>3</sup>Note, that we can also choose  $T_p = T/r$  for an integer  $r$ . However, for simplicity we restrict attention to the case  $r = 1$ .

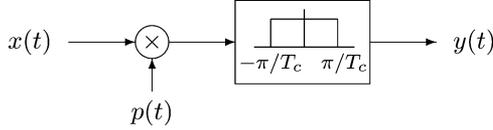


Fig. 4. Mixing prior to lowpass filtering of  $x(t)$ .

where

$$b_k = \frac{1}{T} \int_{-T/2}^{T/2} p(t) e^{-j2\pi kt/T} dt, \quad k \in \mathbb{Z} \quad (22)$$

are the Fourier coefficients of  $p$ . The sum (21) is assumed to converge in  $L^2$  which implies that the sequence  $\{b_k\}_{k \in \mathbb{Z}}$  is an element of  $\ell^2$ . The output  $y = P_{\pi/T_c}(px)$  of the LPF is then given in the frequency domain<sup>4</sup> by

$$\hat{y}(\omega) = \sum_{k \in \mathbb{Z}} b_k \hat{x}\left(\omega - k \frac{2\pi}{T}\right), \quad \omega \in \left[-\frac{\pi}{T_c}, \frac{\pi}{T_c}\right]. \quad (23)$$

Using (16) and the fact that  $\hat{a}_n(e^{j\omega T})$  is  $2\pi/T$ -periodic, (23) can be written as

$$\hat{y}(\omega) = \sum_{n=1}^N \hat{a}_n(e^{j\omega T}) \sum_{k \in \mathbb{Z}} b_k \hat{\phi}_n\left(\omega - k \frac{2\pi}{T}\right), \quad (24)$$

for  $\omega \in [-\pi/T_c, \pi/T_c]$ . Defining

$$\hat{\gamma}_n(\omega) := \sum_{k \in \mathbb{Z}} b_k \hat{\phi}_n\left(\omega - k \frac{2\pi}{T}\right), \quad 1 \leq n \leq N \quad (25)$$

and denoting by  $\hat{\gamma}$  the vector whose  $n$ th element is  $\hat{\gamma}_n$ , we express (24) as

$$\hat{y}(\omega) = \hat{\gamma}^T(\omega) \hat{\mathbf{a}}(e^{j\omega T}), \quad \omega \in \left[-\frac{\pi}{T_c}, \frac{\pi}{T_c}\right]. \quad (26)$$

Equation (26) is similar to (17) with  $\hat{\gamma}$  replacing  $\hat{\phi}$ . Therefore, as in the case in which no preprocessing took place (cf. Section IV), we can create  $L-1$  additional equations by evaluating  $\hat{y}(\omega)$  at frequencies  $\omega + 2k\pi/T$  as long as  $\pi/T_c \geq L\pi/T$ . This yields the system of equations

$$\hat{\mathbf{y}}(\omega) = \mathbf{\Gamma}_L(\omega) \hat{\mathbf{a}}(e^{j\omega T}), \quad \omega \in \left[-\frac{\pi}{T}, \frac{\pi}{T}\right] \quad (27)$$

where  $\hat{\mathbf{y}}$  and  $\hat{\mathbf{a}}$  are defined as in (10) and

$$\mathbf{\Gamma}_L(\omega) = \begin{bmatrix} \hat{\gamma}_1(\omega + L_0 \frac{2\pi}{T}) & \dots & \hat{\gamma}_N(\omega + L_0 \frac{2\pi}{T}) \\ \vdots & & \vdots \\ \hat{\gamma}_1(\omega) & \dots & \hat{\gamma}_N(\omega) \\ \vdots & & \vdots \\ \hat{\gamma}_1(\omega - L_0 \frac{2\pi}{T}) & \dots & \hat{\gamma}_N(\omega - L_0 \frac{2\pi}{T}) \end{bmatrix}.$$

Consequently,  $\hat{\mathbf{a}}$  can be recovered from the given measurements as long as the matrix  $\mathbf{\Gamma}_L(\omega)$  has full column rank for all  $\omega \in$

<sup>4</sup>Note that a periodic sampling with sampling period  $T$  of the signal  $x$  constitutes a special case of multiplying  $x$  with a periodic sequence [18]. In this case, the coefficients will have the special form  $b_k = e^{-jk\alpha 2\pi/T}$  where  $\alpha$  is an arbitrary delay.

