ANCIENT AND CLASSIC WORLS

Until the 5-th Century

GEOMETRY

Collected and edited by Prof. Zvi Kam,
Weizmann Institute, Israel

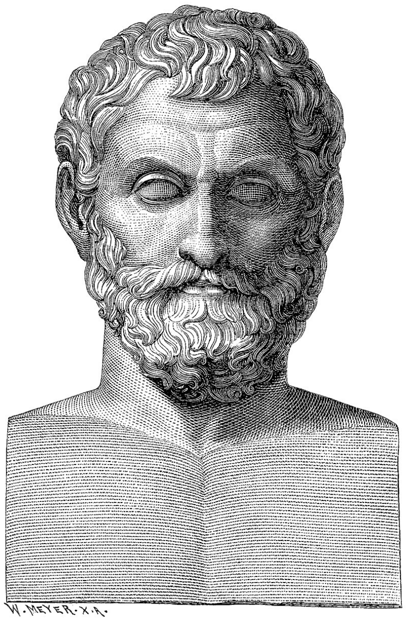
Geometry served as an important tool in the ancient world:
It was required for measurements of land, for buildings and for positioning stars and planets motion.
Geometry provided a sense of symmetry, aesthetics and order of the universe.

Geometry was the first mathematical field that was organized based on a set of axioms with systematic use of logics, building a layered structure of proofs for theorems and lemmas (Euclid).

Last, geometric constructions were used for calculations, replacing numerical operations, especially in Greek science. A number of geometrical laws provided such ability to perform multiplications, squares and square roots and ratios by measuring lengths.

1900-1600 BC Sumer & Babylon in Mesopotamia – Tables of wedges of right angle triangles, e.g. (3,4,5) (5,12,13), used as multiplication tables.

Ahmes in Egypt 1680-1620 BC – writes the Rhind papyrus. Division algorithm. 87 mathematical problems, including solution of unknowns from equations, and formulae for volumes.



Thales of Miletus 624–547 BC

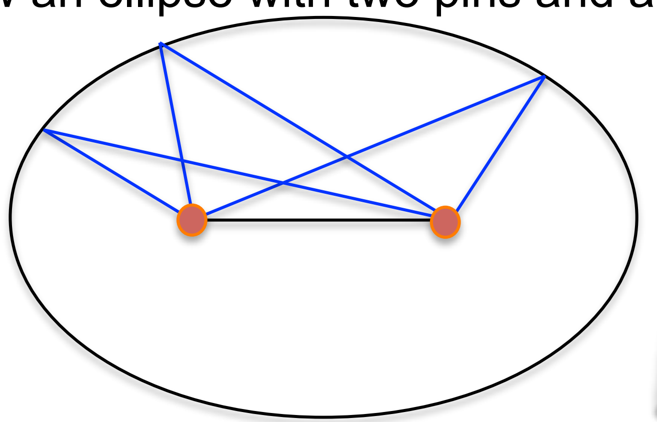
Triangle geometry.

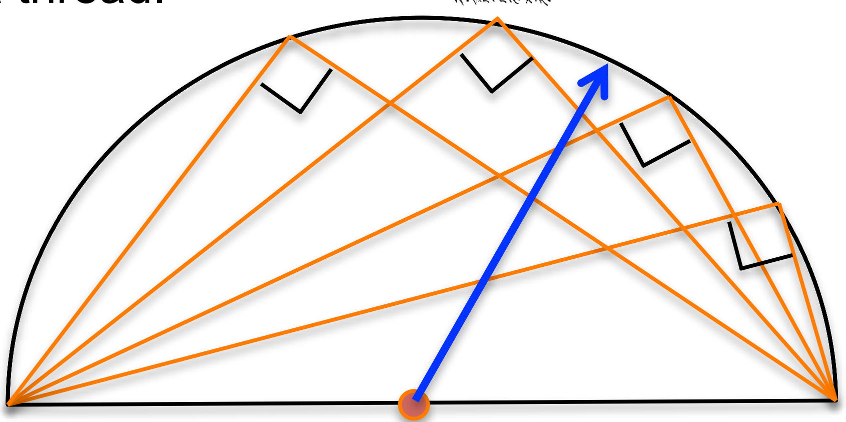
A circle is the assembly of all points of equal distance (=radius) from the center.

Triangle with one wedge as the diameter of a circle, and opposite edge on its perimeter is a right angle triangle.

Ellipse is the assembly of points with equal sum of distances from Two points (the ellipse foci):

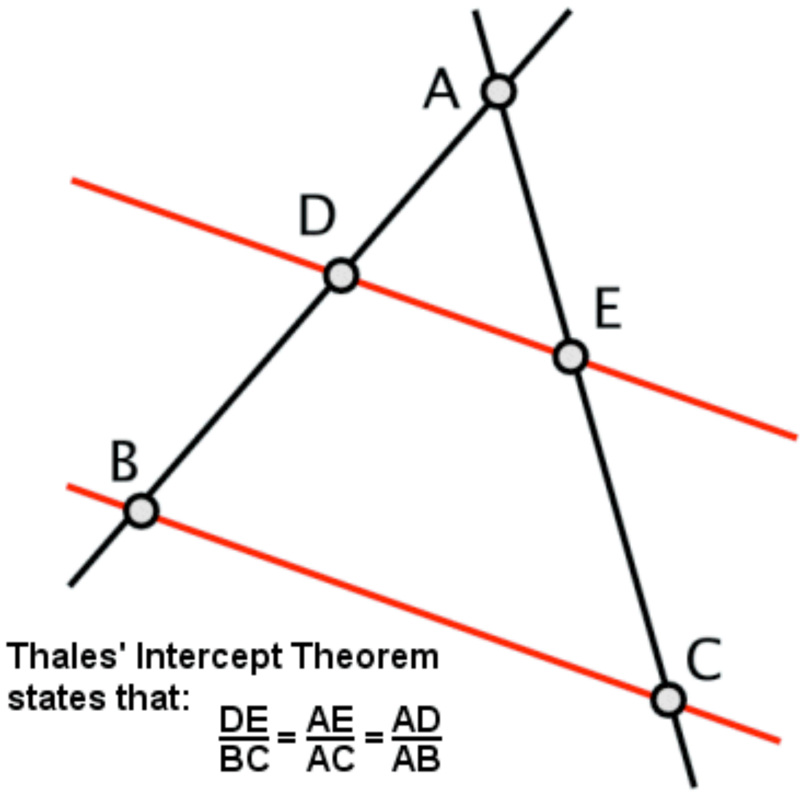
EXERCISE: Draw an ellipse with two pins and a thread.



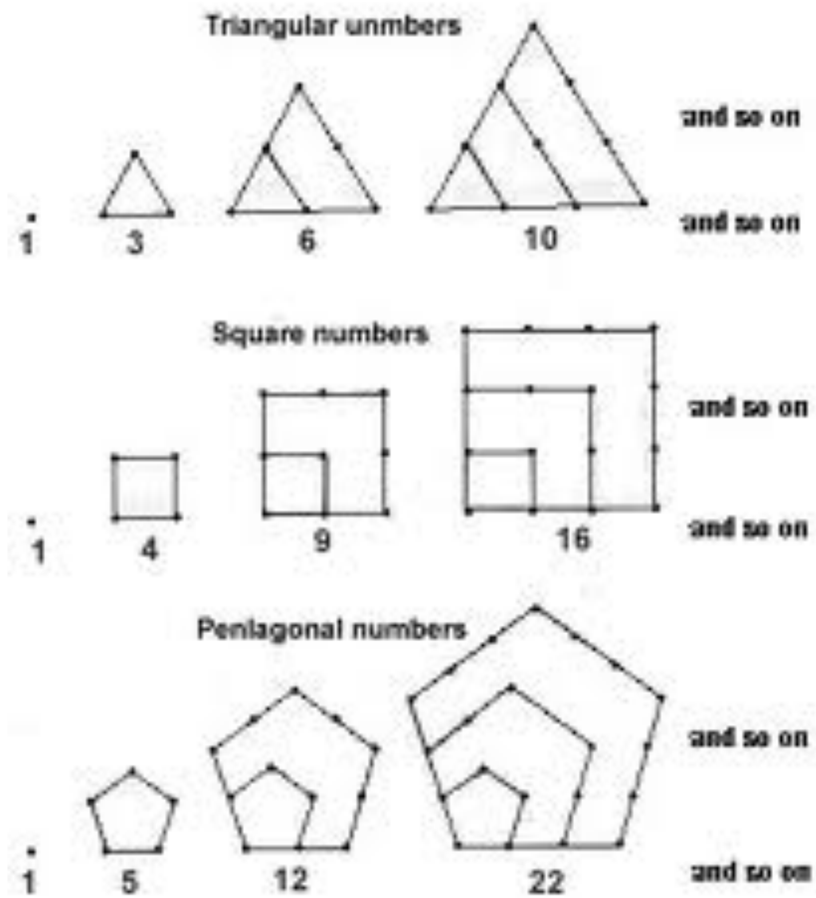


Thales intercept theorem

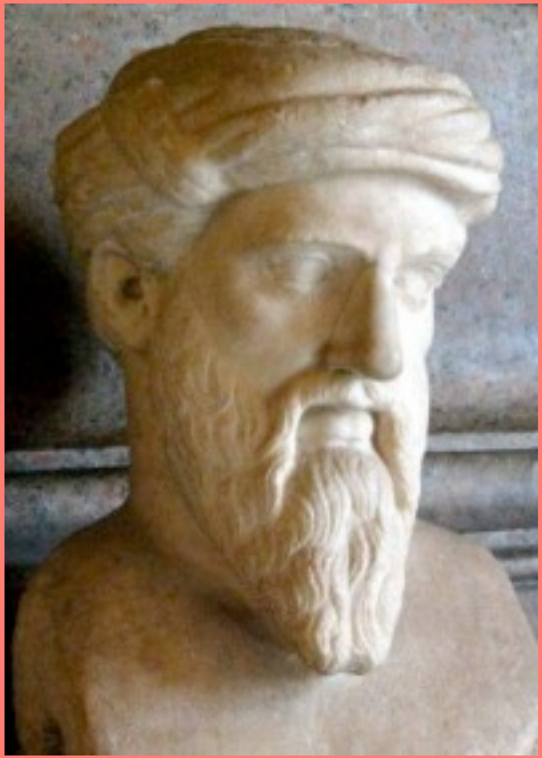
The ratio between wedges of similar triangles: if DE and BC are parallel lines (therefore the triangles ABC and ADE have the same angles), then:
DE/BC = AE/AC = AD/AB

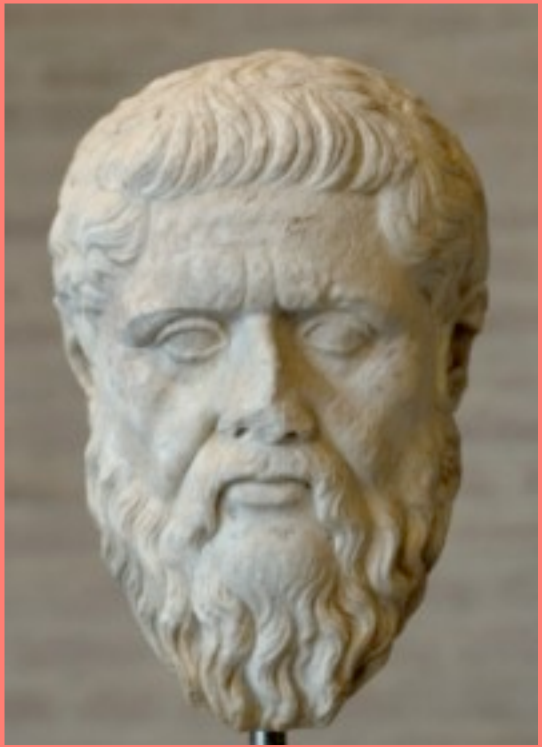


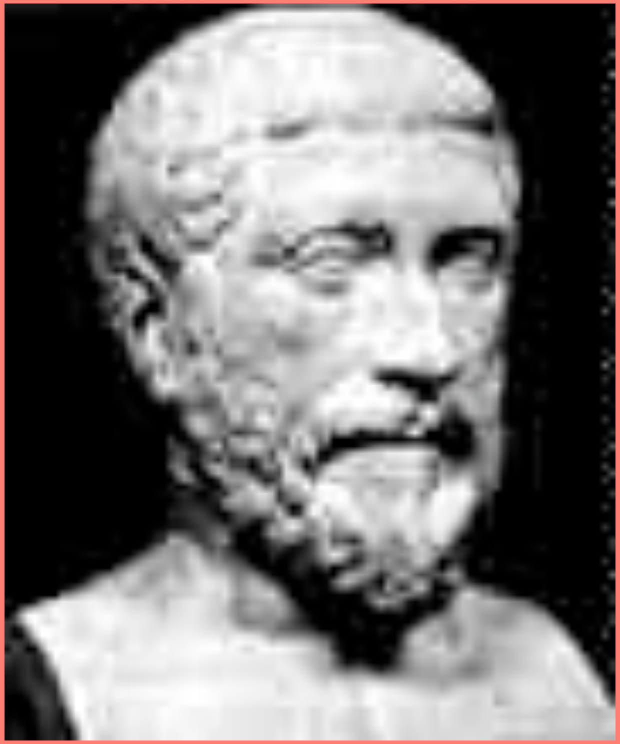
Similarity of geometrical shapes



Pythagoras of Samos 569-475 BC



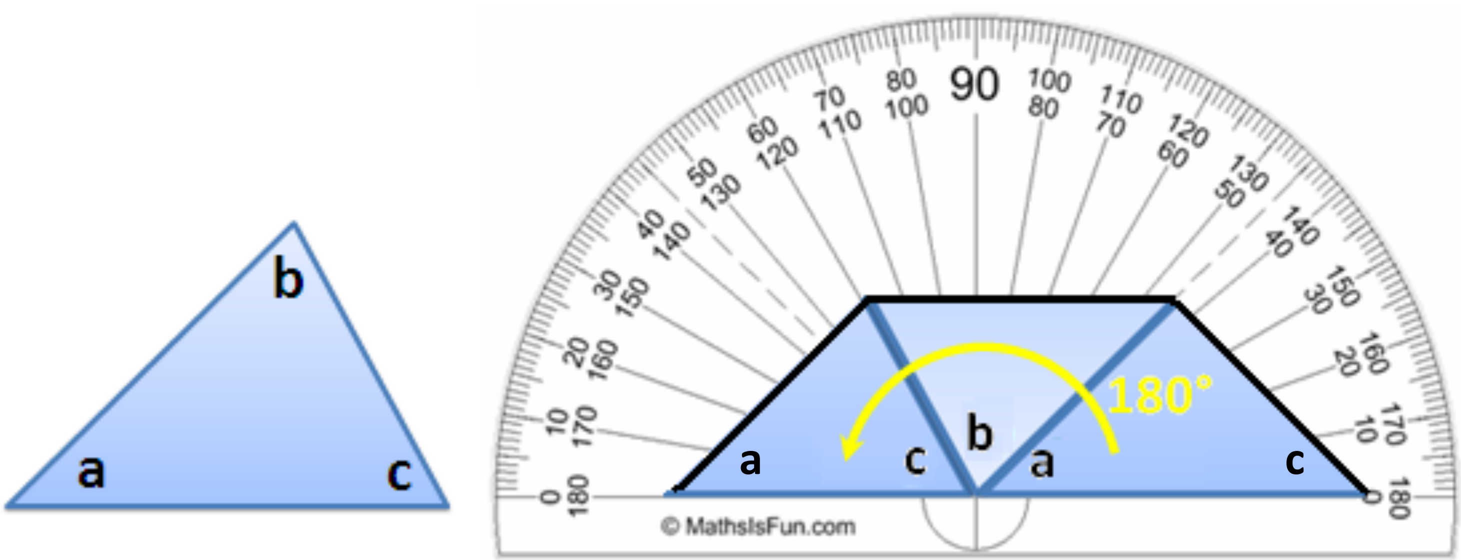




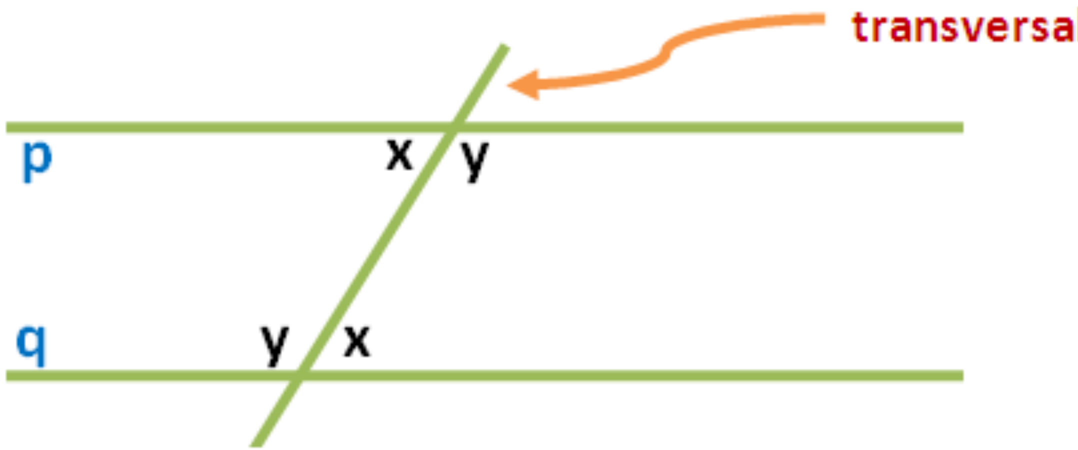
Pythagoras was a student of Thales. He believed in harmony and beauty of integers and rational numbers. He deduces geometrical laws based on elementary principles – <u>Geometrical proofs</u>.
The most famous Pythagoras law for right-angle triangle: $a^2+b^2=c^2$ (see following).

Not les known law of Pythagoras: the sum of angles in all triangles is 180°

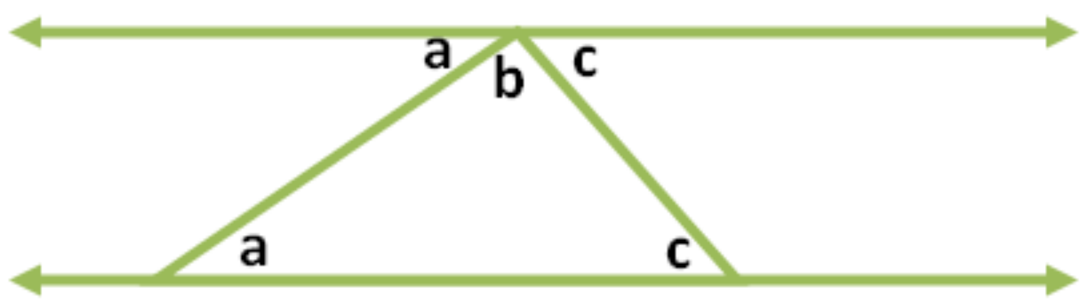
A simple visual proof:



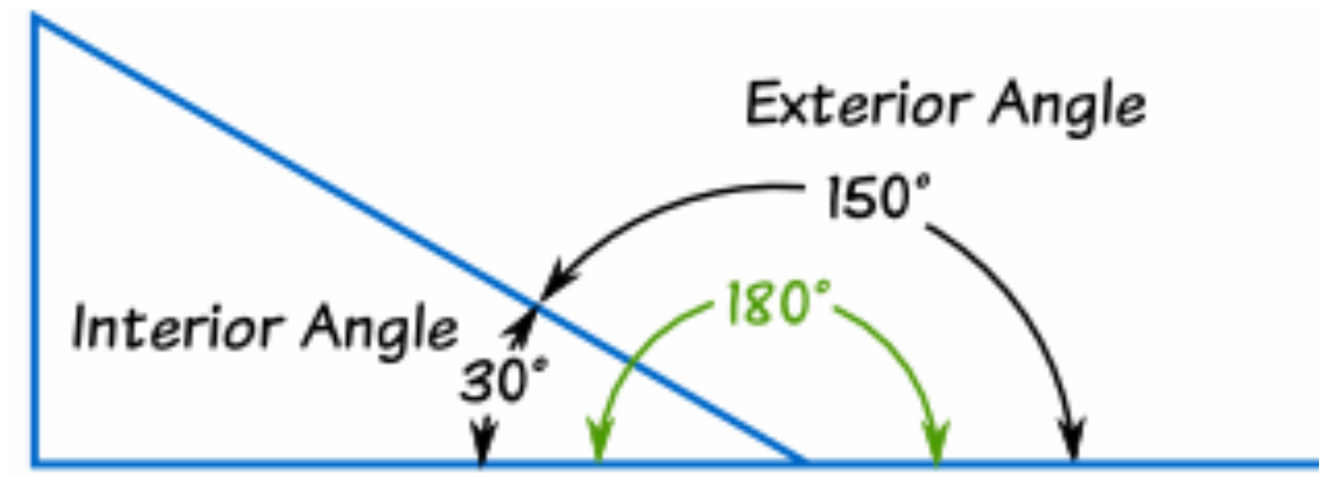
More formally: applying Euclid's 5th axiom for equality of interchanging angles of a transversal line with two parallels, p, q:

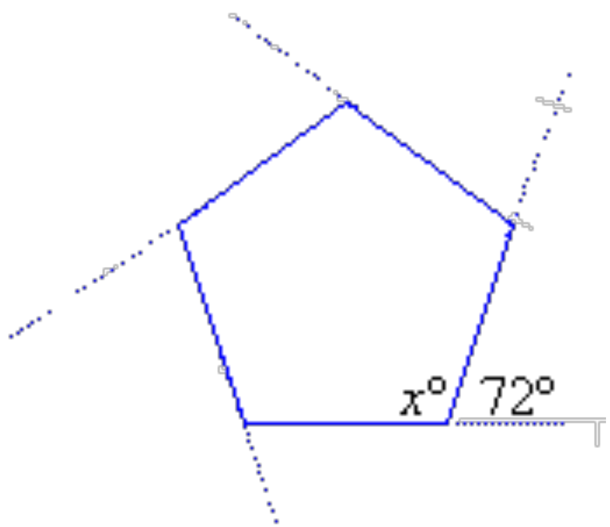


Drawing the triangle and a line parallel to its base passing through the opposite edge: interchanging angles a and c equal, and therefore $a+b+c=180^o$



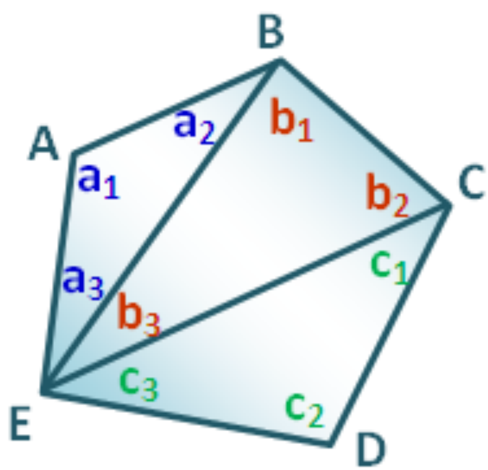
The sum of interior angles of a polygon with N wedges is 180*(N-2)
The sum of exterior angles is 360°
Why independent on N ? (The more wedges the smaller the exterior angles).