$[-\pi/T, \pi/T]$ . To this end it is necessary that  $\pi/T_c \geq N\pi/T$ , i.e., that  $L \geq N$ .

Due to the mixing of the signal, the coefficient matrix  $\mathbf{\Phi}_L(\omega)$  in (18) is changed to  $\mathbf{\Gamma}_L(\omega)$  in (27). This new coefficient matrix is constructed out of the new generators  $\{\hat{\gamma}_n\}_{n=1}^N$  in exactly the same way as  $\mathbf{\Phi}_L(\omega)$  is constructed from the original generators  $\{\phi_n\}_{n=1}^N$ . Equation (25) shows that the Fourier transform  $\hat{\gamma}_n$  of each new generator lies in a shift invariant space

$$\mathcal{S}_{2\pi/T}(\hat{\phi}_n) = \text{span}\{S_{k2\pi/T} \hat{\phi}_n : k \in \mathbb{Z}\}$$

spanned by shifts of  $\hat{\phi}_n$ . The coefficients  $\{b_k\}_{k \in \mathbb{Z}}$  of the mixing sequence are then the ‘‘coordinates’’ of  $\hat{\gamma}_n$  in  $\mathcal{S}_{2\pi/T}(\hat{\phi}_n)$ .

We now show that the invertibility condition of  $\mathbf{\Gamma}_L(\omega)$  is in general easier to satisfy than the analogous condition on the matrix  $\mathbf{\Phi}_L(\omega)$  of (10). To this end, we write  $\mathbf{\Gamma}_L(\omega)$  as

$$\mathbf{\Gamma}_L(\omega) = \mathbf{B}_L \mathbf{\Phi}(\omega) \quad (28)$$

where  $\mathbf{\Phi}(\omega)$  denotes the matrix consisting of  $N$  columns and infinitely many rows  $\hat{\phi}^T(\omega + k2\pi/T)$  with  $k \in \mathbb{Z}$ . Note that  $\mathbf{\Phi}(\omega)$  has the form (10) with  $L \rightarrow \infty$ , i.e.,  $\mathbf{\Phi}(\omega) = \mathbf{\Phi}_\infty(\omega)$ . The matrix  $\mathbf{B}_L$  with  $L = 2L_0 + 1$  rows and infinite columns contains the Fourier coefficients  $\{b_k\}_{k \in \mathbb{Z}}$  of the mixing sequence (21) and is given by

$$\mathbf{B}_L = \begin{bmatrix} \dots & b_{L_0-1} & b_{L_0} & b_{L_0+1} & \dots \\ & & \vdots & & \\ \dots & b_0 & b_1 & b_2 & \dots \\ \dots & b_{-1} & b_0 & b_1 & \dots \\ \dots & b_{-2} & b_{-1} & b_0 & \dots \\ & & \vdots & & \\ \dots & b_{-L_0-1} & b_{-L_0} & b_{-L_0+1} & \dots \end{bmatrix}. \quad (29)$$

Representation (28) follows immediately from the relation  $\hat{\gamma}_n(\omega - \ell 2\pi/T) = \sum_{k \in \mathbb{Z}} b_{k-\ell} \hat{\phi}_n(\omega - k2\pi/T)$  for the entries of the matrix  $\mathbf{\Gamma}_L(\omega)$ .

The Grammian  $\mathbf{M}_\phi(\omega)$  of the generators  $\phi$ , defined in (4), may be written as  $\mathbf{M}_\phi(\omega) = \mathbf{\Phi}^*(\omega) \mathbf{\Phi}(\omega)$ . Therefore, under our assumption (3) on the generators,  $\mathbf{\Phi}(\omega)$  has full column rank for all  $\omega \in [-\pi/T, \pi/T]$ . The question then is whether we can choose the sequence  $\{b_k\}_{k \in \mathbb{Z}} \in \ell^2$ , and consequently the function  $p$ , so that  $\mathbf{B}_L \mathbf{\Phi}(\omega)$  has full-column rank i.e., such that the matrix  $\mathbf{\Gamma}_L^*(\omega) \mathbf{\Gamma}_L(\omega) = \mathbf{\Phi}^*(\omega) \mathbf{B}_L^* \mathbf{B}_L \mathbf{\Phi}(\omega)$  is invertible for all  $\omega \in [-\pi/T, \pi/T]$ .

If we choose the mixing sequence  $p(t) \equiv 1$  then  $b_0 = 1$  and  $b_k = 0$  for all  $k \neq 0$ . Consequently  $\mathbf{B}_L \mathbf{\Phi}(\omega)$  is comprised of the first  $L$  rows of  $\mathbf{\Phi}(\omega)$ , so that  $\mathbf{\Gamma}_L(\omega) = \mathbf{\Phi}_L(\omega)$ . However, by allowing for general sequences  $\{b_k\}_{k \in \mathbb{Z}}$ , we have more freedom in choosing  $\mathbf{B}_L$  such that the product  $\mathbf{B}_L \mathbf{\Phi}(\omega)$  may have full column-rank, even if  $\mathbf{\Phi}_L(\omega)$  does not.

We next give a simple example which demonstrates that preprocessing by an appropriate mixing function can enable the recovery of the signal.

*Example 3:* We continue Example 1 with the single generator  $\phi_1$  given by (7). Here we assume that the parameter  $D$  satisfies the relation  $1 < D/T < 3/2$  and that the cutoff frequency of the lowpass filter is  $\pi/T_c = \pi/T$ . In this case, recovery of  $x$  from its lowpass component  $y = P_{\pi/T_c} x$  is not possible, as discussed

in Example 1. However, we will show that there exist mixing functions  $p$  so that  $x$  can be recovered from  $y = P_{\pi/T_c}(px)$ .

One possible mixing function is

$$p(t) = 1 + 2 \sin\left(2\pi \frac{t}{T}\right)$$

whose Fourier coefficients (22) are given by  $b_{-1} = -j$ ,  $b_0 = 1$ ,  $b_1 = j$ , and  $b_k = 0$  for all  $|k| \geq 2$ . With this choice, the "new generator" (25) becomes

$$\hat{\gamma}_1(\omega) = \frac{\sin(\omega D)}{\omega D} + j \left[ \frac{\sin(\omega D - 2\pi \frac{D}{T})}{\omega D - 2\pi \frac{D}{T}} - \frac{\sin(\omega D + 2\pi \frac{D}{T})}{\omega D + 2\pi \frac{D}{T}} \right].$$

Since  $\pi/T_c = \pi/T$ , the matrix  $\Gamma_L(\omega)$  reduces to the scalar  $\hat{\gamma}_1(\omega)$  and we have to show that  $0 < |\hat{\gamma}_1(\omega)| < \infty$  for all  $\omega \in [-\pi/T, \pi/T]$ . The upper bound is trivial; for the lower bound, it is sufficient to show that the real and imaginary part of  $\hat{\gamma}_1$  have no common zero in  $[-\pi/T, \pi/T]$ . This fact is easily verified by noticing that the only zeros of the real part of  $\hat{\gamma}_1(\omega)$  are at  $\omega_1 = \pi/D$  and  $\omega_2 = -\pi/D$ . Evaluating the imaginary part  $\Im\{\hat{\gamma}_1\}$  of  $\hat{\gamma}_1$  at these zeros gives

$$|\Im\{\hat{\gamma}_1(\omega_{1,2})\}| = \frac{1}{2\pi} \left| \frac{\sin\left(\frac{2\pi D}{T}\right)}{\left(\frac{D}{T}\right)^2 - \frac{1}{4}} \right|$$

which is nonzero under the assumption made on  $D/T$ .

The general question whether for a given set  $\phi = \{\phi_1, \dots, \phi_N\}$  of generators there exists a matrix  $\mathbf{B}_L$  such that (28) is invertible for all  $\omega \in [-\pi/T, \pi/T]$ , or under what conditions on the generators  $\phi$  such a matrix can be found seems to be an open and nontrivial question. The major difficulty is that according to (28), we look for a constant (independent of  $\omega$ ) matrix  $\mathbf{B}_L$  such that  $\mathbf{B}_L \Phi(\omega)$  has full column rank for all  $\omega \in [-\pi/T, \pi/T]$ . Moreover, the matrix  $\mathbf{B}_L$  has to be of the particular form (29) with a sequence  $\{b_k\}_{k \in \mathbb{Z}} \in \ell^2$ .