Proof: split the polygon into n-2 triangles, by joining one vertex to the n-3 opposite vertices. The sum of angles of all these triangles equals the sum of interior angles of the polygon.

$$\text{Pentagon Angle Sum} = \overset{A}{a_1} + \overbrace{(a_2 + b_1)}^{B} + \overbrace{(b_2 + c_1)}^{C} + \overset{D}{c_2} + \overbrace{(c_3 + a_3 + b_3)}^{E}.$$



$$= (a_1 + a_2 + a_3) + (b_1 + b_2 + b_3) + (c_1 + c_2 + c_3)$$

$$= 180° + 180° + 180°$$

$$= 540°$$

Aryabhata 500 AC - the Indian mathematician, describes trigonometric functions, Sin θ, cost θ
the cosine law extends Pythagoras law for any triangles

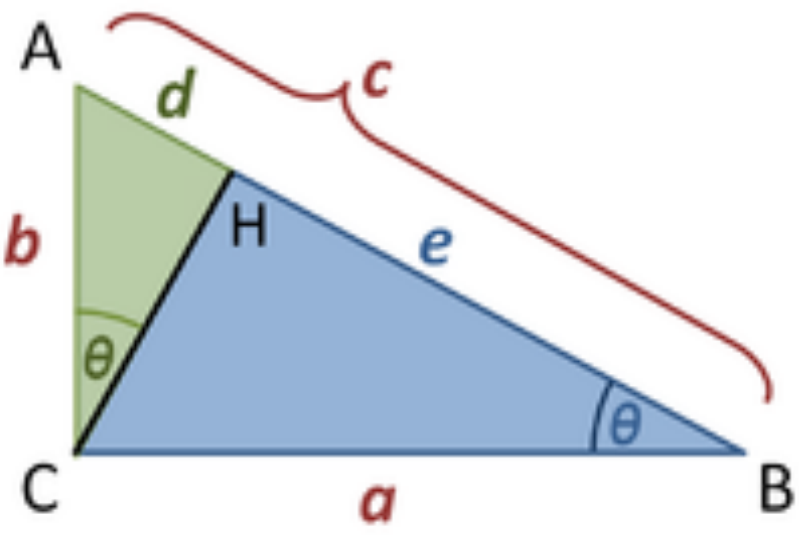
$$a^2+b^2-2ab\cos(\theta)=c^2$$

Pythagoras 550 BC for right angle triangle: $a^2+b^2=c^2$
And vise versa: if above, the triangle is right angled.

There is a large number of proofs for Pythagoras law.
Here are some

1. Based on ratio of wedges in similar triangles:

$$a/c=e/a \quad b/c=d/b$$
$$a^2=c*e \quad b^2=c*d$$
$$a^2+b^2=c*(e+d)=c^2$$



2. Euclid's proof (book 1 Proposition 47):

The two blue triangles have two equal wedges: a=AB, b=AC and equal angle in between: ABD =~FBC

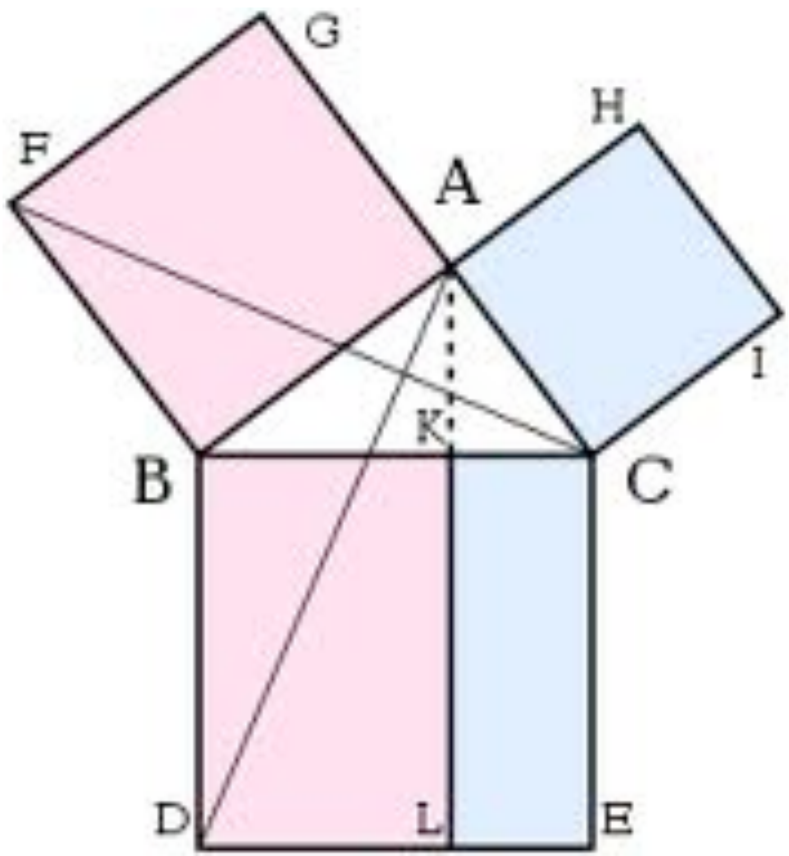
Area of ABD is half the area of BDLK since they have equal base BD and height BK

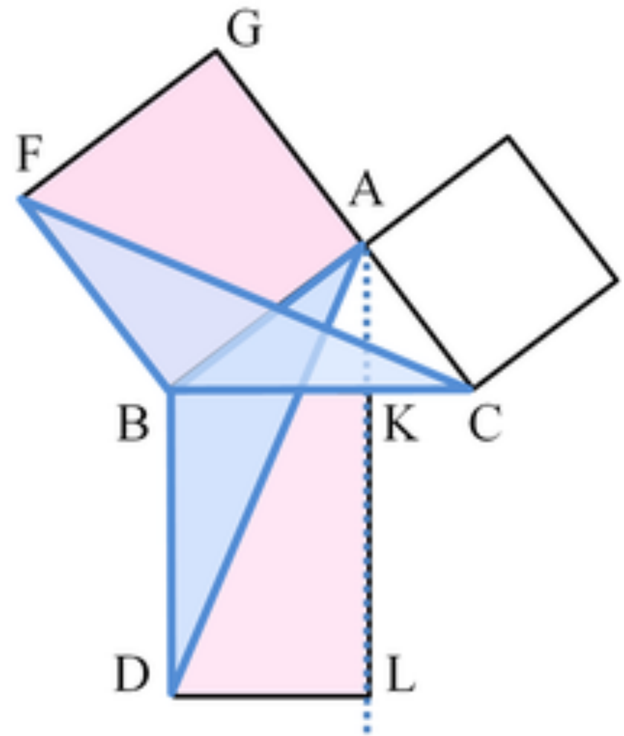
Also area of FBC is half the area BAGF, therefore: $BDLK = BAGF = AB^2 = a^2$

and also $CKLE = ACIH = AC^2 = b^2$

Therefore: $AB^2 + AC^2 = BDLK + CKLE = BC^2$

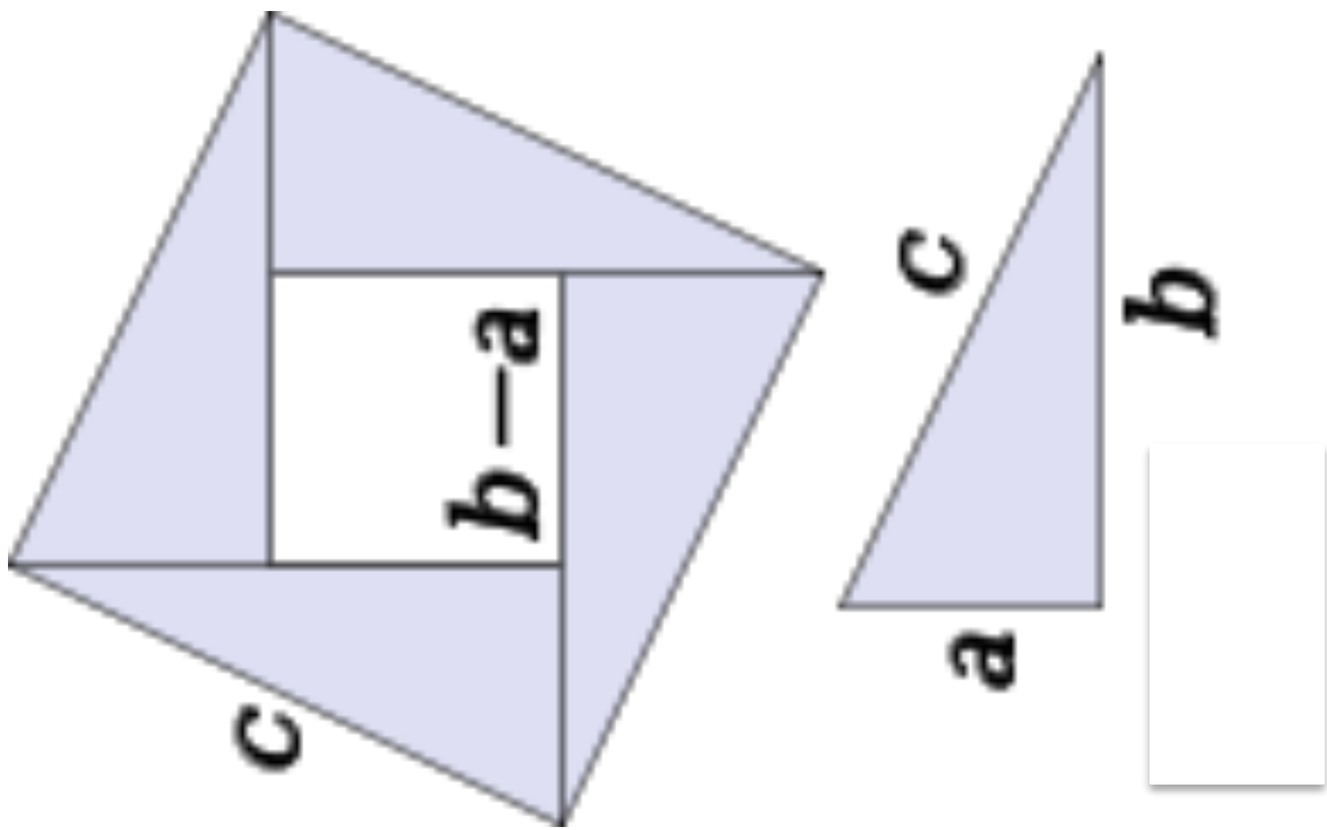
or: $a^2 + b^2 = c^2$



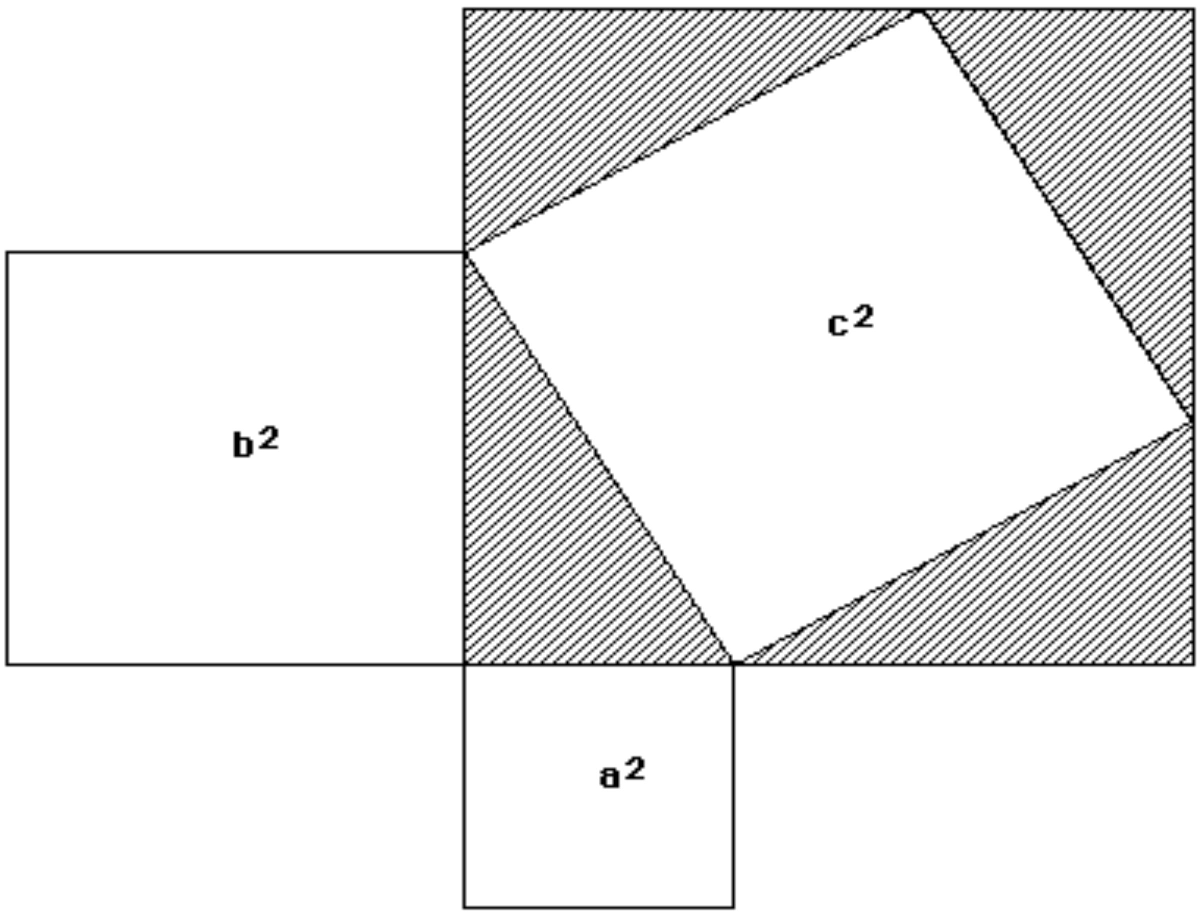


3. Algebraic proof: Area of the square = sum of its parts:

$$c^2 = (b - a)^2 + 4 * ab / 2 = a^2 + b^2$$



4. Area of the large square: $(a+b)^2$
It consists of 4 triangles with area ab/2 each, and a square c^2
Therefore $(a+b)^2 = c^2 + 4ab/2$ or: $a^2+b^2= c^2$



Pythagoras law exists for integers, e.g.: 3,4,5 5,12,13
Are there an infinite number of such triplets (primitive triplets)?

)3, 4, 5(	)5, 12, 13(	)8, 15, 17(	7 , 24, 25)
(20, 21, 29)	(12, 35, 37)	(9, 40, 41)	(28, 45, 53)
(11, 60, 61)	(16, 63, 65)	(33, 56, 65)	(48, 55, 73)
(13, 84, 85)	(36, 77, 85)	(39, 80, 89)	(65, 72, 97)

Euler showed that any two integers n,m can create a primitive triplet:

$$a = m^2 - n^2,\ \ b = 2mn,\ \ c = m^2 + n^2$$

Moreover, the above, multiplied by an integer, generated all primitive triplets.

Next we ask, are there triplets that for n>2 obey the relation:

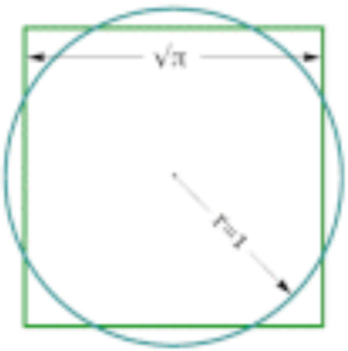
$$a^n + b^n = c^n$$

Fermat 1601–1665 noted at the margin of a book he read that he found a proof that no such numbers exist. Many false proofs were proposed since, but only recently, 1995, using modern mathematical tools, Wiles proposed a proof.

It is still puzzling if Fermat had indeed a correct proof.

Three problems posed by Pythagoras that were not solved

squaring the circle



construct a square with an area exactly equal to that of a given circle

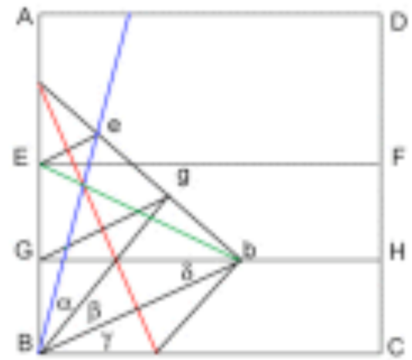
proved impossible by Ferdinand von Lindeman in 1882

doubling the cube

construct (duplicate) a cube with exactly twice the volume of a given cube

proved impossible by Pierre Wantzel in 1837

trisecting the angle

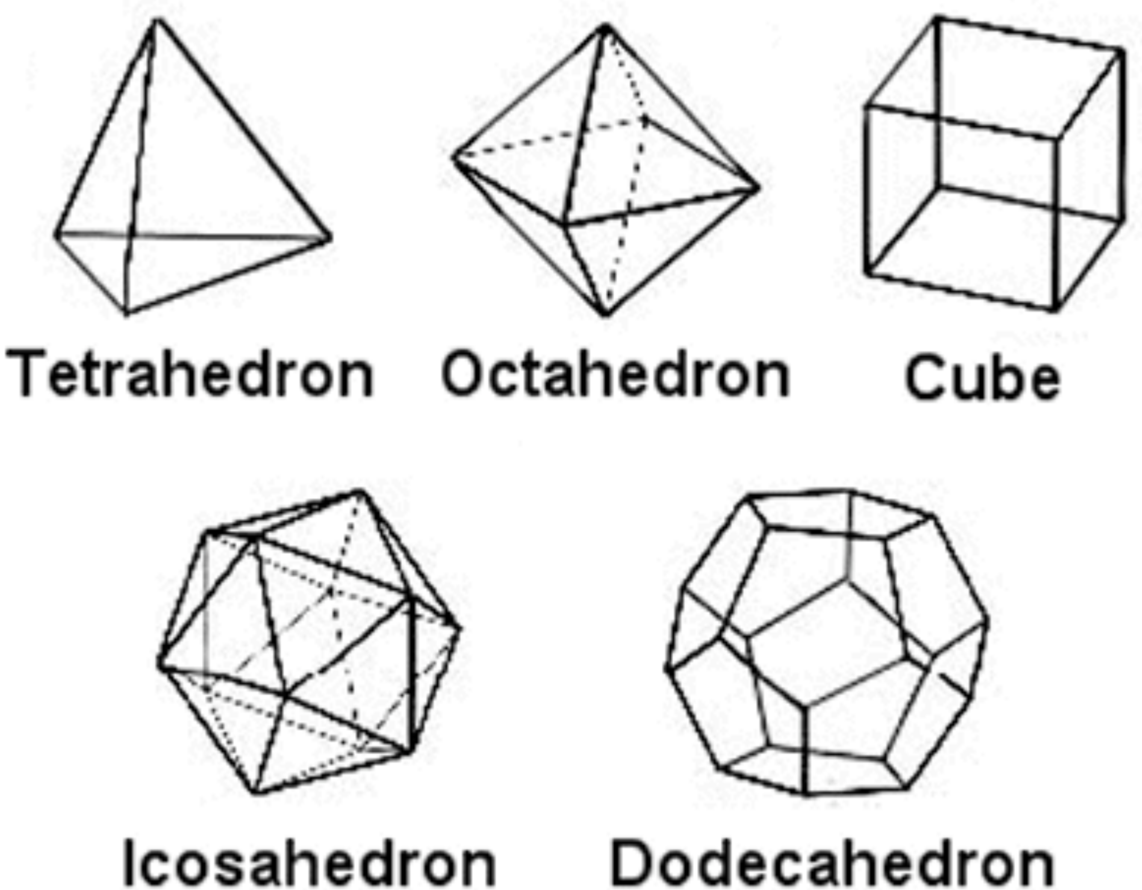


construct an angle exactly one-third of any given angle

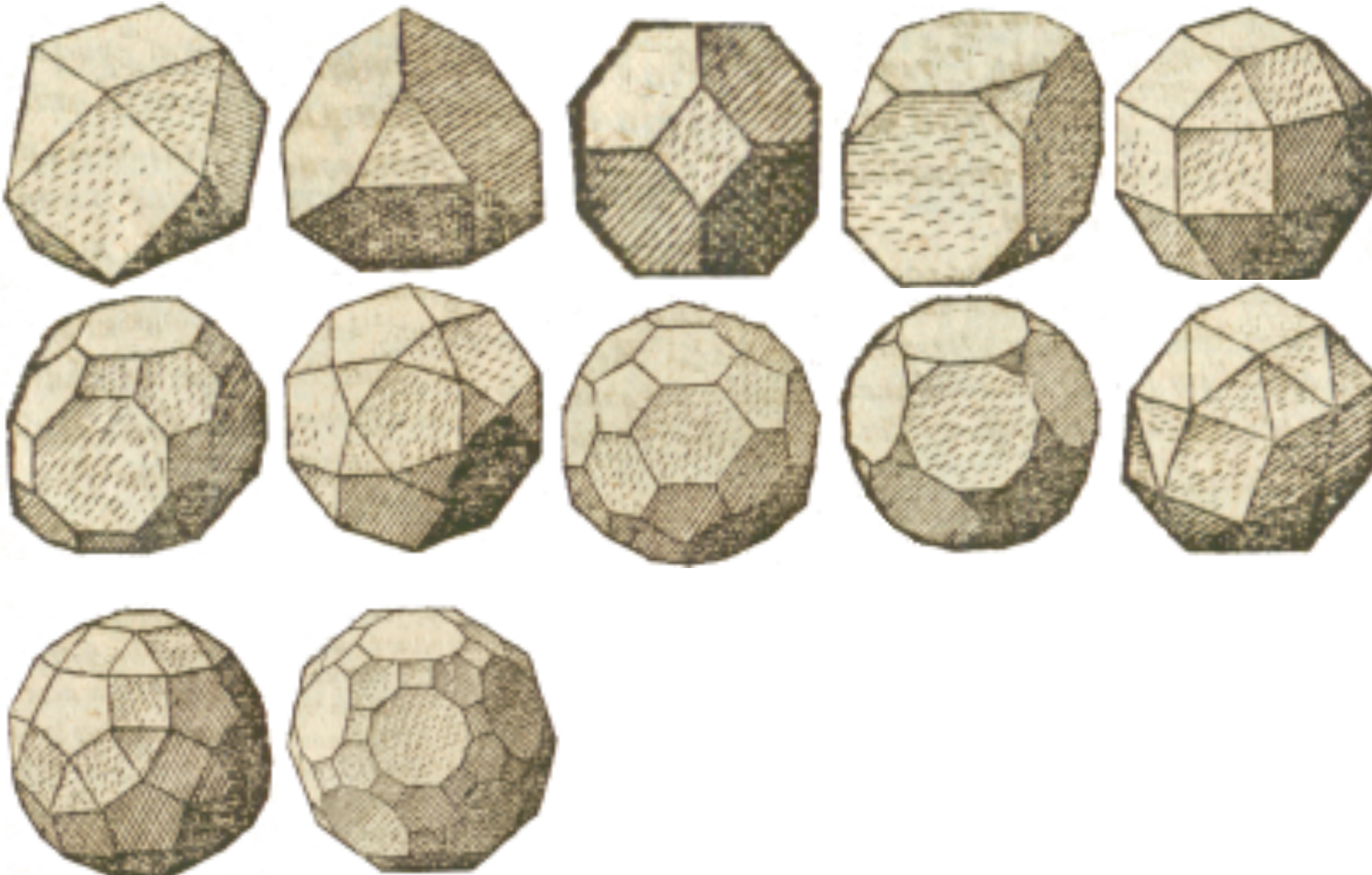
proved impossible by Pierre Wantzel in 1837

The solutions Pythagoras sought for depended on geometrical construction with a compass and a ruler alone. Today we know at any precision the values of π, $\sqrt[3]{2}$ or $1/3$, and can build the corresponding shapes from measurement.