The next example characterizes a class of generators for which a simple (trivial) mixing sequence always exist.

*Example 4 (Generators With Compact Support):* Consider the case of a single generator ( $N = 1$ ) and assume that  $\pi/T_c = \pi/T$ , i.e.,  $L = N = 1$ . Our problem then reduces to finding a function  $\hat{\gamma}_1 \in \mathcal{S}_{2\pi/T}(\hat{\phi}_1)$  such that  $\hat{\gamma}_1(\omega) \neq 0$  for all  $\omega \in [-\pi/T, \pi/T]$ .

We treat the special case of a generator  $\phi_1$  with finite support of the form  $[-D, D]$  for some  $D \in \mathbb{R}$ , i.e., we assume that  $\phi_1(t) = 0$  for all  $t \notin [-D, D]$ . This means that its Fourier transform  $\hat{\phi}_1$  is an element of the Paley–Wiener space  $\widehat{PW}(D)$  and so are all linear combinations of the shifts  $S_{2\pi/T}\hat{\phi}_1$ . It follows that  $\mathcal{S}(\hat{\phi}_1) \subset \widehat{PW}(D)$ .

Let now  $\hat{\gamma} \in \mathcal{S}(\hat{\phi}_1)$  be arbitrary and let  $\{\omega_k\}_{k \in \mathbb{Z}}$  be the ordered sequence of real zeros of  $\hat{\gamma}$  with  $\omega_n \leq \omega_{n+1}$ . Then a theorem of Walker [19] states that

$$\sup_{n \in \mathbb{Z}} |\omega_{n+1} - \omega_n| > \frac{\pi}{D}.$$

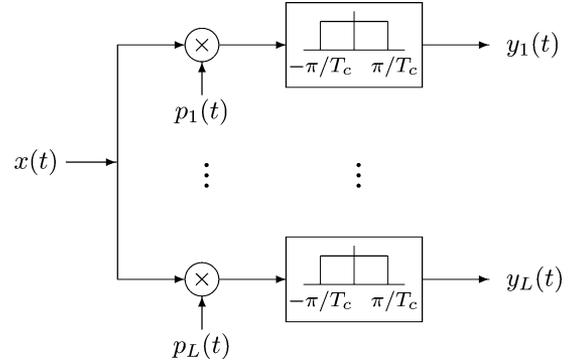


Fig. 5. Bank of mixing functions.

Thus, there exists at least one interval of the real line of length  $\pi/D$  such that  $\hat{\gamma}$  has no zeros in this interval. Consequently, if  $\pi/D > 4\pi/T$  then there always exists a  $k_0 \in \mathbb{Z}$  such that

$$\hat{\gamma}\left(\omega - k_0 \frac{2\pi}{T}\right) \neq 0 \quad \text{for all } \omega \in \left[-\frac{\pi}{T}, \frac{\pi}{T}\right]. \quad (30)$$

This holds in particular for the generator  $\hat{\phi}_1$  itself.

We conclude that if the support of the generator  $\phi_1$  satisfies  $\text{supp}(\phi_1) < T/4$ , then there always exists a  $k_0 \in \mathbb{Z}$  such that  $\hat{\gamma}_1(\omega) = \hat{\phi}_1(\omega - k_0 2\pi/T) \neq 0$  for all  $\omega \in [-\pi/T, \pi/T]$ . The corresponding mixing sequence is given by  $b_{k_0} = 1$  and  $b_k = 0$  for all  $k \neq k_0$ .

### B. Multiple Channels

In the single channel case, it was necessary that the cutoff frequency  $\pi/T_c$  of the LPF is at least  $N$  times larger than the bandwidth of the desired signal  $\hat{\mathbf{a}}$  in order to be able to recover the signal. We will now show that using several channels can reduce the cutoff frequency  $\pi/T_c$  of the filter in each channel, from which the original signal  $x$  is still recoverable.

Suppose that we have  $L \geq N$  channels, where each channel uses a different mixing sequence, as in Fig. 5. Since  $L \geq N$ , we expect to be able to reduce the cutoff in each channel. We therefore consider the case in which  $T_c = T$ . The output  $y_\ell = P_{\pi/T}(p_\ell x)$  of the  $\ell$ th channel in the frequency domain is then equal to

$$\hat{y}_\ell(\omega) = \hat{\gamma}_\ell^T(\omega) \hat{\mathbf{a}}(e^{j\omega T}), \quad \omega \in \left[-\frac{\pi}{T}, \frac{\pi}{T}\right]$$

where  $\hat{\gamma}_\ell^T(\omega)$  is the vector with  $n$ th element

$$[\hat{\gamma}_\ell(\omega)]_n = \hat{\gamma}_n^\ell(\omega) := \sum_{k \in \mathbb{Z}} b_k^\ell \hat{\phi}_n \left(\omega + k \frac{2\pi}{T}\right),$$

and  $\{b_k^\ell\}_{k \in \mathbb{Z}}$  are the Fourier coefficients associated with the  $\ell$ th sequence  $p_\ell$ . Defining by  $\hat{\mathbf{y}}(\omega)$  the vector with  $\ell$ th element  $\hat{y}_\ell(\omega)$  we conclude that

$$\hat{\mathbf{y}}(\omega) = \mathbf{\Gamma}_L(\omega) \hat{\mathbf{a}}(e^{j\omega T}), \quad \omega \in \left[-\frac{\pi}{T}, \frac{\pi}{T}\right]$$

where  $\mathbf{\Gamma}_L(\omega)$  is the matrix whose entry in the  $\ell$ th row and  $n$ th column is  $[\mathbf{\Gamma}(\omega)]_{\ell,n} = \hat{\gamma}_n^\ell(\omega)$ . Now, all we need is to choose the  $L$  sequences  $\{b_k^\ell\}_{k \in \mathbb{Z}} \in \ell^2$  such that  $\mathbf{\Gamma}_L(\omega)$  has full column rank. More specifically, as before we can write

$$\mathbf{\Gamma}_L(\omega) = \mathbf{B}_L \mathbf{\Phi}(\omega) \quad (31)$$

where  $\mathbf{B}_L$  is a matrix with  $L$  rows and infinitely many columns whose  $\ell$ th row is given by the coefficient sequence  $\{b_k^\ell\}_{k \in \mathbb{Z}}$ , i.e.,

$$\mathbf{B}_L = \begin{bmatrix} \dots & b_{-2}^1 & b_{-1}^1 & b_0^1 & b_1^1 & b_2^1 & \dots \\ \dots & b_{-2}^2 & b_{-1}^2 & b_0^2 & b_1^2 & b_2^2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & b_{-2}^L & b_{-1}^L & b_0^L & b_1^L & b_2^L & \dots \end{bmatrix}.$$

By our assumption  $\mathbf{\Phi}(\omega)$  has full column rank and so it remains to choose  $\mathbf{B}_L$  such that  $\mathbf{\Gamma}_L(\omega)$  is invertible for every  $\omega \in [-\pi/T, \pi/T]$ .

It should be noted that we used the same notation as in the previous subsection although the definition of the particular matrices and vectors differ slightly in both cases. Nevertheless, the formal approach is very similar. In the previous subsection, we observed the output signal in different frequency channels  $1 \leq \ell \leq L$  whereas in this subsection the channels  $1 \leq \ell \leq L$  are characterized by different mixing sequences.<sup>5</sup>

As in the previous subsection, the general question whether for a given system  $\phi = \{\phi_1, \dots, \phi_N\}$  of generators there always exists an appropriate system of mixing sequences  $p = \{p_1, \dots, p_L\}$  such that  $\mathbf{\Gamma}_L(\omega)$  has full column rank for all frequencies  $\omega$  seems to be nontrivial. The formal difficulty lies in the fact that we look for a constant (independent of  $\omega$ ) matrix  $\mathbf{B}_L$  such that (31) has full column rank for each  $\omega \in [-\pi/T, \pi/T]$ . However, compared with the previous section, where only one mixing sequence was applied, the problem of finding an appropriate matrix  $\mathbf{B}_L$  becomes simpler: In the former case  $\mathbf{B}_L$  has to have the special (diagonal) form (29), whereas here its entries can be chosen (almost) arbitrarily. The sequences  $\{b_k^\ell\}_{k \in \mathbb{Z}}$  only have to be in  $\ell^2$ .