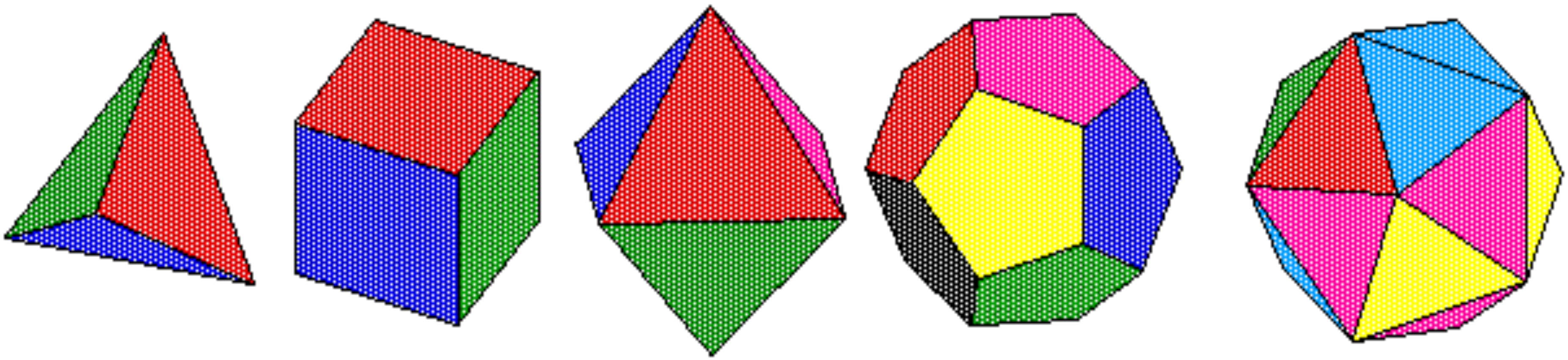
Pythagoras describe three symmetrical polyhedrons: Tetrahedron, Octahedron and Cube. (regular polytops)
Plato describe two additional: Icosahedron and Dodecahedron
Euclid proved 200 years later that these are the only bodies with equal faces



Archimedes (probably studies with Euclid**)** Describes 13 polytops with equal edge lengths, and equal number of them meeting in all vertices, but different kinds of faces



Properties of Plato’s 5 regular polytops

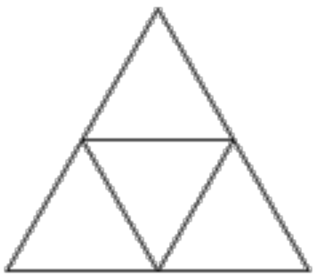


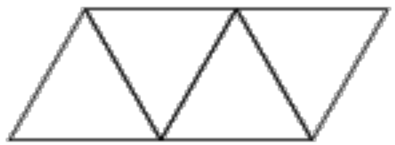
The Tetrahedron **The Cube** **The Octahedron** **The Dodecahedron** **The Icosahedron**

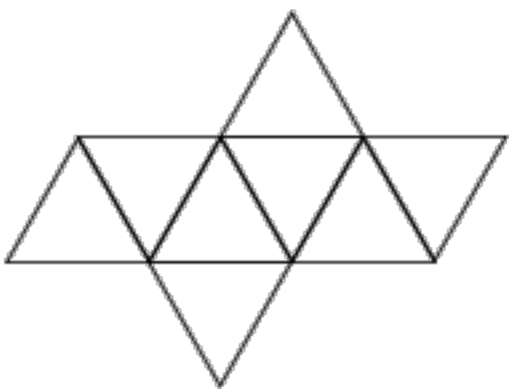
The five regular solids discovered by the Ancient Greek mathematicians are:

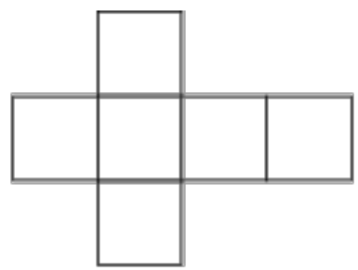
The **Tetrahedron**:	4 vertices	6 edges	4 faces	each with 3 sides
The **Cube**:	8 vertices	12 edges	6 faces	each with 4 sides
The **Octahedron**:	6 vertices	12 edges	8 faces	each with 3 sides
The **Dodecahedron**:	20 vertices	30 edges	12 faces	each with 5 sides
The **Icosahedron**:	12 vertices	30 edges	20 faces	each with 3 sides

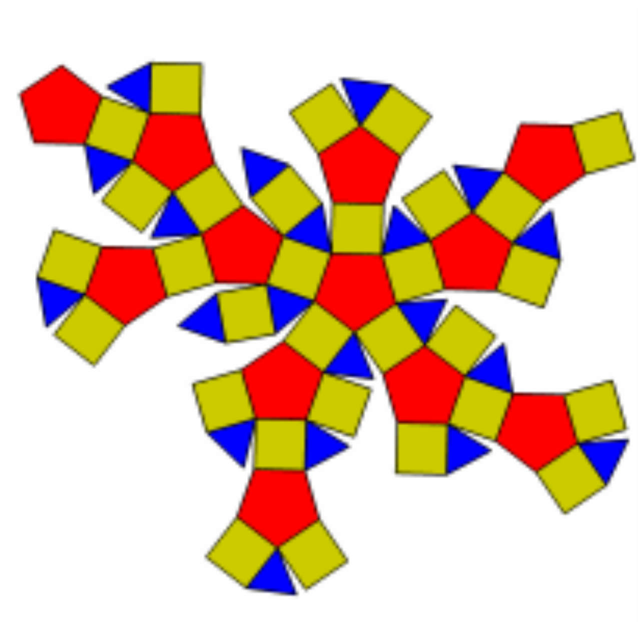
How to build these bodies from flat paper?

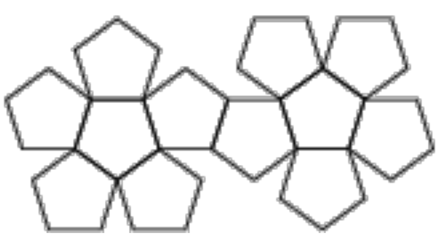


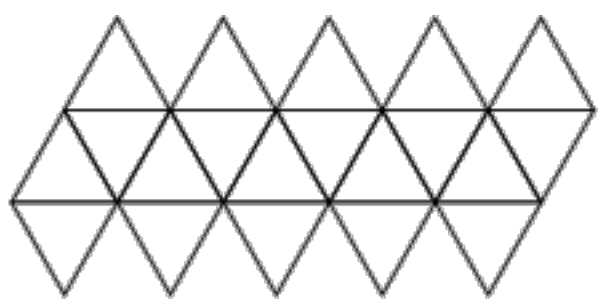








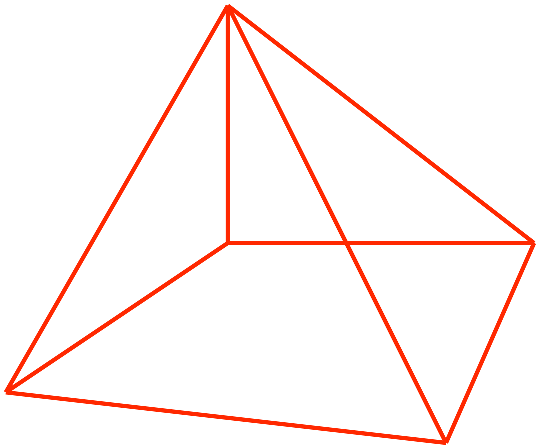




Leonhard Euler 1707-1783 describes a formula relating the number of faces, f, edges, n, and vertices, v for convex polyhedron:

F+v-n=2

For regular polytops, m faces meet at all vertices, the number of edges=f*n/2 and the number of vertices=f*n/m, applying Euler's formula:



$$f + f \cdot \frac{n}{m} = f \cdot \frac{n}{2} + 2$$

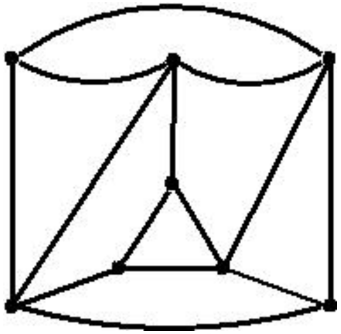
$$f = \frac{4m}{2m - (m-2)n}$$

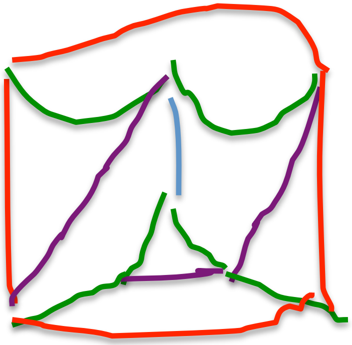
This is a Diophantus equation, with the only 5 integer solutions (v,n,f)= (4,6,4),(8,12,6),(6,12,8),(20,30,12),(12,30,20)

Extension of this formula:

Number of faces + number of vertices - number of edges =

= 2 – number of holes





Euler's formula also holds for planar graph with no Intersecting edges, that is fully connected (a line from every vertex to any other Vertex): Number of arcs + number of faces – number of vertices = 2

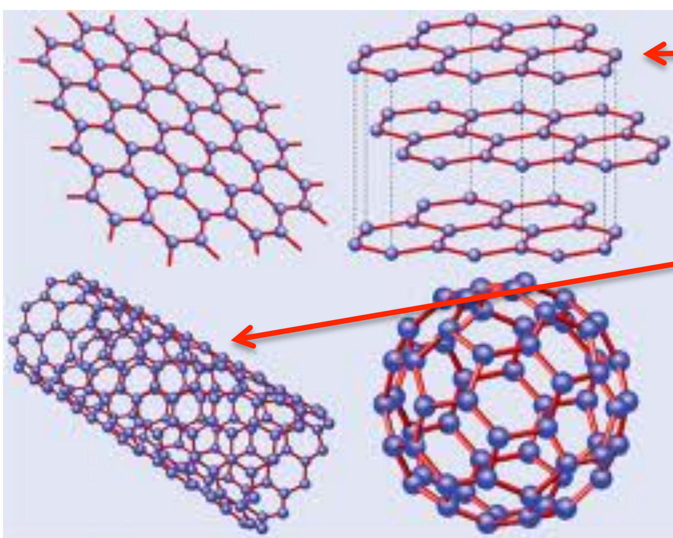
Proof: Erase the most external arcs (red), each reduce number of faces by 1. We continue with the purple arcs. Eventually we are left with a branched tree (green). We erase a vertex and an arc until left with one arc and one vertex.

Structures base on regular shapes:

Football

Geodesic dome

Buckminster Fuller



Graphene

Nobel, 2010 A.Geim and K.Novoselov

Carbon Nano-tubes

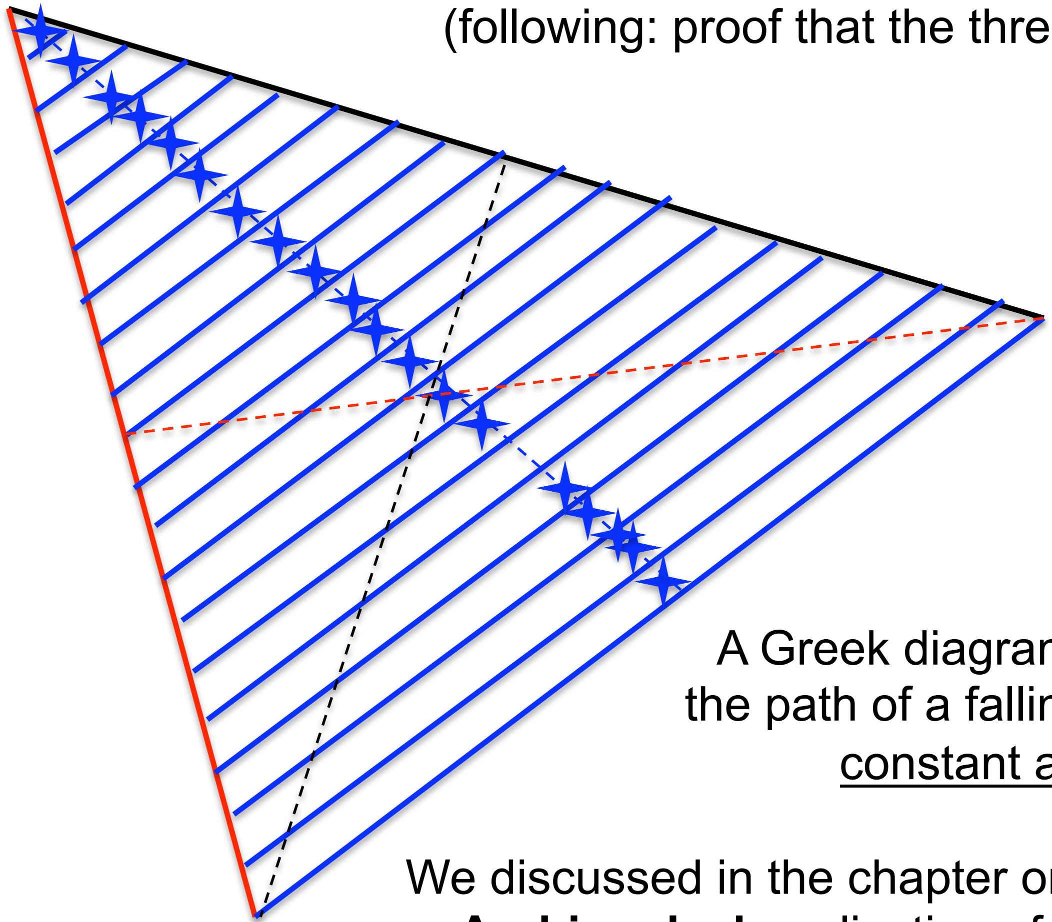
C_{60} Buckball (Fullerine)

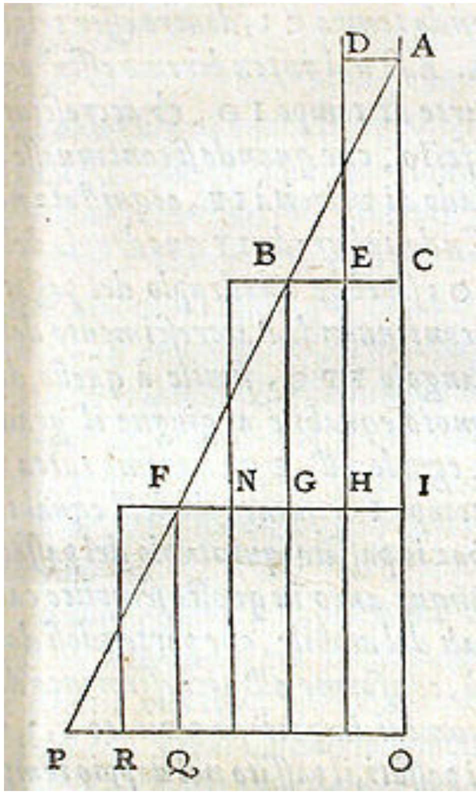
H.Kroto, R.Curl and R.Smalley **Nobel 1996**

Seed of infinitesimal calculus

The center of mass of a triangle is the meeting point of its edge bisectors.
Proof: by dividing into infinitesimally narrow sections, the center of mass for each is at its center.

(following: proof that the three edge bisectors meet at one point)





A Greek diagram explaining the path of a falling body with constant acceleration.

We discussed in the chapter on “numbers” **Archimedes’** application of infinitesimal methods for the first time to determine the circumference, area and volume of circles and spheres, and setting upper and lower rational limits to the value of π

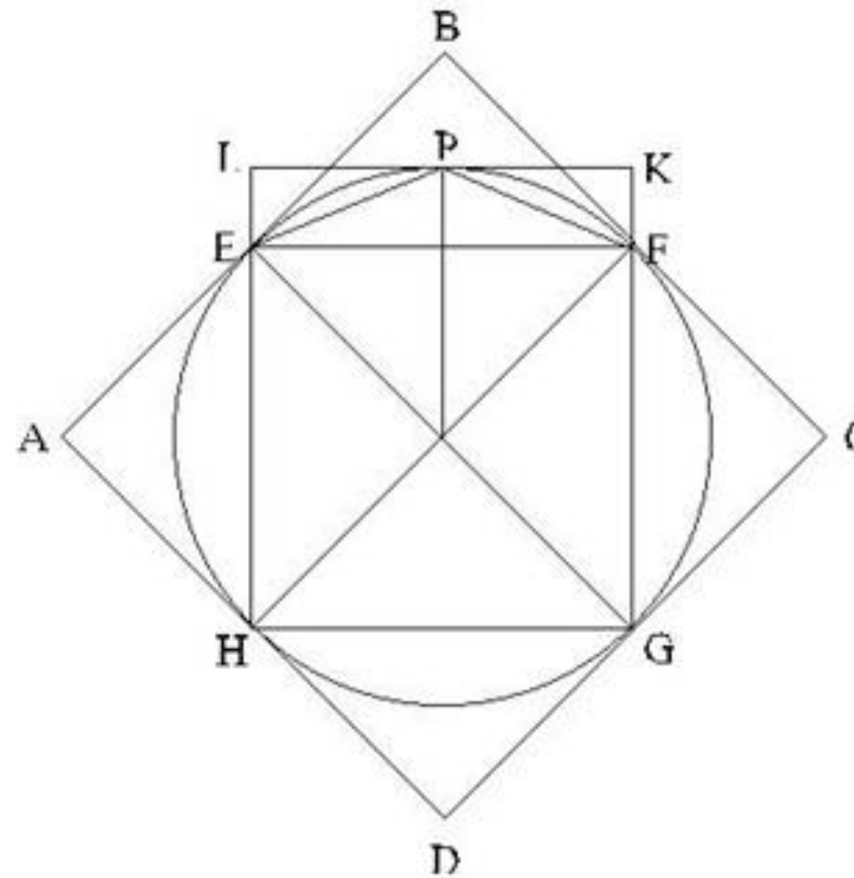
Beginning of infinitesimal calculus concepts

Zeno - discussion of ever decreasing intervals (Achilles and the turtle)
Eudoxus, Euclid and Archimedes – continued along Zeno's line of thought.
Socrates (was probably Zeno's student):
Asking a question and searching for an answer via systematic research logics. Socrates theories were considered too rebellious and spoiled the youth. He was sentenced to death, after refusing to deny his beliefs, by drinking poison.
Archimedes use infinitesimal logics to compute π
Egyptians found the value of π from measurements, and probably did not know that the same value appearing in the circumference and the area of a circle [they used $A=(8/9\ d)^2 = 3.1605\ r^2$]

Eudoxus of Cnidus 408–355 BC

Describes his Exhaustion method in his “theory of ratios”:
re-invented as Dedekind cut 1872 which was the modern basis of differential and integral calculus.
Eudoxus contribution to the successive approximations and incremental convergence (i.e. of polygons to a circle) was to set lower and upper limits that indicate at any stage of the approximation how far we are from the result of infinite steps.

For example, in the figure the areas of the Inscribed and the circumscribed square ABCD & EFGH, are bigger and smaller than the area of the circle.



Archimedes applied Eudoxus exhaustion method to prove that the area of a circle is half its perimeter times its radius:
If we assume the area of the circle is smaller than the triangle, we build a series of polygons with 4,8,16,32,… edges, and get areas larger than the triangle area, contradicting our assumption. If we assume the triangle are is bigger, we build a series of polygons with area smaller than the triangle, again contradicting our assumption, therefore the area of the circle and the triangle are exactly equal.

Archimedes also used “integrals” to find areas and volumes of shapes, that he split into a large number of segments (i.e. rings in a circle).

Example of non-converging sum of an infinite series:… + 1+1/2+1/3+1/4

A picture of a sphere inscribed inside a cylinder:
The drawing **Archimedes** asked to put on his grave, emphasizing the importance he attributed to this finding (according to **Cicero**, 75 BC).

Archimedes attributes the proof of the volume of the sphere to **Eudoxus (**exhaustion method).
Here is the proof: Volume of a cone is 1/3 of the volume of a cylinder with the same base.
This extension from prisms and Daltons can be deduced by splitting the cone basis into infinitesimal triangles. The area of a section of the sphere at height y is:

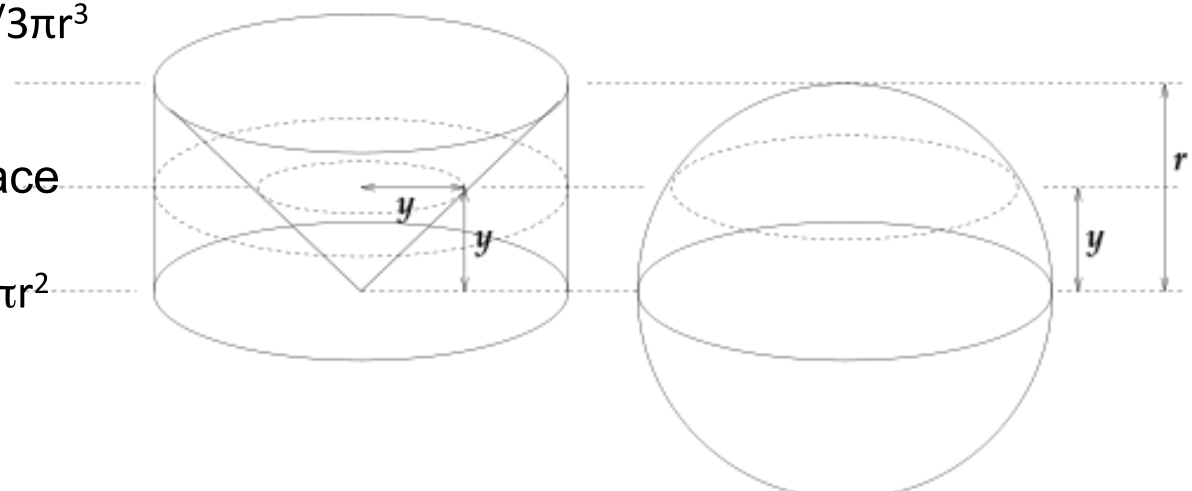
$$\pi\,(r^2 - y^2)$$

But this is also the area of the cylinder section, $\pi\, r^2$
Minus the area of the code section,
$\pi\, y^2$. The cone volume is $\pi r^3/3$
therefore the sphere volume is

$$r^3(1-1/3)2=4/3\pi r^3$$

Similarly, the surface area of a sphere is 2/3 of the cylinder surface area (including its two bases):

$$2/3\ (4\pi r^2+2\pi r^2) = 4\pi r^2$$



Bonaventura Francesco Cavalieri 1598-1647

Cavalieri's principle is applying infinitesimal principles, and "reinvented" Archimedes in a more general form:

If every section through two bodies create equal areas then their volumes are equal.

and

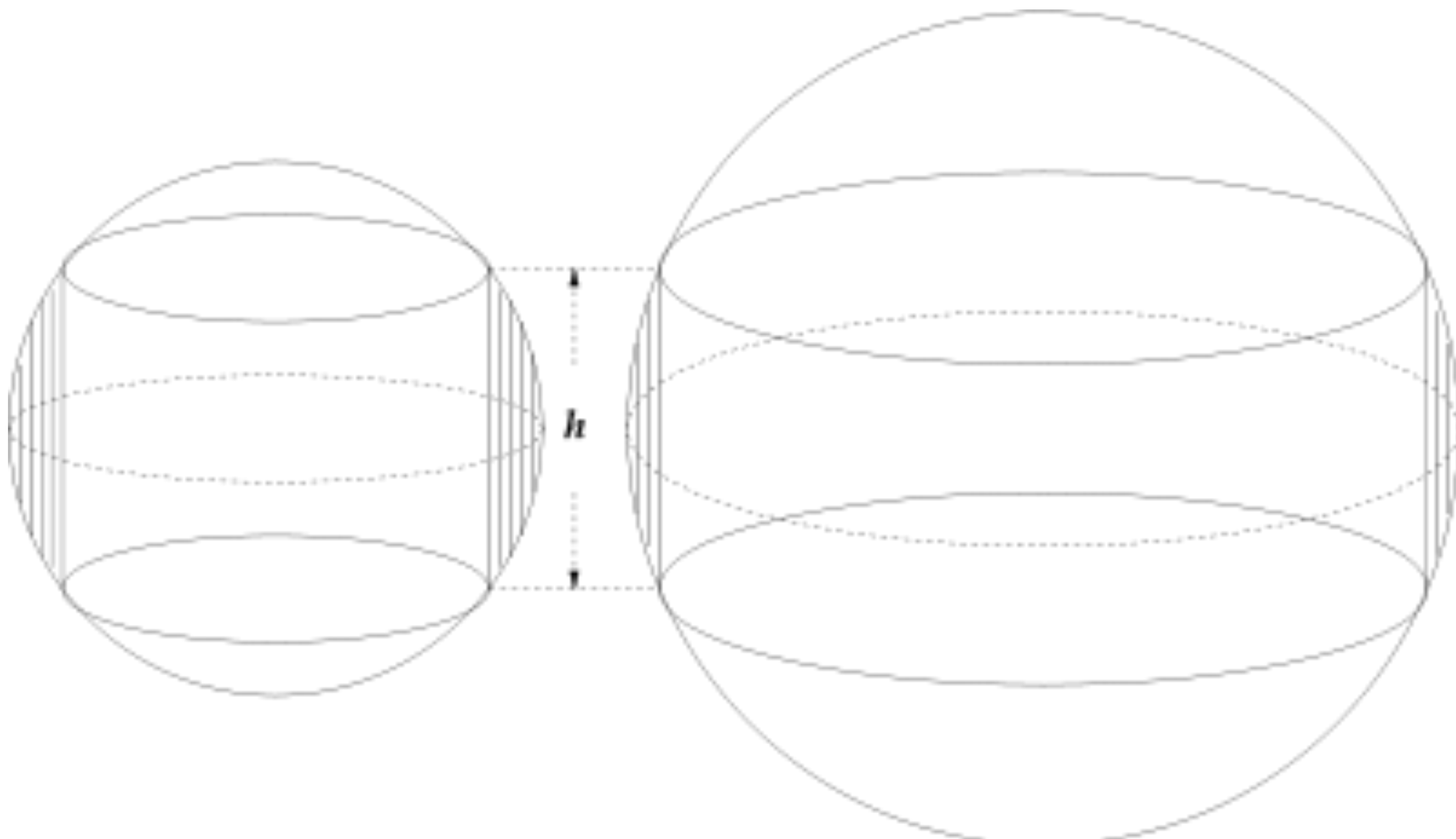
If every section through two bodies create equal circumferences then their surface areas are equal.