A special choice of periodic functions that are easy to implement in practice are binary sequences. This example was studied in [14] in the context of sparse multiband sampling. More specifically,  $p_\ell$ ,  $1 \leq \ell \leq L$  are chosen to attain the values  $\pm 1$  over intervals of length  $T/M$  where  $M$  is a given integer. Formally,

$$p_\ell(t) = \alpha_n^\ell, \quad n \frac{T}{M} \leq t < (n+1) \frac{T}{M}, \quad 0 \leq n \leq M-1 \quad (32)$$

with  $\alpha_n^\ell \in \{+1, -1\}$ , and  $p_\ell(t+kT) = p_\ell(t)$  for every  $k \in \mathbb{Z}$ . In this case, we have

$$\begin{aligned} b_k^\ell &= \frac{1}{T} \int_0^T p_\ell(t) e^{-j2\pi/Tkt} dt \\ &= \frac{1}{T} \int_0^{T/M} \sum_{n=0}^{M-1} \alpha_n^\ell e^{-j2\pi/Tk(t+nT/M)} dt \\ &= \frac{1}{T} \sum_{n=0}^{M-1} \alpha_n^\ell e^{-j2\pi/Mnk} \int_0^{T/M} e^{-j2\pi/Tkt} dt. \end{aligned}$$

<sup>5</sup>In the first case, we perform ‘‘frequency multiplexing’’ whereas the second case resembles ‘‘code multiplexing’’.

Evaluating the integral gives

$$b_0^\ell = \frac{1}{M} \hat{\alpha}_0^\ell \quad \text{and} \quad b_k^\ell = \frac{1 - e^{-j\omega_0 k}}{j2\pi k} \hat{\alpha}_k^\ell, \quad k \neq 0$$

where  $\omega_0 = 2\pi/M$ , and  $\{\hat{\alpha}_k^\ell\}_{k \in \mathbb{Z}}$  denotes the discrete Fourier transform (DFT) of the sequence  $\{\alpha_n^\ell\}_{n=0}^{M-1}$ . Note that  $\{\hat{\alpha}_k^\ell\}_{k \in \mathbb{Z}}$  is  $M$ -periodic so that  $\hat{\alpha}_k = \hat{\alpha}_{k+M}$ .

With these mixing sequences, the infinite matrix  $\mathbf{B}_L$  can be written as

$$\mathbf{B}_L = \mathbf{Q} \mathbf{F}^* \mathbf{W} \quad (33)$$

where  $\mathbf{Q}$  is a matrix with  $M$  columns and  $L$  rows, whose  $\ell$ th row is given by the sequence  $\{\hat{\alpha}_k^\ell\}_{k \in \mathbb{Z}}$ ,  $\mathbf{F}$  is the  $M \times M$  Fourier matrix, and  $\mathbf{W}$  is a matrix with  $M$  rows and infinitely many columns consisting of block diagonal matrices of size  $M \times M$  whose diagonal values are given by the sequence  $\{w_k\}_{k \in \mathbb{Z}}$  defined by  $w_0 = 1/M$  and  $w_k = (1 - \exp[-j\omega_0 k]) / (j2\pi k)$  for  $k \neq 0$ . Applying these binary mixing sequences, the problem is now to find a finite  $L \times N$  matrix  $\mathbf{Q}$  with values in  $\{+1, -1\}$  such that  $\mathbf{Q} \mathbf{F}^* \mathbf{W} \mathbf{\Phi}(\omega)$  has full column rank for every  $\omega \in [-\pi/T, \pi/T]$ .

The next example shows how to select  $\mathbf{Q}$  in the case of bandlimited generators.

*Example 5 (Bandlimited Generators):* We consider the case where each generator  $\phi_n$  is bandlimited to the interval  $[-K_0\pi/T, K_0\pi/T]$  for some  $K_0 \in \mathbb{N}$ , and  $N = 2K_0 + 1$ . In this case,  $\mathbf{\Phi}(\omega) = \mathbf{\Phi}_N(\omega)$  is essentially an  $N \times N$  matrix (all other entries are identically zero). This matrix is invertible for every  $\omega \in [-\pi/T, \pi/T]$  according to assumption (3).

We now apply  $L = N$  different mixing sequences  $\{p_\ell\}_{\ell=1}^L$  having the special structure (32), and choose  $M = N$ . According to (31) and (33) the matrix  $\mathbf{\Gamma}_L(\omega)$  then becomes

$$\mathbf{\Gamma}_L(\omega) = \mathbf{Q} \mathbf{F}^* \mathbf{W} \mathbf{\Phi}(\omega) \quad (34)$$

where  $\mathbf{Q} \mathbf{F}^*$  and  $\mathbf{W} \mathbf{\Phi}(\omega)$  are matrices of size  $N \times N$ . The matrix  $\mathbf{W} \mathbf{\Phi}(\omega)$  may be considered as the product of the invertible  $N \times N$  matrix  $\mathbf{\Phi}(\omega) = \mathbf{\Phi}_N(\omega)$  with an  $N \times N$  diagonal matrix consisting of the central diagonal matrix of  $\mathbf{W}$ , i.e.,

$$\mathbf{W} \mathbf{\Phi}(\omega) = \text{diag}(w_0, \dots, w_{N-1}) \mathbf{\Phi}_N(\omega).$$

Since this diagonal matrix is invertible also  $\mathbf{W} \mathbf{\Phi}_N(\omega)$  is invertible for every  $\omega \in [-\pi/T, \pi/T]$ . Therefore, using the fact that the Fourier matrix  $\mathbf{F}$  is invertible,  $\mathbf{\Gamma}_L(\omega)$  is invertible for each  $\omega \in [-\pi/T, \pi/T]$  if the values  $\{\alpha_n^\ell\}_{n=1}^N$  of the mixing sequences  $p_\ell$  are chosen such that  $\mathbf{Q}$  is invertible. This can be achieved by choosing  $\mathbf{Q}$  as a Hadamard matrix of order  $N$ . It is known that Hadamard matrices exist at least for all orders up to 667 [20].

In the previous example,  $\mathbf{\Phi}_N(\omega)$  was an  $N \times N$  invertible matrix for all  $\omega \in [-\pi/T, \pi/T]$ . According to Proposition 3 recovery of the signal  $x$  is therefore possible if the bandwidth of the LPF is larger than  $N\pi/T$ . However, the example shows that pre-processing of  $x$  by applying the binary sequences in  $L = N$  channels allows recovery of the signal already from its signal components in the frequency range  $[-\pi/T, \pi/T]$ .

For simplicity of the exposition, we assumed throughout this subsection that the bandwidth  $2\pi/T_c$  of the lowpass filter is

equal to the signal bandwidth  $2\pi/T$  and that the number of channels  $L$  is at least equal to the number of generators  $N$ . However, it is clear from the first subsection that in cases where  $L < N$ , recovery of the signal may still be possible if the bandwidth of the LPF is increased.

## VII. CONNECTION WITH SPARSE ANALOG SIGNALS

In this section we depart from the subspace assumption which prevailed until now. Instead, we would like to incorporate sparsity into the signal model  $x(t)$  of (1). To this end, we follow the model proposed in [9] to describe sparsity of analog signals in SI spaces. Specifically, we assume that only  $K$  out of the generators  $\phi_n(t)$  are active, so that at most  $K$  of the sequences  $a_n[k]$  have positive energy.

In [9], it was shown how such signals can be sampled and reconstructed from samples at a low rate of  $2K/T$ . The samples are obtained by pre-processing the signal  $x(t)$  with a set of  $2K$  sampling filters, whose outputs are uniformly sampled at a rate of  $1/T$ . Without the sparsity assumption, at least  $N$  sampling filters are needed where generally  $N$  is much larger than  $K$ . In contrast to this setup, here we are constrained to sample at the output of a LPF with given bandwidth. Thus, we no longer have the freedom to choose the sampling filters as we wish. Nonetheless, by exploiting the sparsity of the signal we expect to be able to reduce the bandwidth needed to recover  $x(t)$  of the form (1), or in turn, to reduce the number of branches needed when using a bank of modulators.