The piles of pennies demonstrate this principle:

Another example of application of Cavalieri’s principle:
If we drill a hole at the center of a sphere, so that we are left with height h, what is the volume of the drilled sphere material ?
The area of the ring created by a section at height y from the sphere center is:

$$\pi\left(r^2-y^2\right)-\left(r^2-\left(h/2\right)^2\right)=\pi\left(\left(h/2\right)^2-y^2\right)$$

Interestingly independent on r, since for larger r the the diameter of the remaining material grows, but its thickness becomes smaller.
Thus the volume is: $\pi h\left[\left(h/2\right)^2-y^2\right]$



Another late theorem that Archimedes and later Pappus implicitly applied to determine center of mass for bodies:

Pappus-Guldin theorem

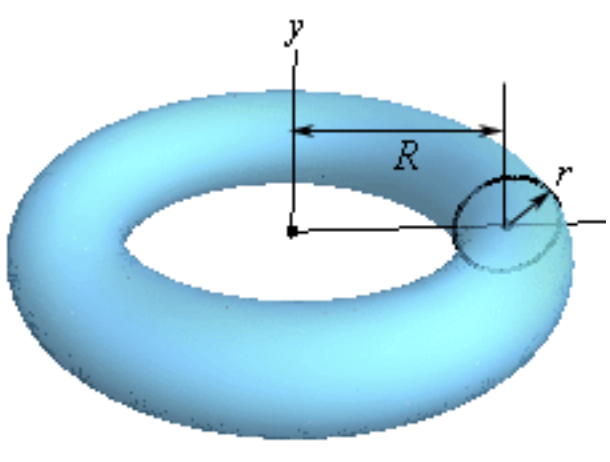
Paul -Habakkuk Guldin 1577-1643
Pappus of Alexandria 290–350

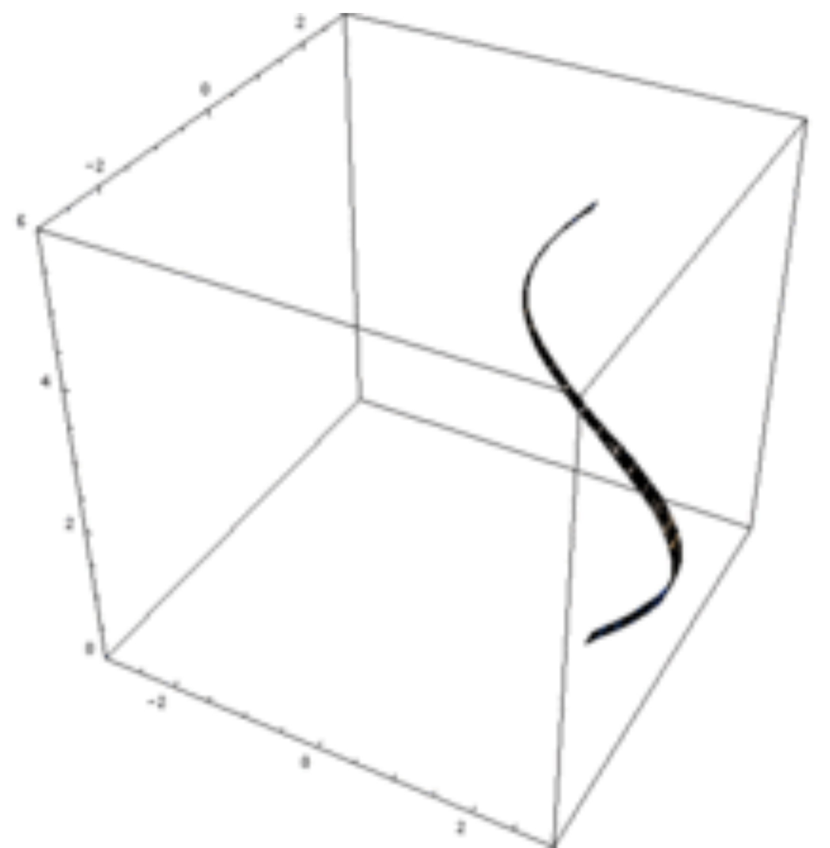
If a planar closed shape with area A and perimeter S is rotated around an axis in this plane, and the axis does not cross the shape, we get a rotation body.
If the center of mass of the shape is at distance R from the axis, the rotation body volume is V=2πRA and its surface area A=2πRS

An example for a torus:

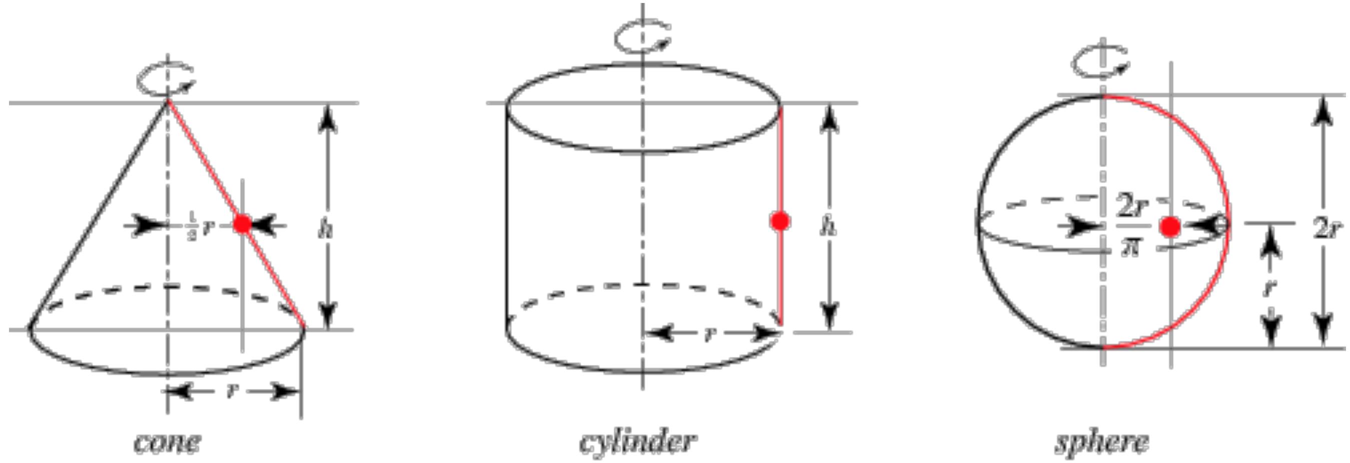
$$V = (\pi r^2)(2\pi R) = 2\pi^2 R r^2.$$

$$A = (2\pi r)(2\pi R) = 4\pi^2 R r.$$





Pappus's Centroid Theorem

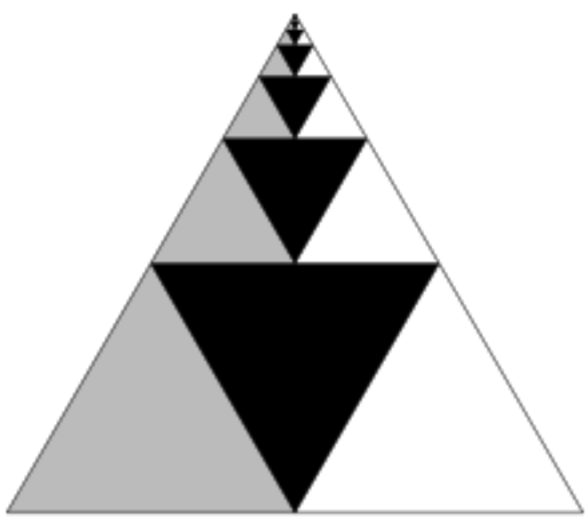


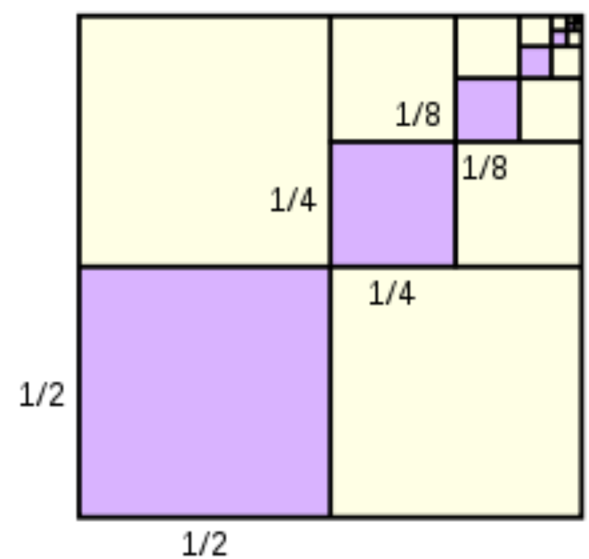
	Cone	Cylinder	Sphere
Center	r/2	r	2r/π
Line length	$\sqrt{r^2+h^2}$	h	πr
Surface area	$\pi r\sqrt{r^2+h^2}$	$2\pi r h$	$4\pi r^2$
Cenetroid	r/3	r	4r/3π
Area	rh/2	rh	$\pi r^2/2$
Volume	$\pi r^2 h/3$	$2\pi r^2 h$	$4/3\pi r^3$

INFINITE SERIES

$$1/4 + 1/4^2 + 1/4^3 + \ldots = ?$$

The calculation of infinite sums were based on geometrical constructions:
The large square area = 1. splitting it sequentially into halves (purple squares) create the left over 2 yellow squares of equal area to the last purple square. Altogether the purple squares area is 1/3.
A similar argument holds for the triangles.

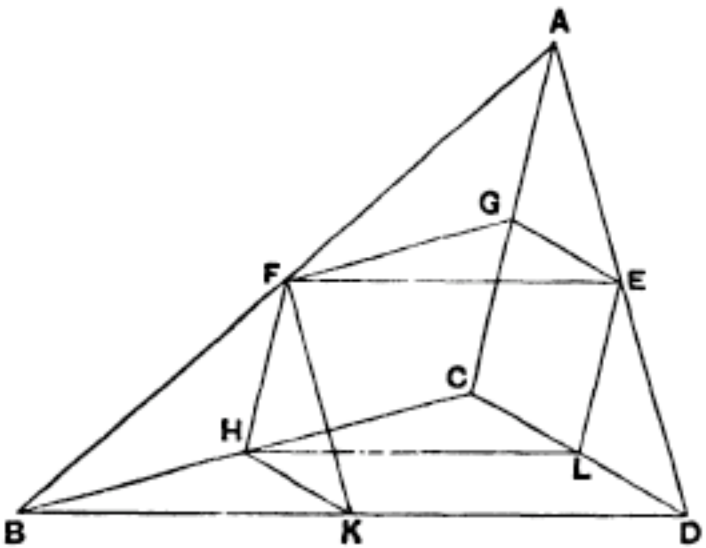


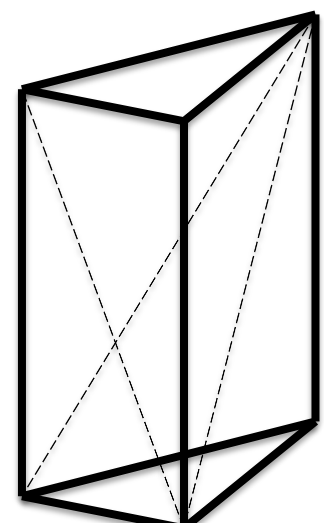


Volume of pyramids:

Triangular Pyramids (Daltons), when their edges are split to halves, create 8 equal volume Daltons: BKHF AEGF plus three in each of the prisms HFGELC and HKFLDE.
Right drawing: Prism splitting into 3 equal volume Daltons (equal base and height).

Therefore, the volume of a pyramid is 1/3 of the volume of a prism with the same base.

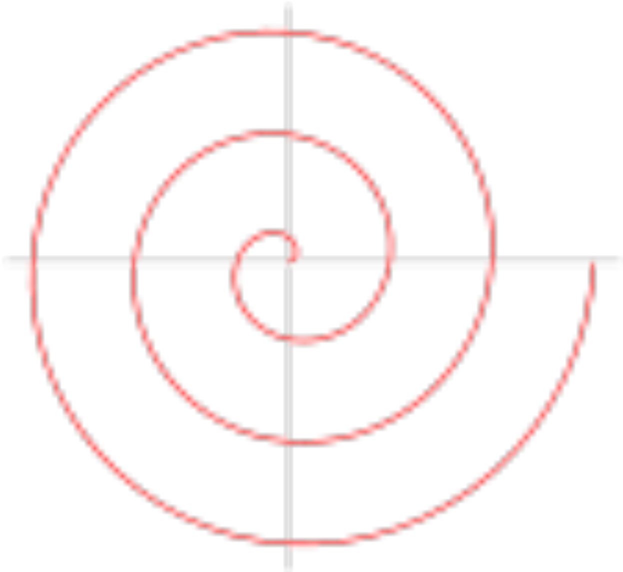




Archimedes spiral

Can be expressed most simply in cylindrical coordinates: $r=a+b\theta$

SPIRAL PUMP: Fast rotation pulls water to the center.
Spirals appear in nature, due to continuous birth and growth of segments.



Stomachion game (cited in Euclid's "Elements"**)**

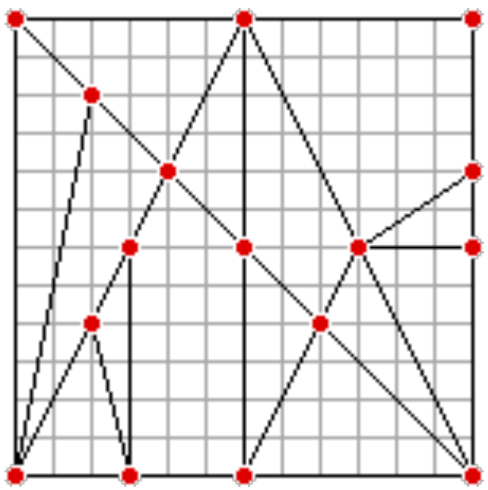
If we draw a 12x12 square grid, and divide this square into triangles with vertices sitting on the grid points (e.g. the red points) we get triangles with area

3, 3, 6, 6, 6, 6, 9, 12, 12, 12, 12, 12, 21, 24

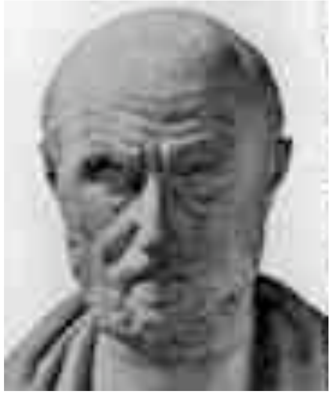
And ratios between their areas:

1, 1, 2, 2, 2, 2, 3, 4, 4, 4, 4, 4, 7, 8

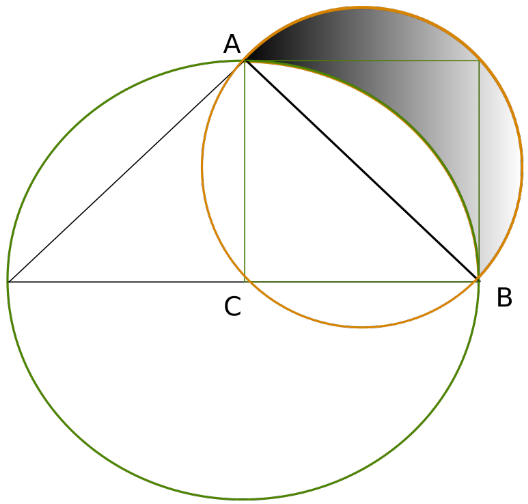
There are 536 ways to fit these triangles into a square. Can you find some?

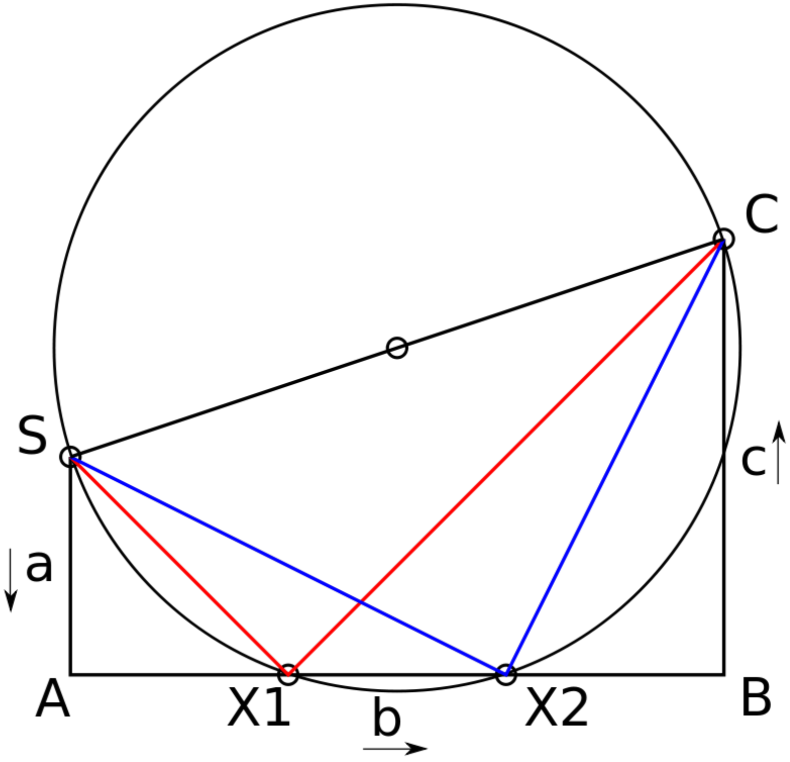


Hippocrates of Chios 470-410 BC ***

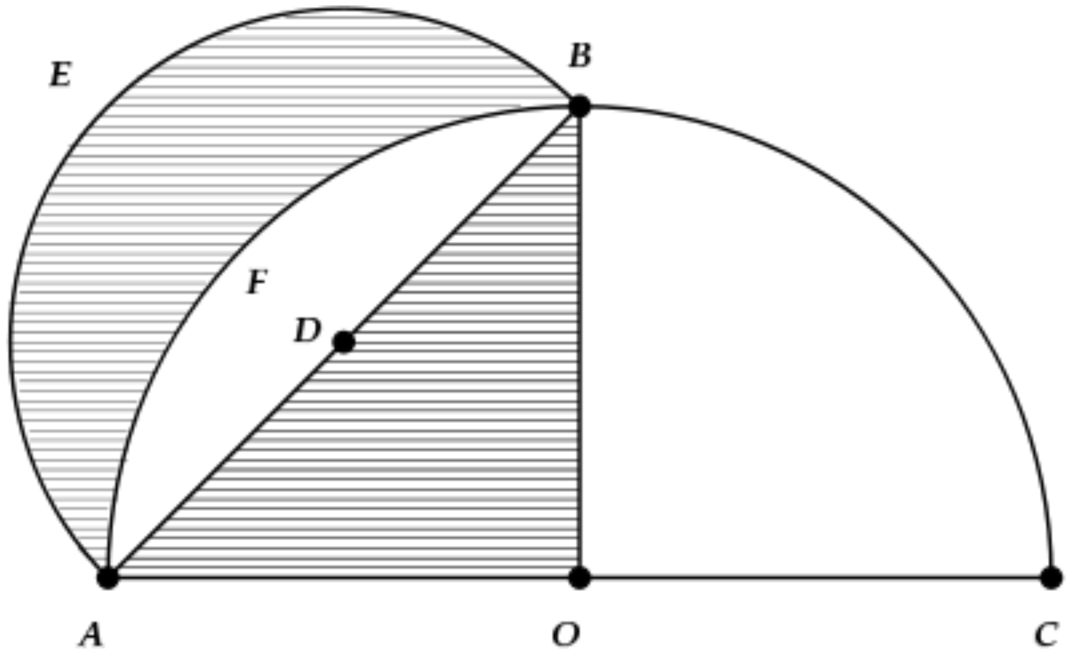


A student of Pythagoras, who wrote “Elements of Geometry”.
Laid the ground for Euclid “Elements”.
He found that ratio of circles area equals the square ratio of their radii.
He solved the solutions x_1 x_2 to quadratic equations $ax^2+bx+c=0$
By geometrical construction, see figure below:
Lune of Hippocrates is a geometrical construction he built attempting to “Squaring the circle”: The shaded area equals the area of the triangle ABC.
Proof: see next slide





(*** this **Hippocrates is** different than **Hippocrates of Cos 460-370** BC, to whom the physician’s oath is attributed)

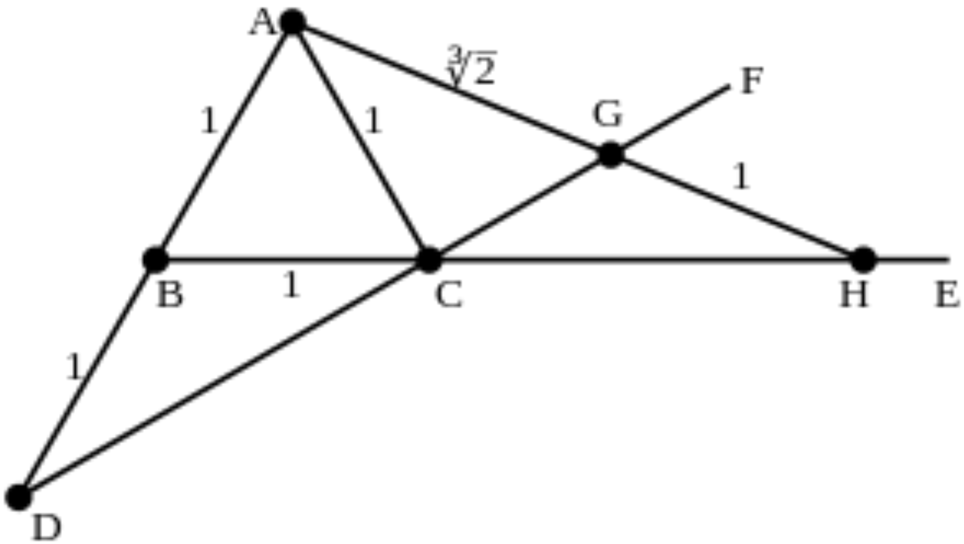


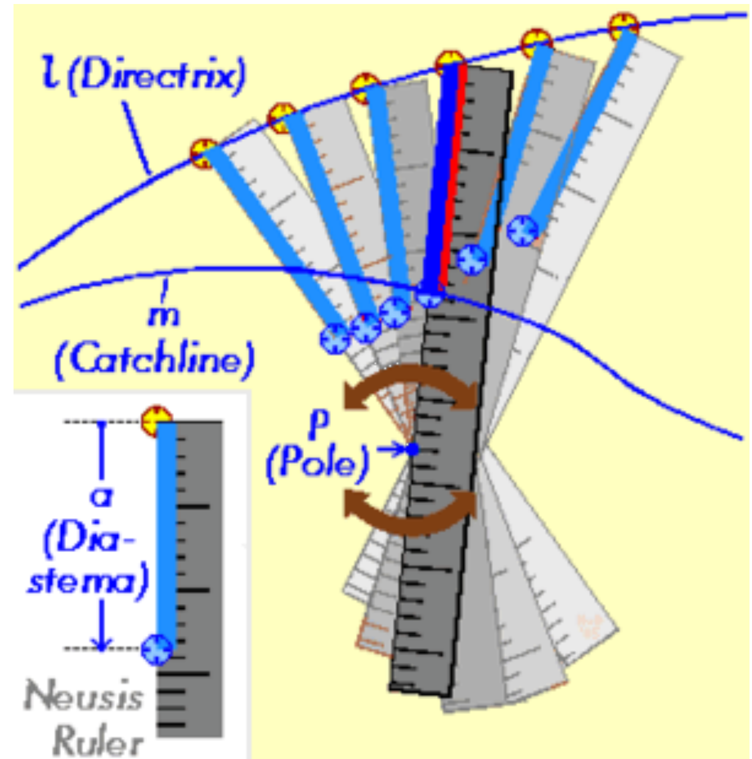
$R=AO$ $r=AD=\mathrm{sqrt}(2R^2)/2=R/\sqrt{2}$ $A=R^2/2$

$a=\pi r^2/2-(\pi R^2/4-A)=\pi R^2/4-\pi R^2/4+A=A$ Q.E.D.

Geometric construction of $\sqrt[3]{2} = 2^{1/3}$

Cannot be constructed using a compass and a ruler alone, but can be constructed with a "rotating ruler" or **Neusis construction:** fitting a line passing through a pole, p, to form a segment between two curves of a given length.





For example: draw length of $\sqrt[3]{2}$

ABC – equilateral triangle, edge=1

Continue AB to distance of 1 till D

Continue BC to direction of E

Continue DC to direction F

Turn the ruler around pole A so that GH=1

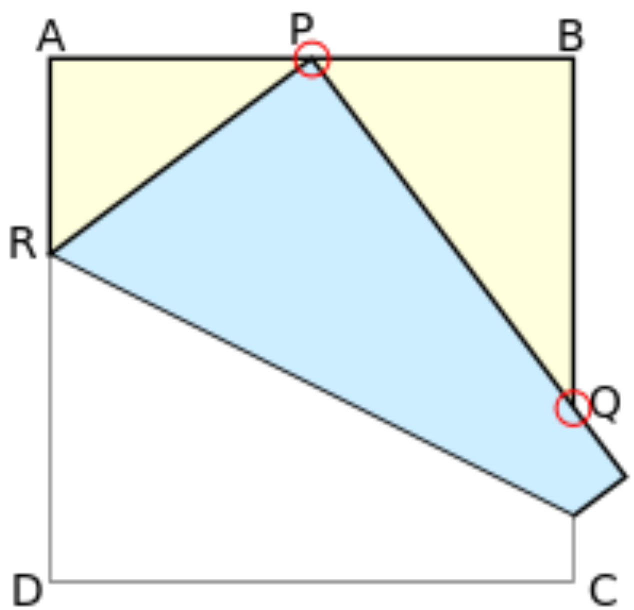
now: AG = $\sqrt[3]{2}$

Neusis construction

Other Neusis constructions solve problems not possible with compass and ruler alone:
Haga's theorems:

Haga's first theorem allows to draw the length of any rational number:
Fold a square as shown, so that vertex D fall on point P on the line AB:

$$BQ = \frac{2AP}{1+AP}.$$



Haga's first theorem

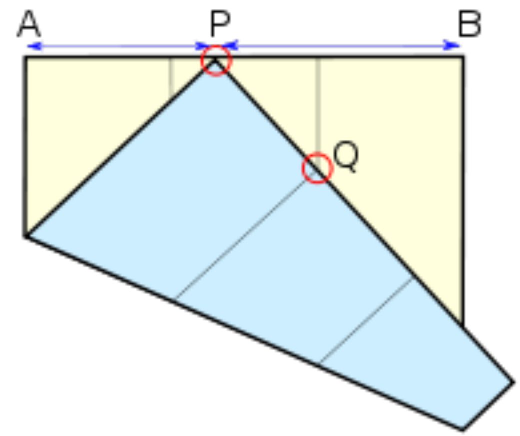
AP	BQ	QC	AR	PQ
x	$\frac{2x}{1+x}$	$\frac{1-x}{1+x}$	$\frac{1-x^2}{2}$	$\frac{1+x^2}{1+x}$
1/2	2/3	1/3	3/8	5/6
1/3	1/2	1/2	4/9	5/6
2/3	4/5	1/5	5/18	13/15
1/5	1/3	2/3	12/25	13/15

Haga's theorem, $\sqrt[3]{2}$ and trisecting an angle

Haga's generalized theorem allows to draw $\sqrt[3]{2}$ ("double the cube") and trisect an angle:

Divide a square into three equal stripes vertically. Slide the point P until Q hits the 1/3 line, then:

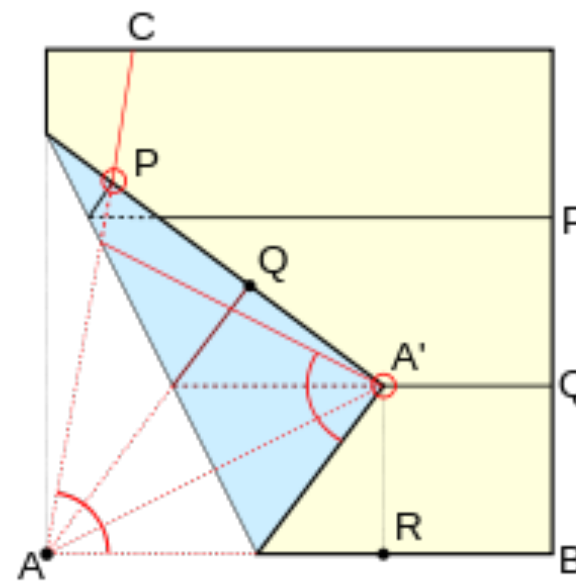
$$PB/PA=\sqrt[3]{2}$$



By similar folding we can trisect an angle, BAC. Divide a square into three equal stripes horizontally by lines P'P Q'Q. Fold A to "slide" on the line, Q'Q, till P fall on the angle line AC.

BAA' = 1/3 BAC

Prove !



Oenopides of Chios 400 BC defines “Axioms” and problems that are soluble based on the axioms

Describes 3 axioms (of the 5 defined by Euclid):

1. Two points define a single line that crosses them
2. A segment of a straight line can be continued by a unique line
3. A center and a radius defines a unique circle
4. All right angles overlap
5. If a line creates with two other lines two angles with sum different than 180^o the two lines meet towards the direction where the sum of inner angles $< 180^o$

5’. An alternative axiom: Through a point off a line one can draw only a unique parallel to the line.

Euclid also defined “common notations”

1. Two things that equal a third thing are equal things
2. If one adds two equal values to a third value, the results are equal
3. If one subtracts two equal values from a third value the results are equal
4. “Reflectivity”: Two overlapping things are equal
5. The whole is bigger than its parts

Proofs can be constructive or conflictive (reject an assumption, or accept its opposite, since it leads to a contradiction).

Euclid measurements in Geometry: distances & angles. Overlap means equal distances and angles. Similarity means equal angles, and constant ratio of corresponding distances.

Theorems for triangles:

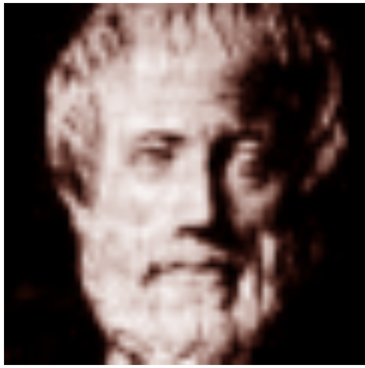
1. If the base angles equal the edges are equal, and vice versa.
2. Sum of angles in a triangle = 180^o
3. Pythagoras law
4. Thales law: Angle lying of the diameter of a circle is a right angle
5. Ratio of area of similar shapes equals the square of the ratio of their (linear) dimensions (Euclid only proved for some shapes)
6. Ratio of volumes of similar shapes equals the cube of the ratio of their dimensions

The Academy in Athens, painted by Raphael. Aristo and Plato are at the center, Aristo points his finger up, symbolizing “Philosophical truth” and Plato down “Experimental proof”

Enlarged detail

Plato 427-347 BC

Mathematical logics and foundations of Geometry.
Established the "Academy" in Athens, the first university.
387 BC published "Phaedo" where he describes "ideal" mathematical objects such as a line with no width. Led the way for Euclid, by making an hypothesis and building proof based on it.
He believed the world is constructed from ATOMS in VACUUM. Atoms Are combined to create all materials, including live.
Plato believed the perfect world can only be studied by logical argumentation, since in reality measurements are never "ideal" (spoiled by errors…).



Aristotle 384-322 BC

A student of Plato, but diverted from his teaching. Established In Athens the "Lyceum" where his theories were taught.
He believed all materials are made of 4 elements (sources):
Fire, Earth, Air and Water. Unlike Plato, he believed truth cannot be found without experiments.

Plato and Aristo were teachers of **Alexander the Great, (Alexander of Macedonia 356-323 BC**)

Theaetetus of Athens 417–369 BC

A student of Plato, who studied the geometry of three-dimensional bodies, rational numbers, and irrational numbers

Theatetus Menaechmus 380–320 BC
Conic sections

All possible planar curves (in XY plane) created by the second order equation:

$$ax^2+by^2+cx+dy+e=0$$

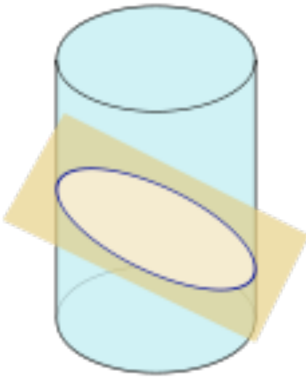
The “canonical forms”:

Circle: $x^2+y^2=r^2$

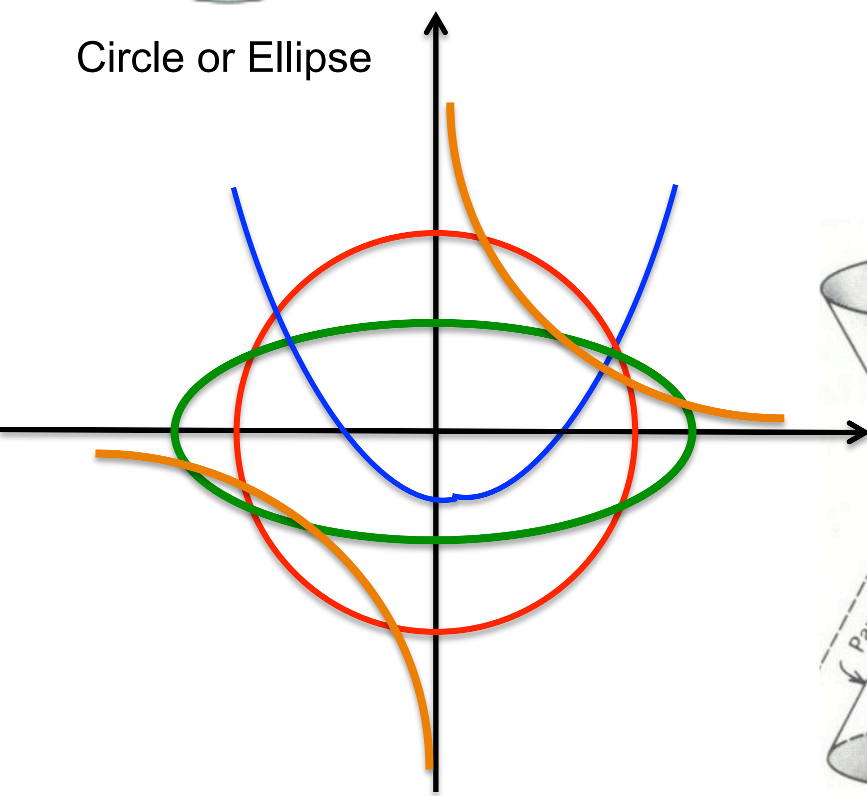
Ellipse: $(x/A)^2+(y/B)^2=r^2$

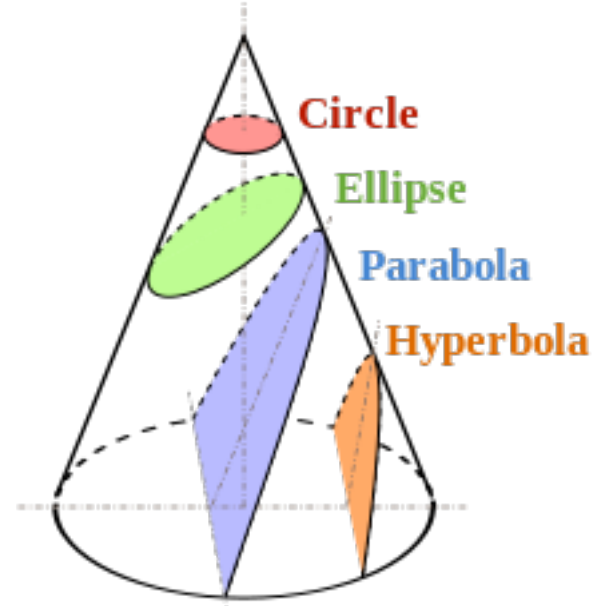
Parabola: $y=Ax^2$

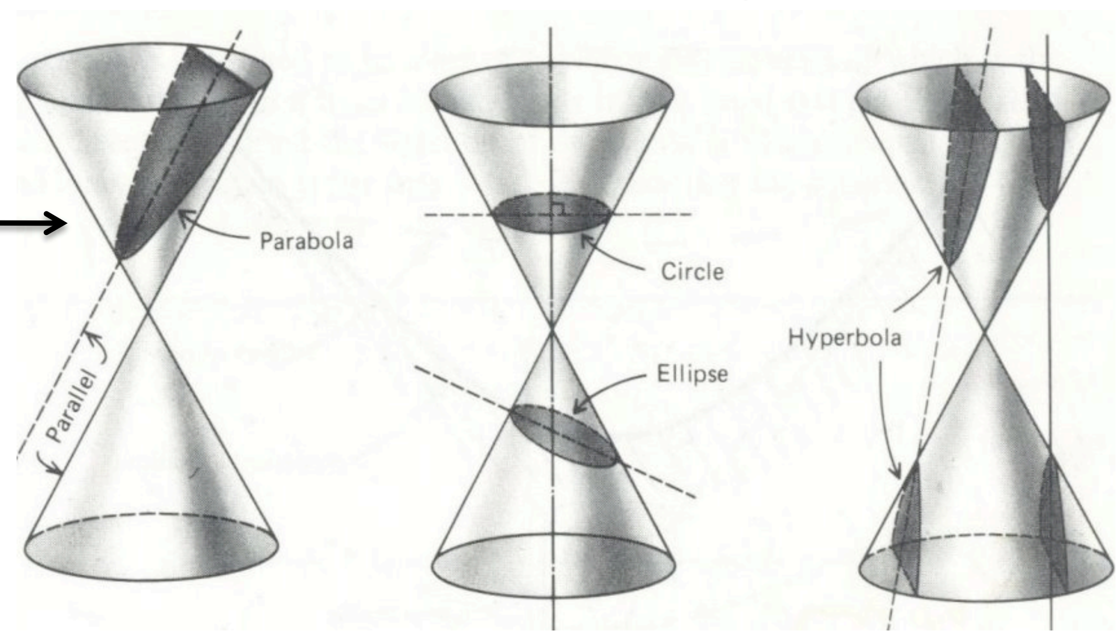
Hyperbola: $y=A/x$



Circle or Ellipse





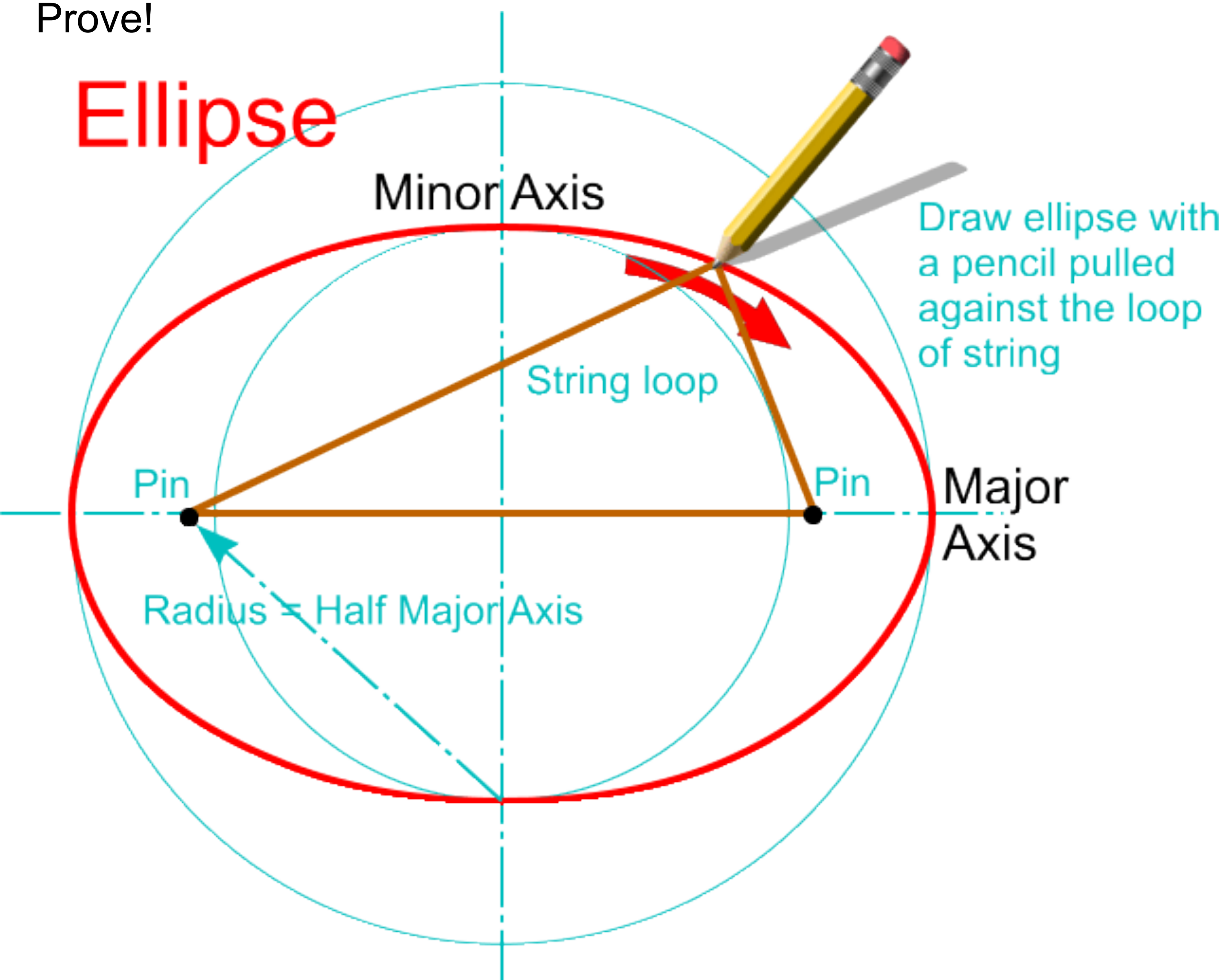


A circle = all points at equal distance from the center

An Ellipse = all points with distances from the two foci adding to the same sum.

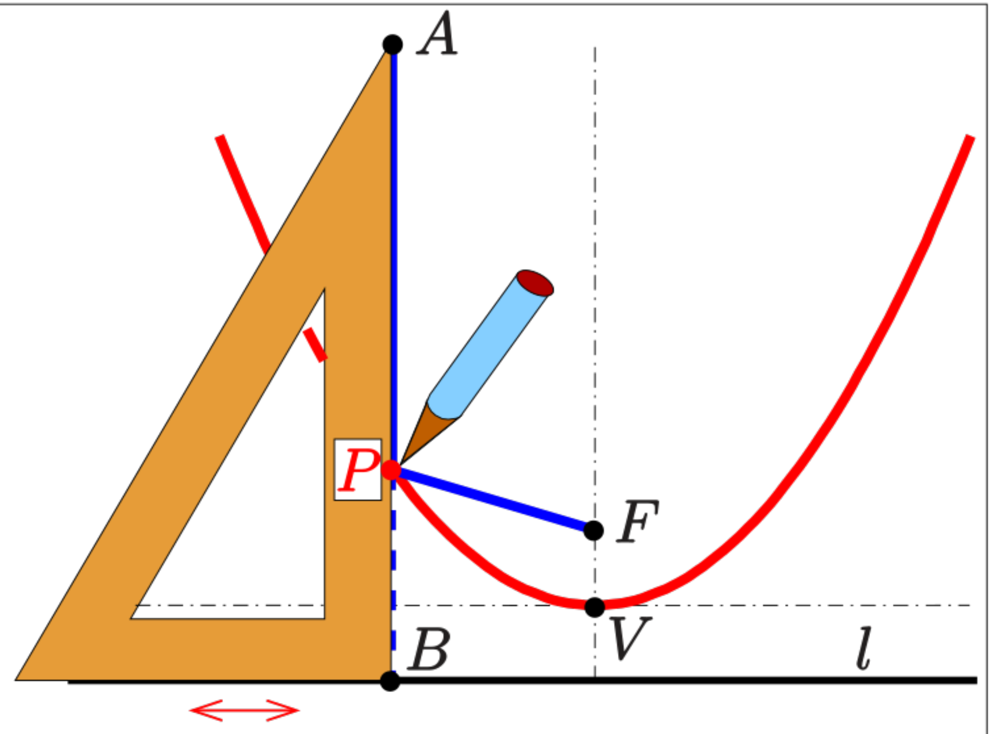
Draw an ellipse by two pins attached to the two ends of a string, and a pen stretching the string and moving on the ellipse perimeter.