We have seen that the ability to recover  $x(t)$  depends on the left invertibility of the matrix  $\Phi_L(\omega)$  (or  $\Gamma_L(\omega)$ ). With appropriate definitions, our problem becomes that of recovering  $\hat{\mathbf{a}}(e^{j\omega T})$  from the linear set of (18) (with  $\Gamma_L(\omega)$  replacing  $\Phi_L(\omega)$  when preprocessing is used). Our definition of analog sparsity implies that at most  $K$  of the Fourier transforms  $\hat{a}_n(\omega)$  have nonzero energy. Therefore, the infinite set of vectors  $\{\hat{\mathbf{a}}(e^{j\omega T}), \omega \in [-\pi/T, \pi/T]\}$  share a joint sparsity pattern with at most  $K$  rows that are not zero. This in turn allows us to recover  $\{\hat{\mathbf{a}}(e^{j\omega T}), \omega \in [-\pi/T, \pi/T]\}$  from fewer measurements. Under appropriate conditions, it is sufficient that  $\hat{\mathbf{y}}(\omega)$  has length  $2K$ , which in general is much smaller than  $N$ . Thus, fewer measurements are needed with respect to the full model (1). The reduction in the number of measurements corresponds to choosing a smaller bandwidth of the LPF, or reducing the number of modulators.

In order to recover the sequences in practice, we rely on the separation idea advocated in [8]: we first determine the support set, namely the active generators. This can be done by solving a finite dimensional optimization problem under the condition that  $\Phi_L(\omega)$  (or  $\Gamma_L(\omega)$ ) are fixed in frequency up to a possible frequency-dependent normalization sequence. Recovery is then obtained by applying results regarding infinite measurement vector (IMV) models to our problem [8]. When  $\Phi_L(\omega)$  does not satisfy this constraint, we can still convert the problem to a finite dimensional optimization problem as long as the sequences  $a_k[n]$  are rich [10]. This implies that every finite set of vectors share the same frequency support. As our focus

here is not on the sparse setting, we do not describe here in detail how recovery is obtained. The interested reader is referred to [8]–[10] for more details.

The main point we wish to stress is that the ideas developed in this paper can also be used to treat the scenario of recovering a sparse SI signal from its lowpass content. The difference is that we can relax the requirement for invertibility of  $\Phi_L(\omega), \Gamma_L(\omega)$ . Instead, it is enough that these matrices satisfy the known conditions from the compressed sensing literature. This in turn allows in general reduction of the LPF bandwidth, or the number of modulators, in comparison with the nonsparse scenario.

## VIII. CONCLUSIONS AND OPEN PROBLEMS

This paper studied the possibility of recovering signals in SI spaces from their low frequency components. We developed necessary conditions on the minimal bandwidth of the LPF and sufficient conditions on the generators of the SI space such that recovery is possible. We also showed that proper pre-processing may facilitate the recovery, and allow to reduce the necessary bandwidth. Finally, we discussed how these ideas can be used to recover sparse SI signals from the output of a LPF.

An important open problem we leave to future work is the characterization of the class of generators for which the proposed pre-processing scheme can (or cannot) be applied. To this end, the following question has to be answered. We formulate it only for the most simple case of one generator (cf. also the discussion in Example 4).

*Problem 1:* Let  $\phi \in L^2$  be an arbitrary function with Fourier transform  $\hat{\phi}$  whose Grammian satisfies (3). Consider the shift-invariant space spanned by  $\hat{\phi}$ , i.e.,

$$\mathcal{S}_{2\pi/T}(\hat{\phi}) = \text{span} \left\{ S_{k2\pi/T} \hat{\phi} : k \in \mathbb{Z} \right\}.$$

For which functions  $\phi \in L^2$  does there exist a function  $\hat{\gamma} \in \mathcal{S}_{2\pi/T}(\hat{\phi})$  such that  $\hat{\gamma}(\omega) \neq 0$  for all  $\omega \in [-\pi/T, \pi/T]$ .

The interesting case is when every function  $\hat{\phi}(\omega - k2\pi/T)$ ,  $k \in \mathbb{Z}$  has at least one zero in the interval  $[-\pi/T, \pi/T]$ . Then the question is whether it is still possible to find a linear combination of these functions which has no zero in  $[-\pi/T, \pi/T]$ .

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