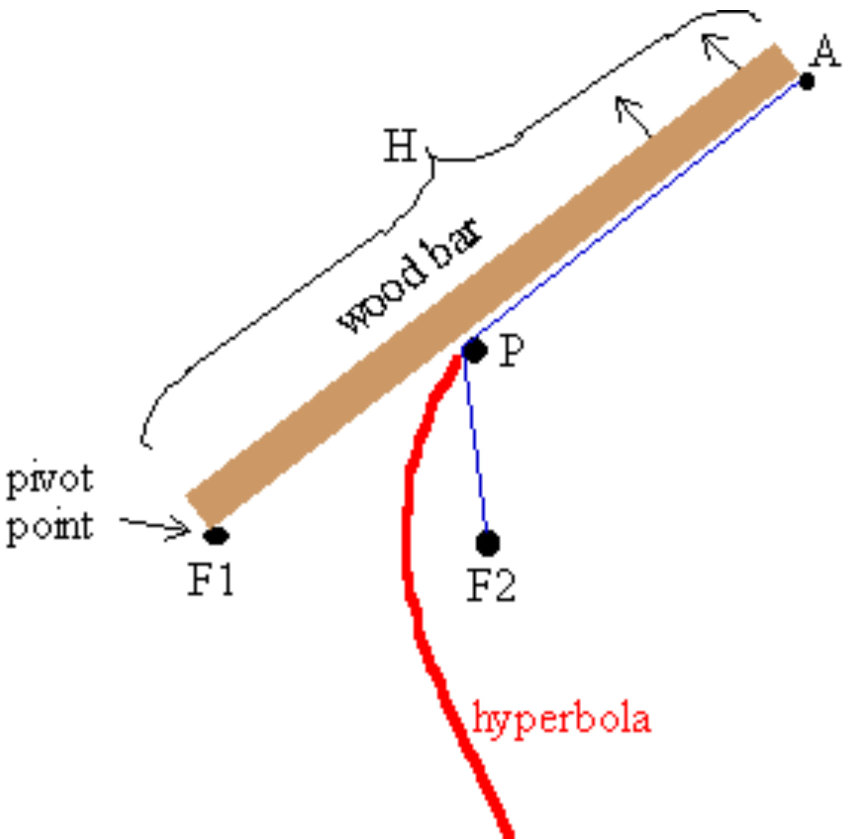
Prove!



Parabola= Draw with a string, a right-angle triangular ruler sliding on a base, and two pins at the ends of the string, one stuck on the ruler, the other in the parabola focus, F.
Hyperbola= Draw with a string, a ruler rotating around one focus, F1, and two pins at the ends of the string, one stuck on the ruler, the other in the hyperbola focus, F2

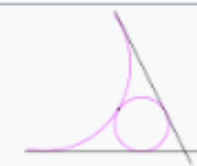
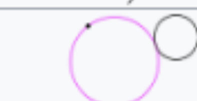
Prove!





Apollonius of Perga 262–190 BC

Cone sections: Constructions of tangential lines, circles, or points passing through them.

Index	Code	Given Elements	Number of solutions (in general)	Example (solution in pink; given objects in black)
1	PPP	three points	1	
2	LPP	one line and two points	2	
3	LLP	two lines and a point	2	
4	CPP	one circle and two points	2	
5	LLL	three lines	4	
6	CLP	one circle, one line, and a point	4	
7	CCP	two circles and a point	4	
8	CLL	one circle and two lines	8	
9	CCL	two circles and a line	8	
10	CCC	three circles (the classic problem)	8	

Euclid of Alexandria
325–265 BC

Euclid of Alexandria 325–265 BC

Wrote 13 volumes of "the elements" of mathematics: Euclidian Geometry
A summary of the knowledge and logical, axiomatic basis for mathematics (rather geometry).

Euclid's "Elements" http://aleph0.clarku.edu/~djoyce/java/elements/toc.html

Book I. The fundamentals of geometry: theories of triangles, parallels, and area.
Definitions Postulates Common Notions Propositions

Book II. Geometric algebra. Definitions Propositions

Book III. Theory of circles. Definitions Propositions

Book IV. Constructions for inscribed and circumscribed figures. Definitions Propositions

Book V. Theory of abstract proportions. Definitions Propositions

Book VI. Similar figures and proportions in geometry. Definitions Propositions

Book VII. Fundamentals of number theory. Definitions Propositions

Book VIII. Continued proportions in number theory. Propositions

Book IX. Number theory. Propositions

Book X. Classification of incommensurables.
Definitions I Propositions 1-47 Definitions II Propositions 48-84 Definitions III Propositions 85-115

Book XI. Solid geometry. Definitions Propositions

Book XII. Measurement of figures. Propositions

Book XIII. Regular solids. Propositions

Other books: *Data, On Divisions of Figures,* the *Phaenomena,* the *Optics*

Lost books: *Surface Loci, Prisms, Conics,* and the *Pseudaria* (that is, the *Book of Fallacies*)

Basis of Mathematical Logics

If …. Then ….
Necessary and Sufficient
Conjecture, Lemma and Proof

The Original Euclid's Postulates (Axioms) (5)

1. For every point *A* and for every point *B* not equal to *A* there exists a unique line that passes through *A* and *B*.
2. For every segment *AB* and for every segment *CD* there exists a unique point *E* such that *B* is between *A* and *E* and such that segment *CD* is congruent to segment *BE*.
3. For every point *O* and every point *A* not equal to *O*, there exists a circle with center *O* and radius *OA*.
4. All right angles are congruent to each other.
5. (Euclid's Parallel Postulate) For every line *l* and for every point *P* that does not lie on *l*, there exists a unique line *m* passing through *P* that is parallel to *l*.

Incidence Axioms (3)

1. Given 2 distinct points there is a unique line incident with them.
2. Given a line there exist at least 2 distinct points incidence with it.
3. There exist 3 distinct points not incident with the same line.

Incidence Propositions

1. If 2 distinct lines are not parallel then they have a unique common point.
2. There exist 3 distinct lines which are not concurrent.
3. For every line there is at least one point not incidence with it.
4. For every point there is at least one line not incidence with it.
5. For every point there exist at least 2 lines incidence with it.

Betweenness Axioms (4)

1. If $A*B*C$ then also $C*B*A$ and A, B, C are distinct collinear points.
2. Given 2 points P and Q there exist 3 points A, B, C such that $P*A*Q$ and $P*Q*B$ and$C*P*Q$.
3. Given 3 collinear points, only one of them can be between the other two.
4. (Plane Separation) For every line l and for every 3 points A, B, C not on l,

(a) If A,B are on the same side of and B,C are on the same side of l, then A,C are on the same side of l.

(b) If A, B are on the opposite sides of l and B, C are on the opposite sides of l, then A, C are on the same side of l. Corollary:

(c) If A,B are on the opposite sides of l and B,C are on the same side of l, then A, C are on the opposite sides of l.

Betweenness Propositions

1. AB> [cut] BA>=AB and AB> [union] BA>=<AB>
2. Every line gives exactly two mutually exclusive half-planes.
3. (a) Given A*B*C and A*C*D then B*C*D and A*B*D
   (b) Given A*B*C and B*C*D then A*B*D and A*C*D
4. Line Separation Property
5. Given A*B*C then
   (a) AB[union]BC=AC
   (b) AB[cut]BC={B}
   (c) BA>[cut]BC>={B}
   (d) AB>=AC>
6. Pasch's Theorem
7. Given <CAB and a point D on the line BC, then D belongs to the interior of <CAB if and only if B*D*C.
8. If D is in the interior of <CAB then
   (a) so is all of ray AD except A itself
   (b) the opposite of ray AD is completely in the exterior
   (c) if C*A*E then B is in the interior of <DAE
9. Crossbar Theorem

Congruence Axioms (6)

1. Given segment *AB* and any ray with vertex *C*, there is a unique point *D* on this ray such that *AB* ≈ *CD*.
2. If *AB*≈*CD* and *AB*≈*DF* then *CD*≈*DF*.
3. Given $A*B*C$ and $D*E*F$, if *AB*≈*DE* and *BC*≈*EF* then *AC*≈*DF*.
4. Given <*D* and any ray *AB* there is a unique ray *AC* on each half-plane of the line *AB* such that <*BAC* ≈ <*D*.
5. If <*A*≈<*B* and <*A*≈<*C* then <*B*≈<*C*.
6. (SAS Criterion) If 2 sides and the included angle of a triangle are congruent to those of another triangle, respectively, then the two triangles are congruent.

Congruence Propositions

1. Segment Subtraction
2. Segment Ordering
3. Supplements of congruent angles are congruent.
4. All vertical angles are congruent to each other.
5. An angle congruent to a right angle is a right angle.
6. Given a line *l* and a point *P* there exists a line through *P* perpendicular to *l*.
7. ASA Criterion
8. Isosceles Triangle Theorem
9. Angle Addition
10. Angle Subtraction
11. Angle Ordering
12. SSS Criterion
13. All right angles are congruent to each other.

Continuity Axioms (2)

1. (Circular Continuity Principle) If a circle has one point inside and one point outside another circle, then the two circles intersect in two points.
2. (Archimedes' Axiom) Given segment *CD* and any ray *AB* there is a number *n* and a point *E* on this ray such that $n \times CD \approx AE >= AB$.

Parallelism Axiom (1)

(Hilbert's Parallel Axiom) Given a line *l* and a point *P* not on *l*, there is at most one line through *P* which is parallel to *l*.

Theorems in Neutral Geometry:

1. Alternate Interior Angle Theorem and its corollaries:
   (a) Two lines perpendicular to another line are parallel.
   (b) Given a line *l* and a point *P* not on *l*, there is a unique line through *P* which is perpendicular to *l*.
   (c) Given a line *l* and a point *P* not on *l*, there exists a line through *P* which is parallel to *l*.
2. SAA Criterion
3. Every segment has a unique midpoint.
4. Every segment has a unique perpendicular bisector.
5. Every angle has a unique bisector.
6. Given $\Delta ABC, AB>BC$ if and only if $<C > <A$.
7. Given ΔABC and ΔDEF with $AB \approx DE$ and $BC \approx EF$, then $AC < DF$ if and only if $<B < <E$.
8. Triangle Inequality Theorem
9. Saccheri-Legendre Theorem
10. If there is one triangle with angle sum = 180^o then a rectangle exists.
11. If a rectangle exists then every triangle has angle sum = 180^o.
12. If there is one triangle with angle sum $< 180^o$ then every triangle has angle sum $< 180^o$.

Note: Using Euclid's Parallel Postulate it can be proved that in Euclidean Geometry the angle sum of any triangle = 180^o. In Hyperbolic Geometry angle sum of any triangle always $< 180^o$ whereas in Elliptic Geometry $> 180^o$.

13. Euclid's Parallel Postulate is equivalent to each of the following statements:

(a) If two lines are cut by a transversal such that two interior angles of the same side have degree sum $< 180^o$ then the two lines intersect on this same side.

(b) Hilbert's Parallel Axiom

(c) If a line intersects *l* then it intersects any line which is parallel to *l*.

(d) The converse of the Alternate Interior Angle Theorem

(e) If $l1 \,//\, l2$ and $m \perp l1$ then $m \perp l2$.

(f) If $l1 \,//\, l2$ and $m1 \perp l1$ and $m2 \perp l2$ then either $m1 = m2$ or $m1 \,//\, m2$.

Hyperbolic Axiom (1)

I There exists a line *l* and a point *P* not on *l* such that there are at least two lines through *P* which are parallel to *l*.

Theorems in Hyperbolic Geometry:

1. There are no rectangles.
2. Universal Hyperbolic Theorem
3. For every line *l* and a point *P* not on *l*, there are infinitely many lines through *P* which are parallel to *l*.
4. The angle sum of any triangle < 180^o.
5. A A A Criterion

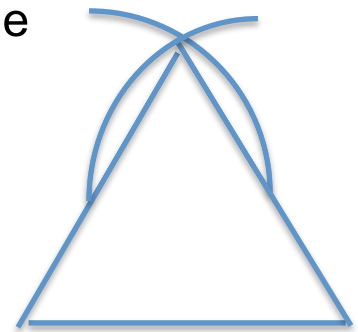
Note: It can be proved that, if Euclidean Geometry is consistent then

(a) so is Hyperbolic Geometry

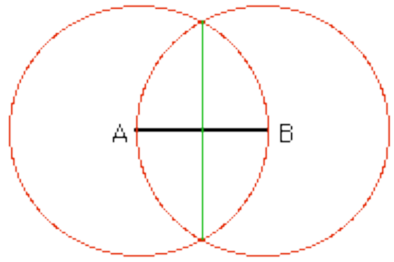
(b) the Parallel Axiom is independent from the other axioms.

Geometric constructions with a ruler and a compass

Equilateral triangle



Bisecting a line segment, and building perpendicular

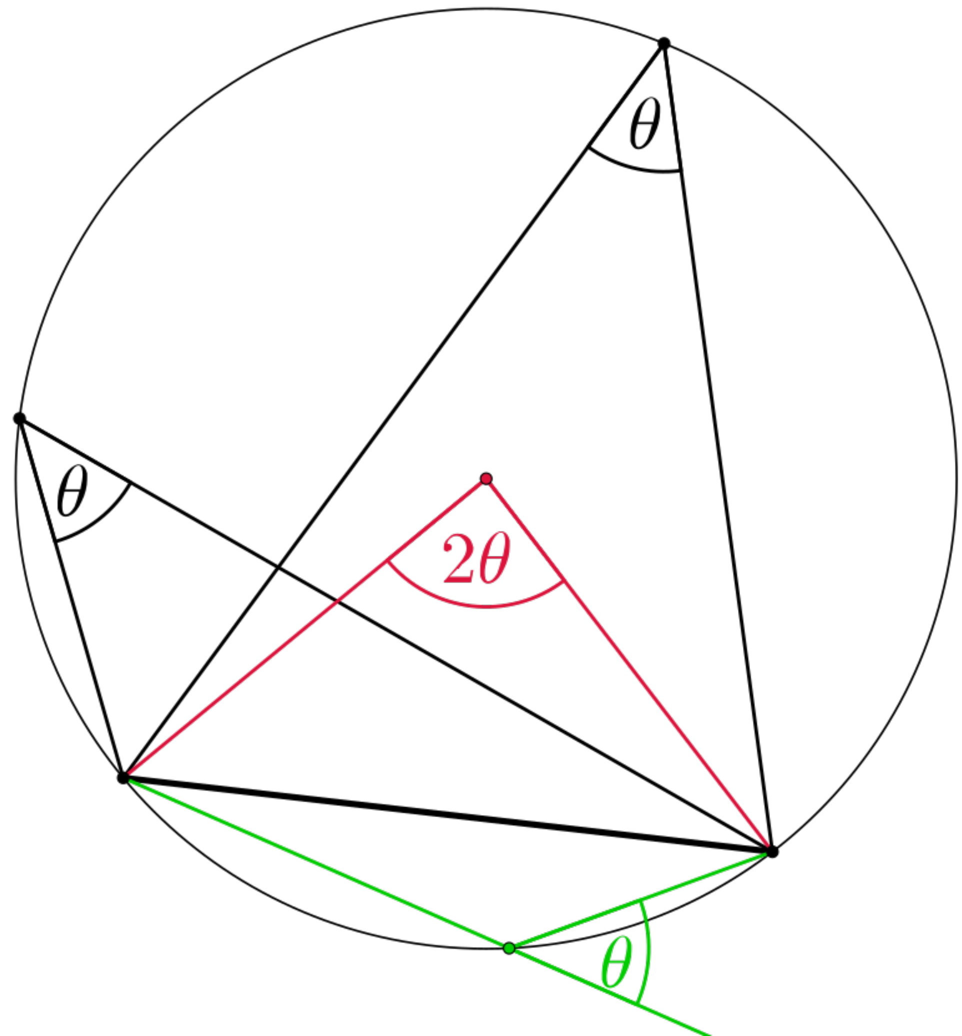


Laws in planar geometry

Angles from points on a circle lying on the same arc on the same side in a circle are equal

Angles from points on a circle on opposite sides of the arc are complementary

Angle lying on an arc from the center of the circle are double the angle from a point on the circle.

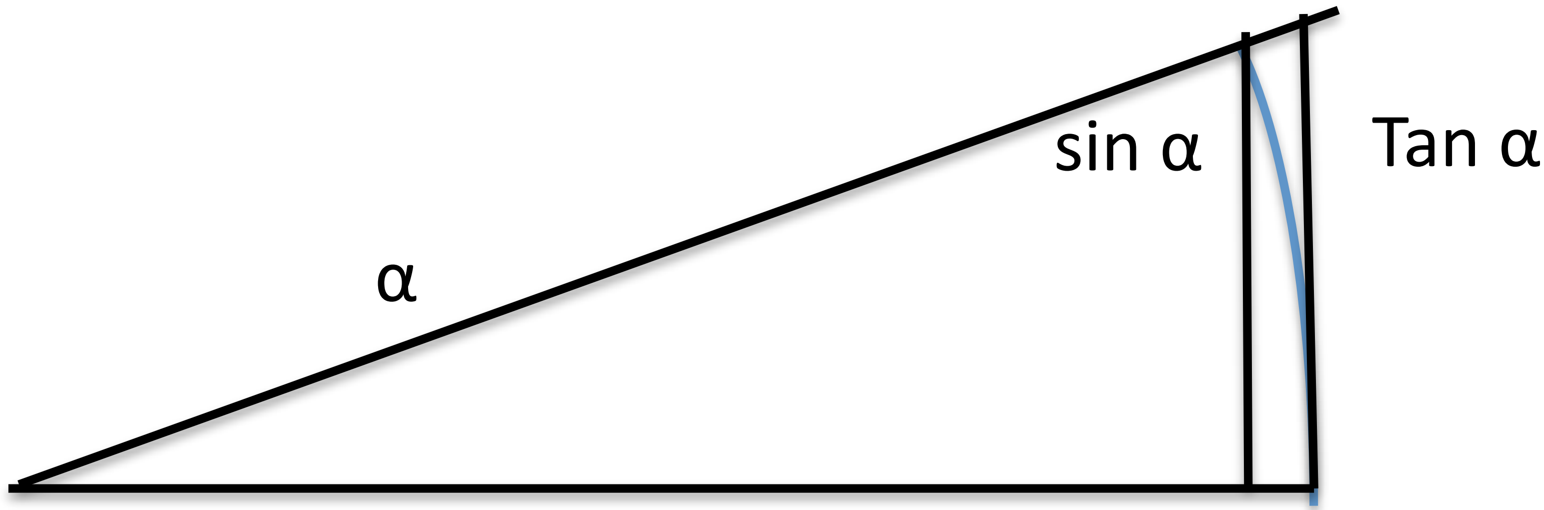


Aristarchus of Samosis 310–230 BC

Was probably first to claim that earth and other planets rotates around the sun,

For small angles: $\sin\alpha < \alpha < \tan\alpha$

$\sin\alpha/\sin\beta < \alpha/\beta < \tan\alpha / \tan\beta \quad 0° < \beta < \alpha < 90°$,



Eratosthenes 276–192 BC

Was a librarian in the Alexandria library.

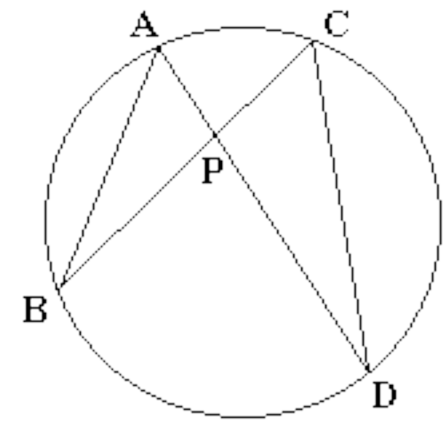
Based on similarity of triangles he measured the radius of earth (see in Astronomy)

Following are collected lemma from Greek geometry that links to Algebra

They show the desire of Greeks to carry computations by Geometric constructions

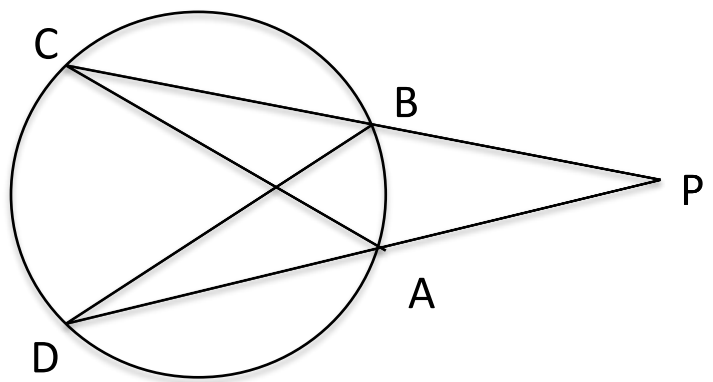
Apollonius intersecting chords theorem: AP*DP = BP*CP

P inside or outside the circle:



Proof: APB and CPD are similar triangles
The angles BAD & BCD equal: lie on the same arc BD
The angles ABC & ADC equal: lie on the same arc AC
Therefore: AP/CP=BP/DP=AB/CD
: AP*DP = BP*CP AP*CD=AB*CP BP*CD=AB*DP.

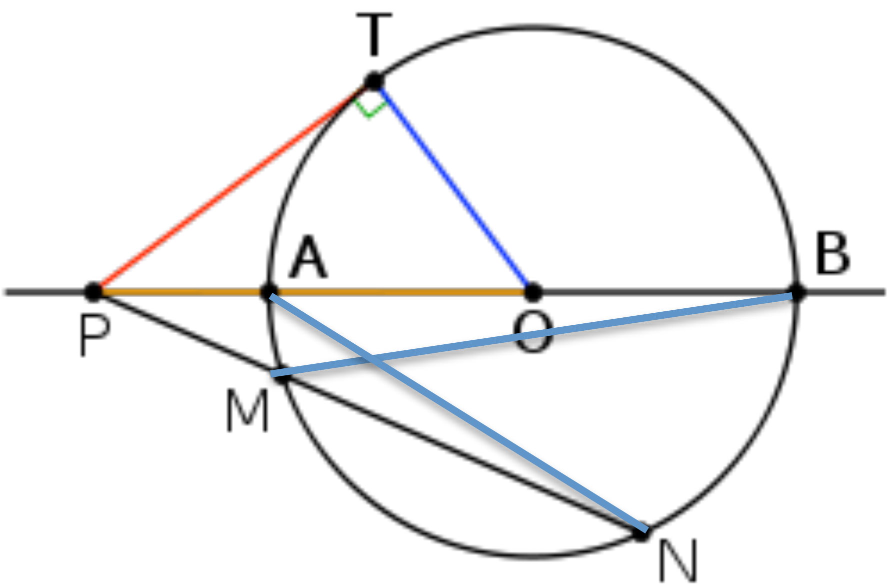
Q.E.D.



The "power", h, of a point P with respect to the circle with radius r

$h = PT^2 \quad s = PO \quad s - r = PA \quad s + r = PB$

$h = s^2 - r^2 = (s-r)*(s+r) = PT^2 = PA*PB = PM*PN$



Hipparchus of Rhodes 190–120 BC

Studies trigonometry. Proved that sqrt(2) is not rational,
(maybe based on Pythagoras)

Heron of Alexandria 10-75 AD

Studies area and volume of bodies

<u>Heron theorem for the area, A, of a triangle with edges a,b,c</u>

A = sort [s(s-a)(s-b)(s-c)] = sqrt [(a+b+c)(-a+b+c)(a-b+c)(a+b-c)] / 2
Where: s=(a+b+c)/2

Menelaus of Alexandria 70-130

Menelaus theorem: Given a triangle ABC and a traversing line EDF

Then:

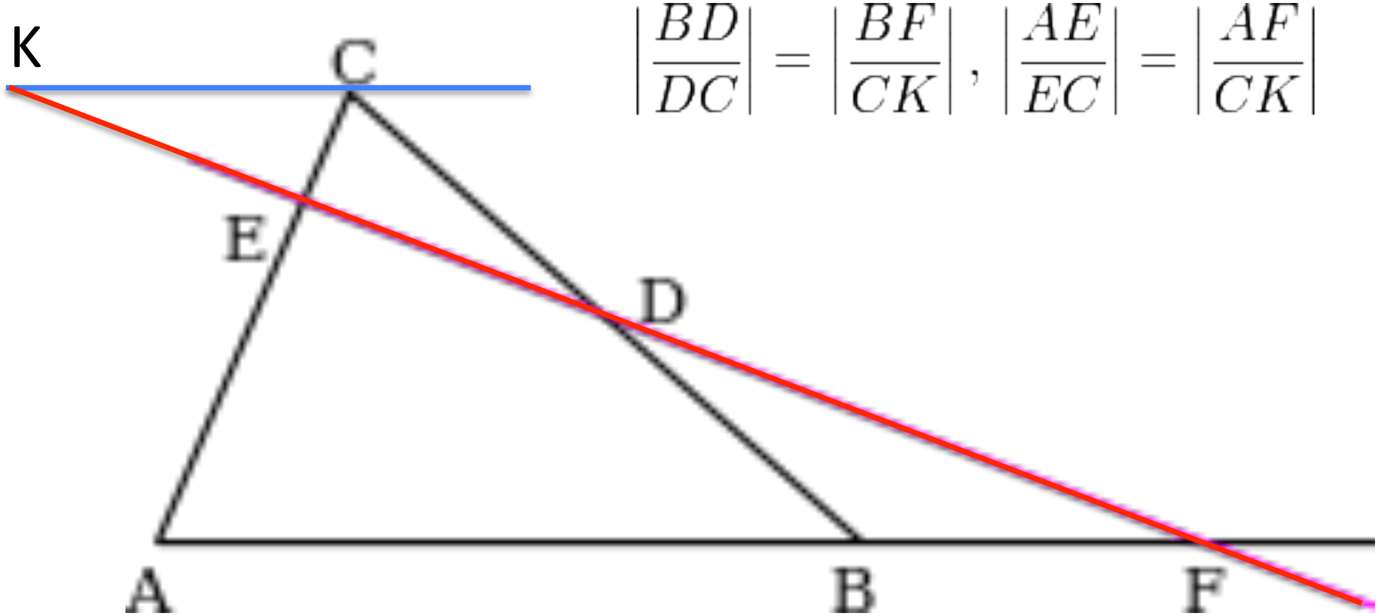
$$\frac{AF}{FB} \times \frac{BD}{DC} \times \frac{CE}{EA} = -1.$$

Or:

$$AF \times BD \times CE = -FB \times DC \times EA.$$

Proof: Build line CK parallel to AB. Triangles DCK and DBF are similar, therefore:

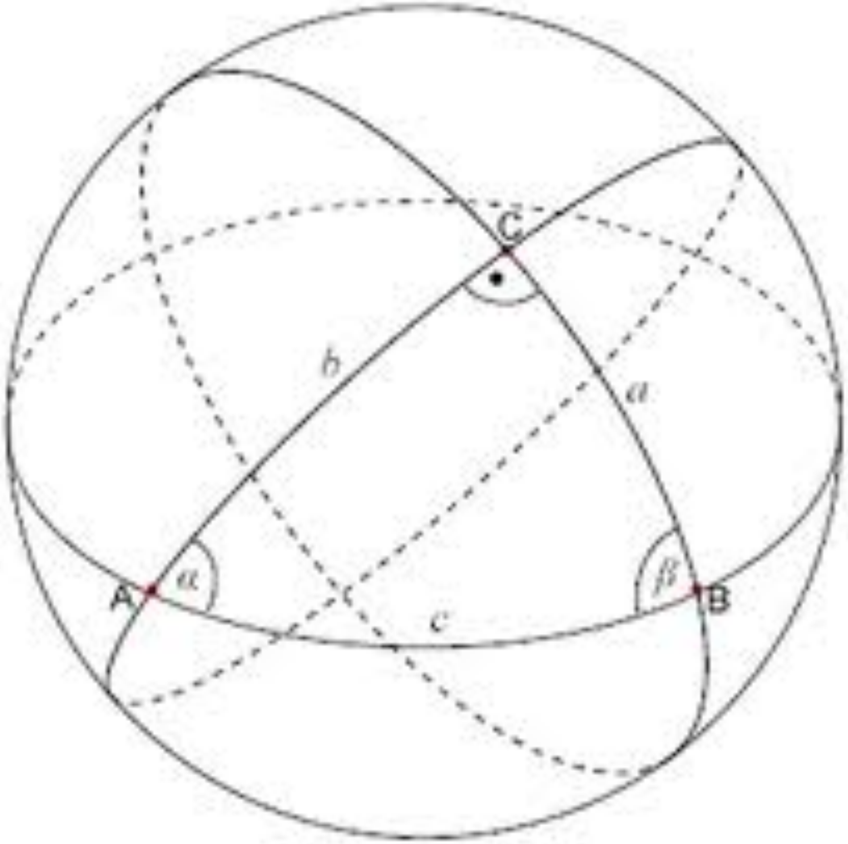
$$\left|\frac{BD}{DC}\right| = \left|\frac{BF}{CK}\right|, \left|\frac{AE}{EC}\right| = \left|\frac{AF}{CK}\right|$$



Menelaus of Alexandria 70-130

Studies spherical Geometry. He proved the planar version
as a lemma for proving the spherical analogue for a large circle cutting
triangle ABC:

$$\sin AD \cdot \sin BE \cdot \sin CF = \sin BD \cdot \sin CE \cdot \sin AF.$$

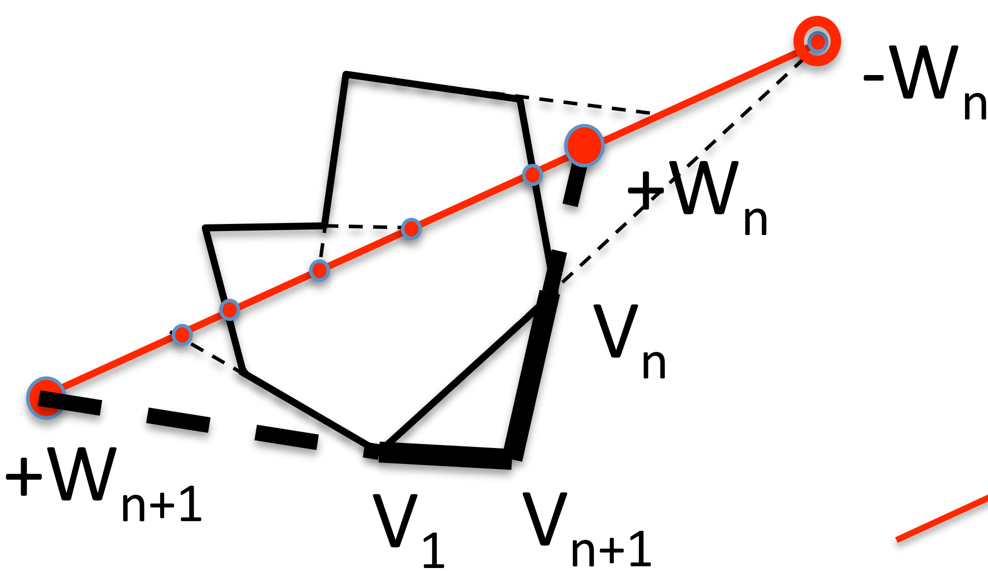


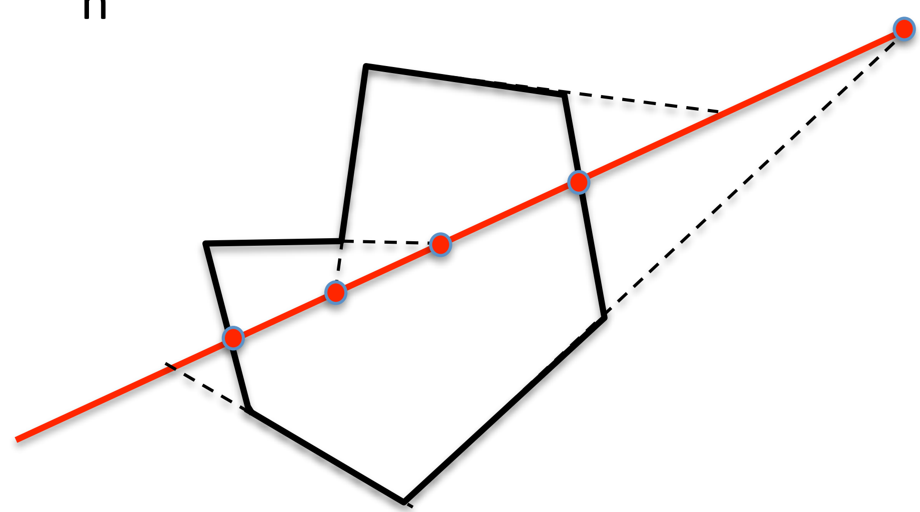
The theorem can be extended to n-sided polygon, vertices V_i and a line cutting it between V_i & V_{i+1} at a point W_i

The sign is positive if $W_i\ V_{i+1}$ & $V_i\ W_i$ are at the same orientation, and negative if opposites.

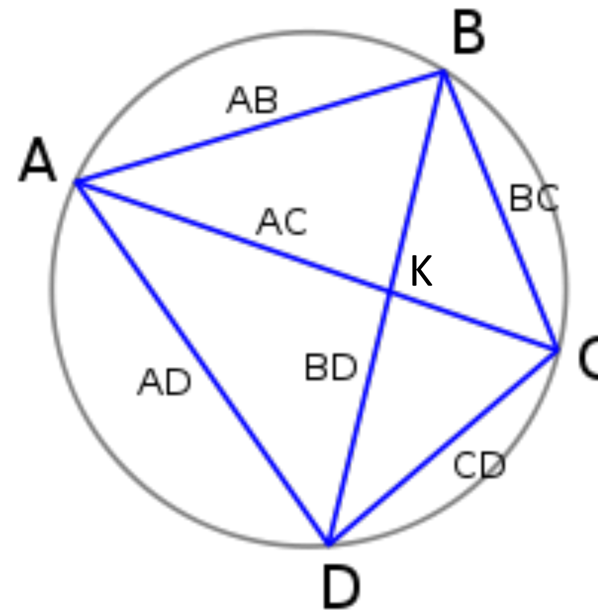
PROOF: BY INDUCTION.

$$\prod_{i=1}^{n}\left[\frac{V_i\ W_i}{W_i\ V_{i+1}}\right]=(-1)^n.$$





Claudius Ptolemy 85–165



Ptolemy's theorem:

$$|\overline{AC}| \cdot |\overline{BD}| = |\overline{AB}| \cdot |\overline{CD}| + |\overline{BC}| \cdot |\overline{AD}|$$

Proof: angles are equal: BAC=BDC   ADB=ACB

Similar triangles, therefore: ABK+CBK=ABC=CBD+ABD   CBK=ABD

Also similarity of ABK & DBC and ABD & KBC

Therefore: AK/AB = CD/BD,   CK/BC = DA/BD;

AK·BD = AB·CD,   CK·BD = BC·DA;

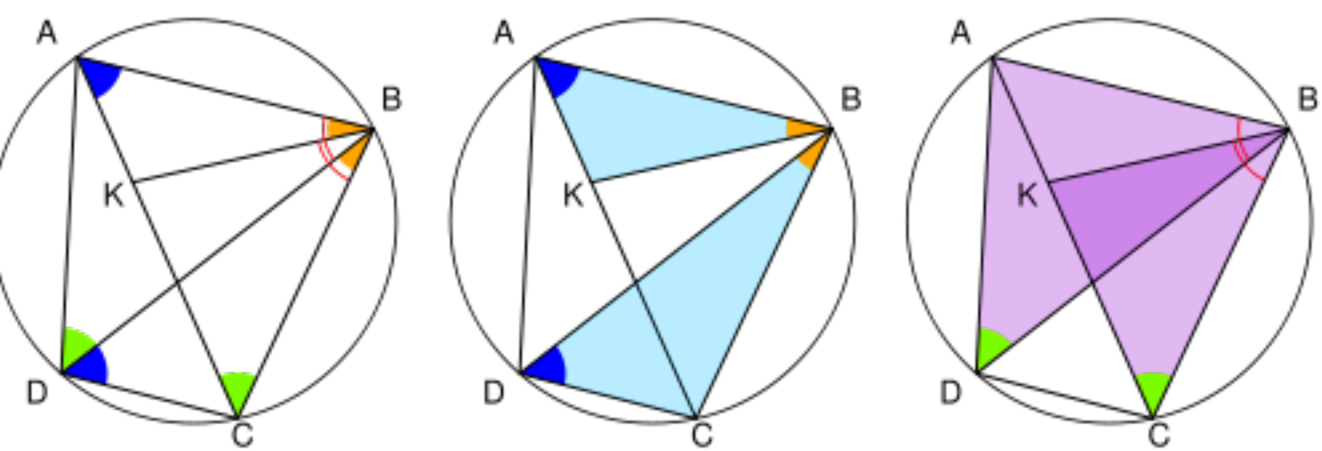
AK·BD + CK·BD = AB·CD + BC·DA;

(AK+CK)·BD = AB·CD + BC·DA;

AK+CK = AC,   -->   AC·BD = AB·CD + BC·DA

AK−CK=±AC

Q.E.D.

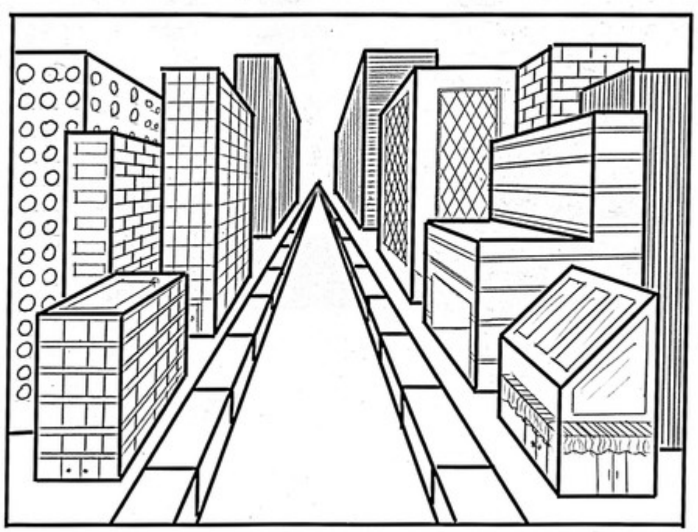


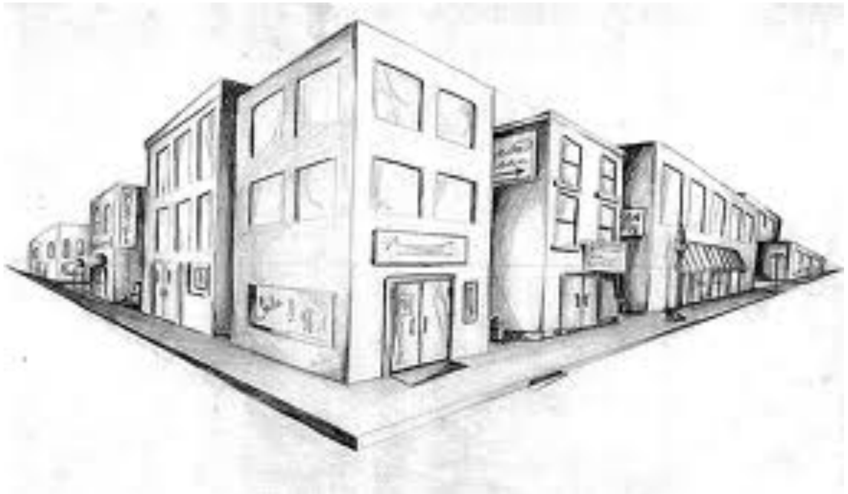
Pappus of Alexandria 290–350 BC

Last of the great Greek geometers
Famous contributions: Projection theorem, **Pappus-Guldin** surface area and volume for bodies of revolution (see above).

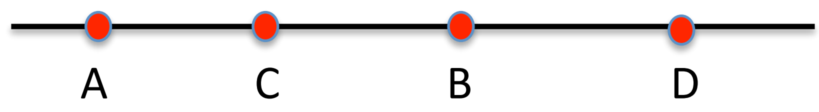
Projections are important for drawing three-dimensional world on two dimensional paper. Perspective: considers shrinking size with distance, and why parallel line meet at the horizon.

There are two major perspectives: 1 and 2 points of infinity where parallel lines meet.:
Perspective was re-discovered by **Brunelleschi** during the renaissance





Harmonic Range



If points A,B on a straight line are divided by in-between point C and external point D with equal ratio:

$$AC/CB = - AD/DB$$

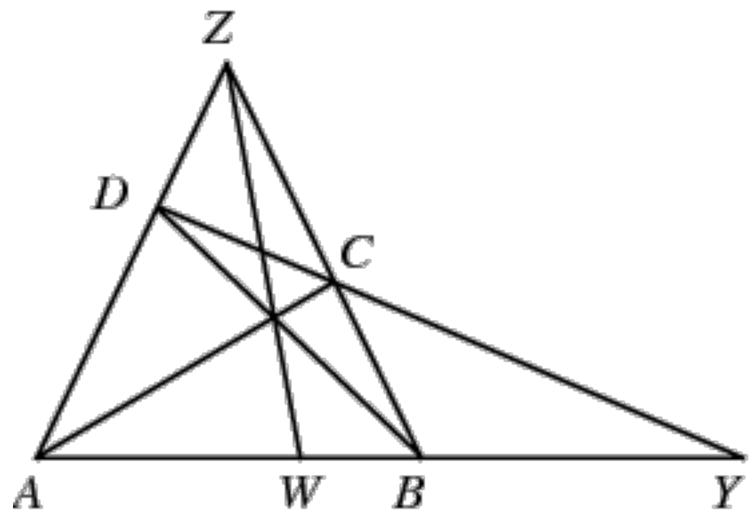
we say that AB are divided by CD Harmonically, since also:

$$CA/AD = CB/BD$$

ABCD are called “harmonic range”.
If O is the center of AB then:

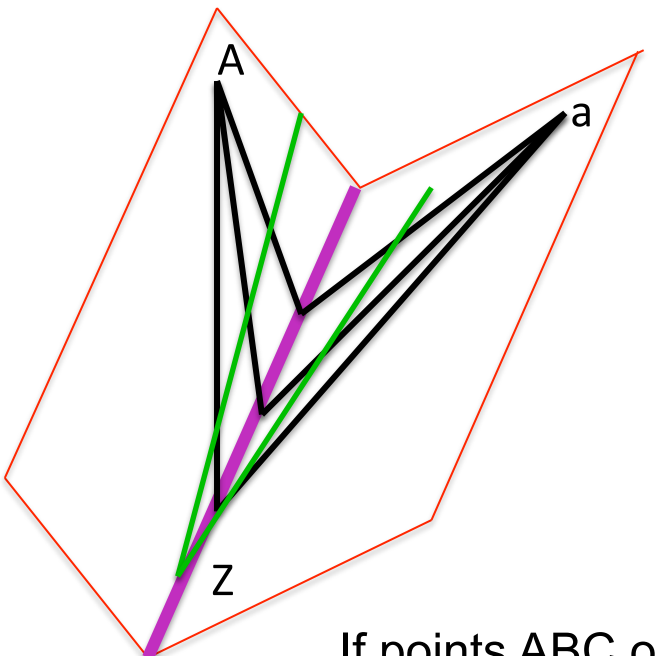
$$OB^2 = OC*OD$$

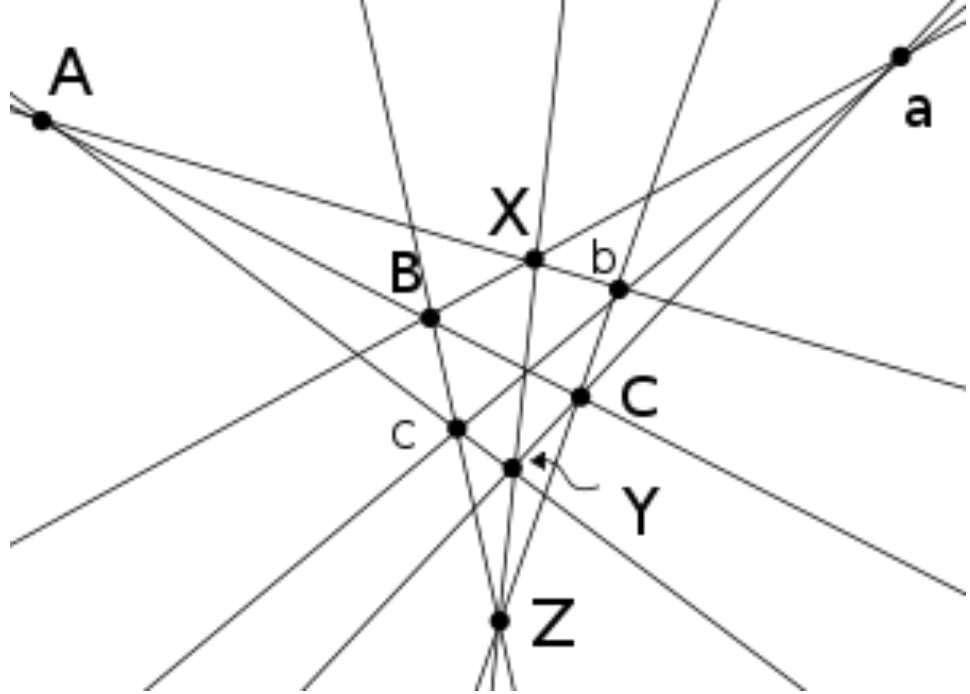
Pappus's Harmonic Theorem



If we join every point within a triangle to its vertices, AZD, the lines thus created cut the triangle edges at WDC. If we continue DC to meet the continuation of AB at Y then the points ABWY create an harmonic Range (see also Menelaus theorem)

Pappus's Hexagon Theorem



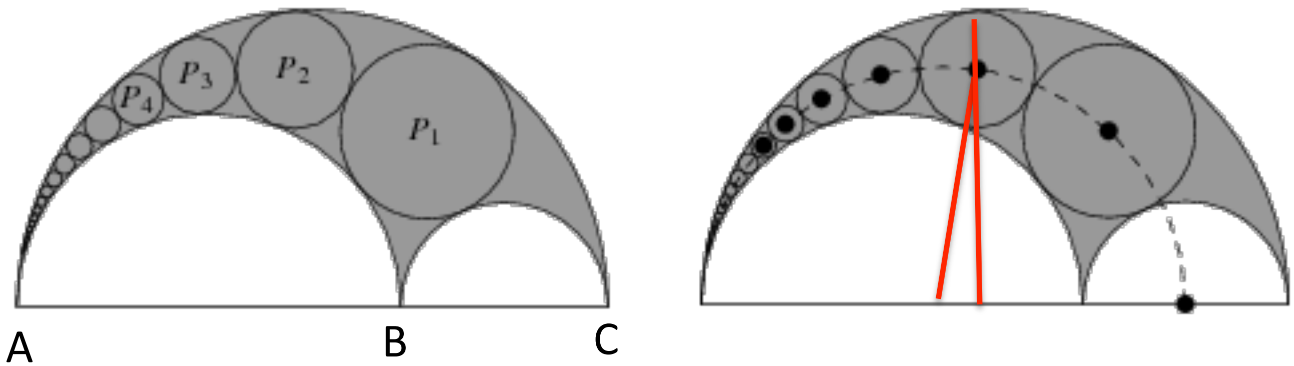


If points ABC on one line and abc on another line, the points XYZ created by the meeting of Ab&aB, Ac&aC Bc&bC are also on one line.
AbCaBc is called **Pappus Hexagon**.
The theorem is "self-dual": if we exchange the 9 points by 9 lines and vice versa.

The proof can be easily done as a projection from the drawing on two planes that dissect at XYZ

This theorem is a special case of Pascal's theorem for the cuts of two cone sections, here degrading into lines.

Pappus Chain



An infinite series of circles, P_n , tangential to the large half circle, U, with diameter AC=2R, the small circle V, with diameter AB=2r, and the previous circle, P_{n-1} . Centers of P_n are on an ellipse with foci at the centers of U and V, since the sum of distances from the two foci is $(R-r_n)+(r+r_n)=R+r$

Distance of the center of p_1 from the base (or its height above the base) is 2r, distance of p_2 is $4r_2$, and for p_n the distance is $2nr_n$.

If AC/AB=R/r=q then the coordinated of center of p_n are:

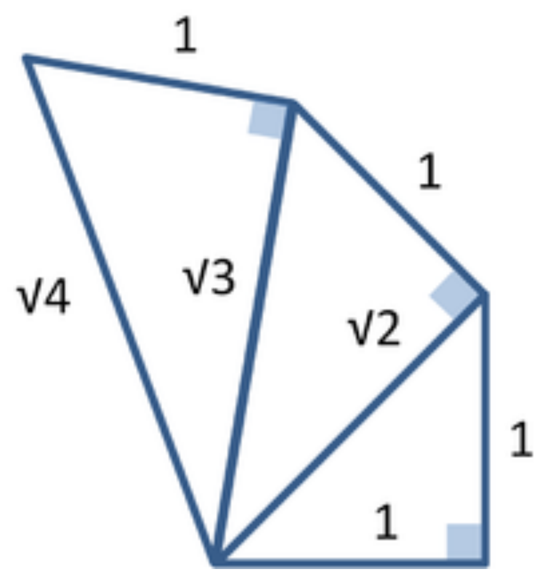
$x_n= q(1+q)/\{2n^2(1-q)^2+2q\}$; $y_n= 2nr_n= nq(1-q)/\{ n^2(1-q)^2+q \}$; $r_n = q(1-q)/\{2n^2(1-q^2)+2q \}$

Hypatia of Alexandria 370-415

Hypatia was the daughter of the mathematician **Theon**, and was the only woman scientist in the classical world. She was an editor of Ptolemy's astronomy book "Almagest" Euclid's "Elements" Diophantus' "Arithmetic" and Apollonius' "Cones".

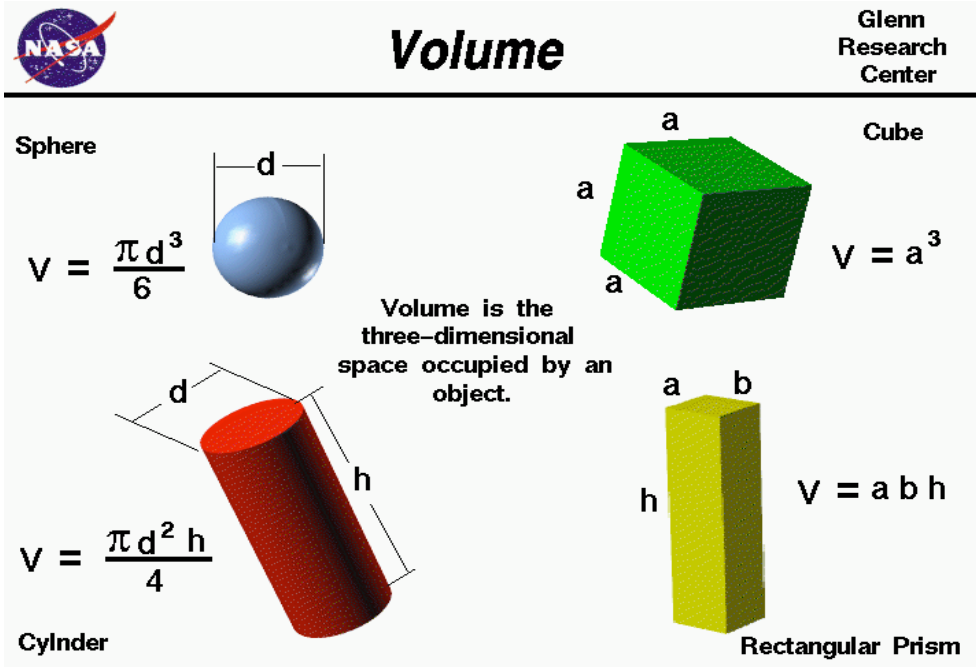
She was an esteemed mathematical figure and became the head of the Platonic school in Alexandria.
She was murdered by a Christian cult that considered science research paganism…

She presented a construction of a spiral based on the

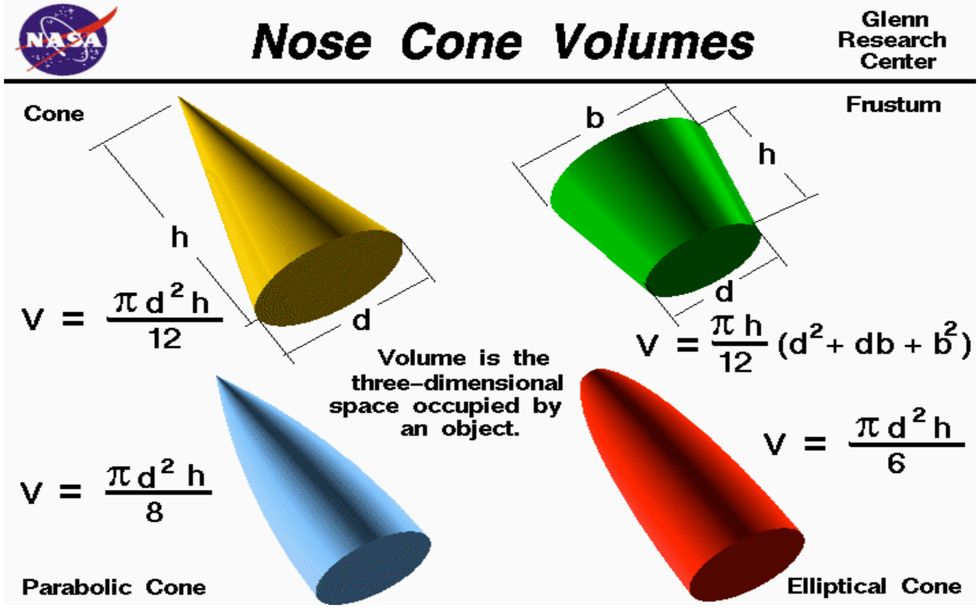


The design of an **astrolabe** is also attributed to Hypatia.

#



Volume
Glenn Research Center
Sphere
d
V = π d 3 / 6
a
Cube
a
a
V = a 3
Volume is the three-dimensional space occupied by an object.
d
h
a
b
h
V = a b h
V = π d 2 h / 4
Cylnder
Rectangular Prism



Nose Cone Volumes
Glenn Research Center
Cone
b
Frustum
h
h
V = π d 2 h / 12
d
d
Volume is the three-dimensional space occupied by an object.
V = π h / 12 (d 2 + db + b 2)
V = π d 2 h / 6
V = π d 2 h / 8
Parabolic Cone
Elliptical Cone

Although we are still in the ancient world, we end this chapter with two “appetizers” from two modern fields in Geometry:

Topology, and Fractals

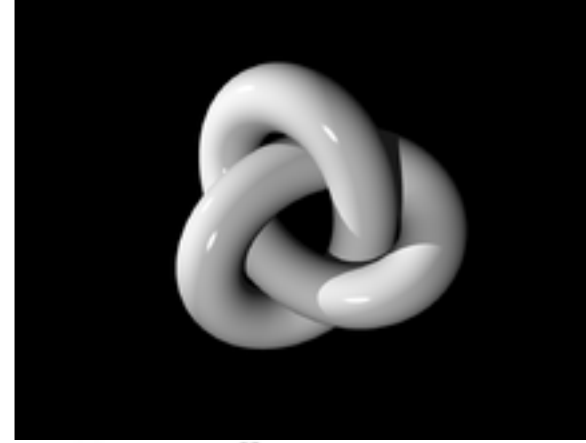
TOPOLOGY

Topology deals with properties of objects that are preserved under continuous distortions that do not join or detach parts of these objects.

Examples of topological properties: the number of holes in the body, and the number of knots.

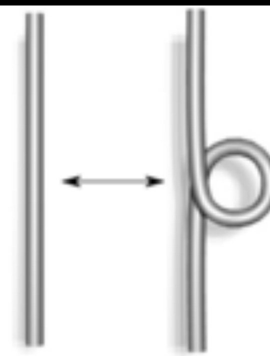
First topological property: **holes**. Here are objects with 0,1,2,and 3 holes.
Can you distort the cup to become a torus ?

Second topological property: knots.
Can we unwind the knot in the picture ?
Can we convert a knitted sweater into a thread?

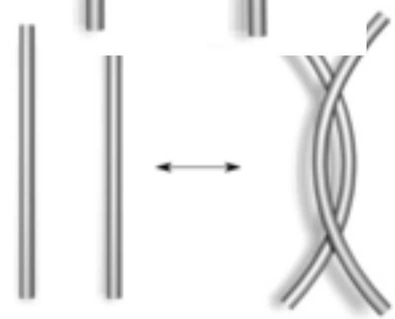


K. Redmeister proposes steps in unwinding knots without using the ends:

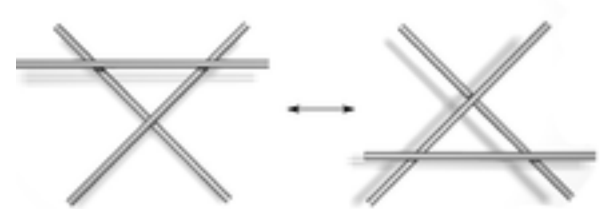
1st step: unwind an loop:



2nd step: unwind pair of junctions



3rd step: move a loop above or below a junction



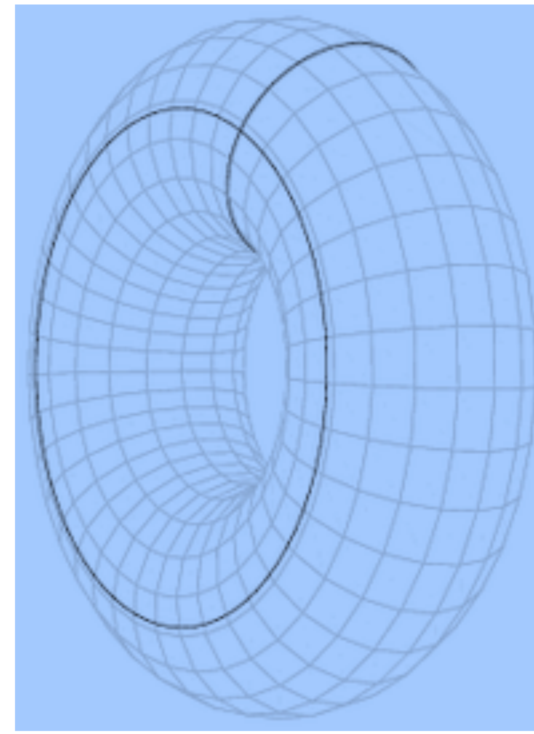
Torus versus a sphere

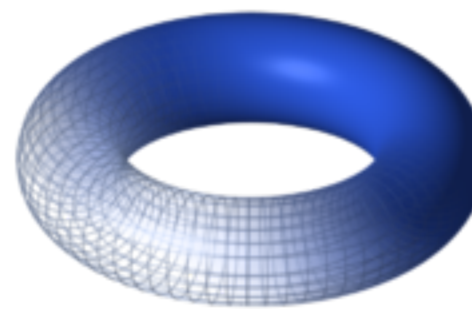
If we are a small ant, how can we discriminate them ?

A trip from the pole at two perpendicular directions: will the roads meet?

They meet for the sphere, but not for the torus (see two blue paths).

Can we scan the entire surface going always in one direction ?

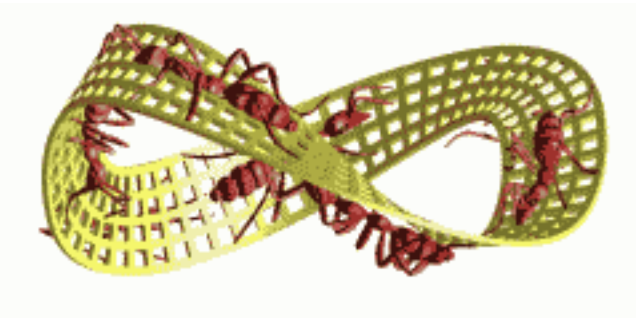




Möbius ring (or strip):

Called after **August Ferdinand Möbius in 1858**,
But described before by **Johann Benedict** Listing.

A trip on the surface <u>scans both sides</u>.
What happens if we cut the ring to half ?
A “straight” paper ring, when cut to half
along its width create two rings.
Möbius ring creates a new Möbius ring,
twice as long and half the width.



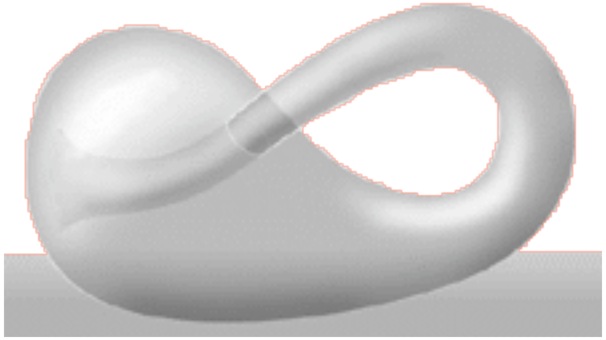
Escher, the painter, drew ants walking on Möbius strip.

Can you think of applications?

1. In old printers the ink ribbon was wound on the spools forming Möbius strip, so that both sides of the ribbon will be equally used.
2. Möbius strip forms a resistor with <u>low induction</u>.

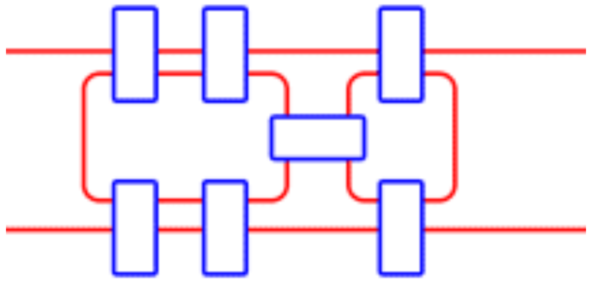
Klein bottle

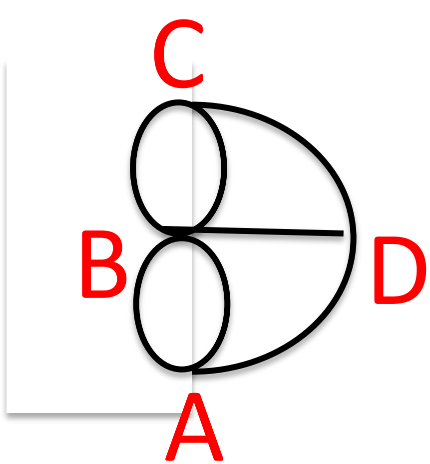
The three dimensional analogue of Möbius ring, because a trip on its surface reaches both surfaces of the bottle.

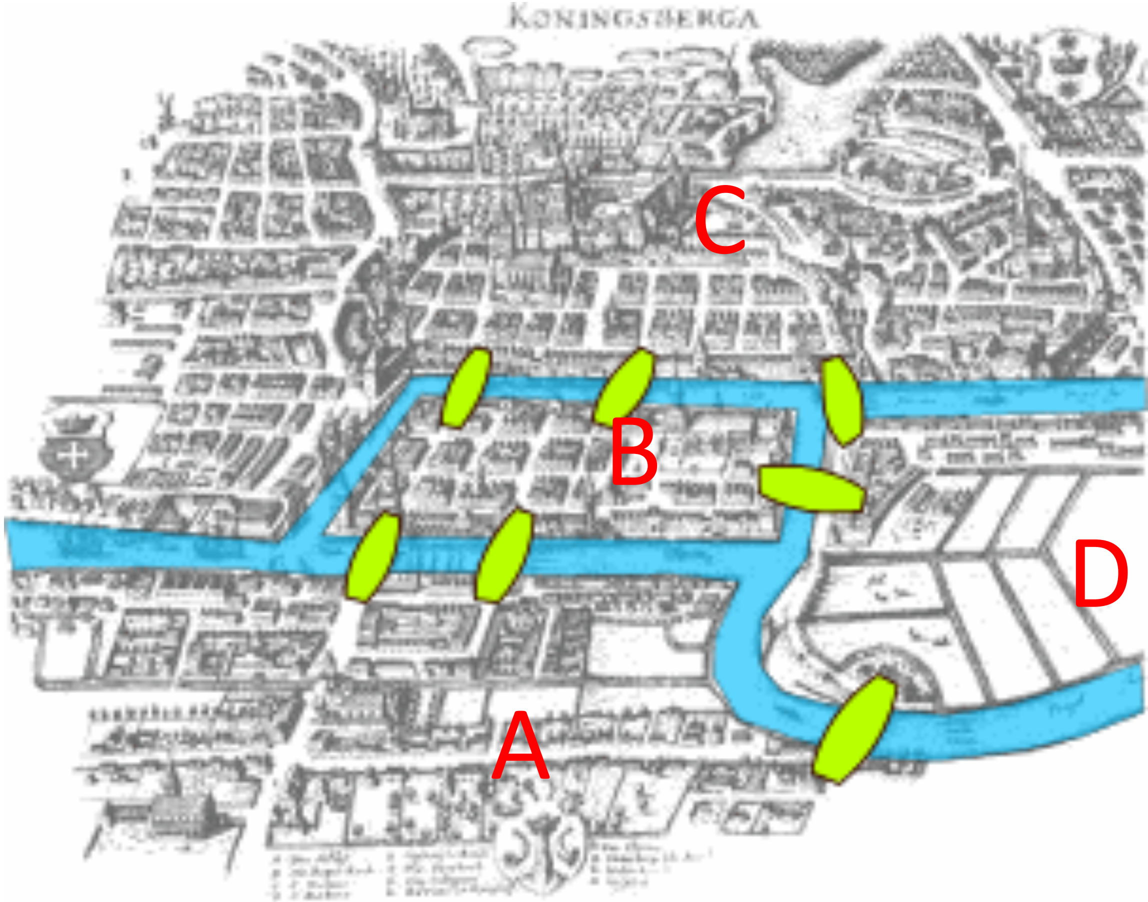


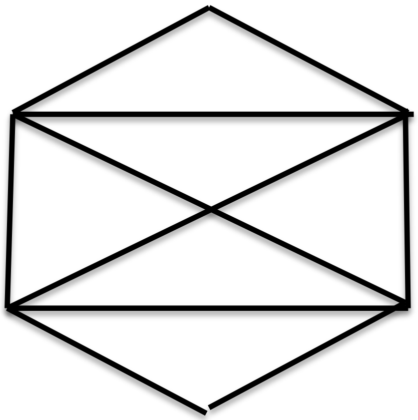
Topological properties of graphs

Euler 1786 solved the problem called "The seven bridges of Königsberg":
Can one visit all the parts of the city without crossing any of the bridges twice ?
Euler converted the problem into a graph, where the city regions are the vertices, and the bridges are arcs connecting them. One can draw a graph without lifting the pen from the paper only if none or two vertices intersect an odd number of arcs, These are the vertices to start and end the drawing. Proof: We must exit from every vertex we enter, and cannot do it on the same arc (bridge), thus other than starting and ending vertex, all must have even number of arcs.

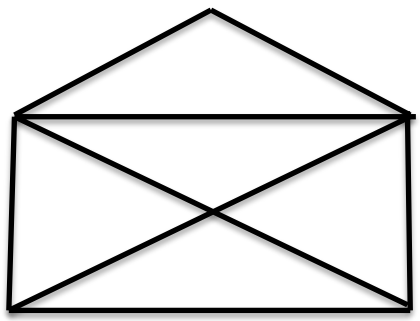
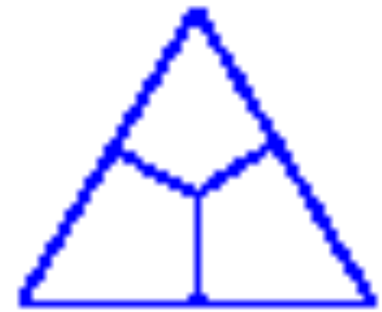
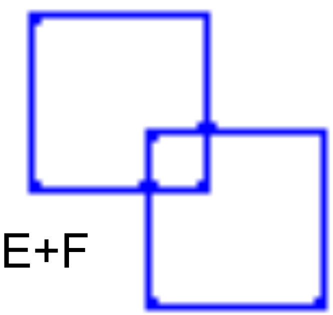




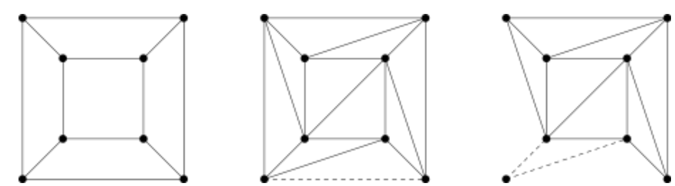






We can define another topological property: χ = V-E+F
Euler characteristic number=
number of vertices−number of edges + number of faces (including external faces).
For a planar graph χ=2.
Proof: by removing successively external faces with edges and vertices that bounds them.

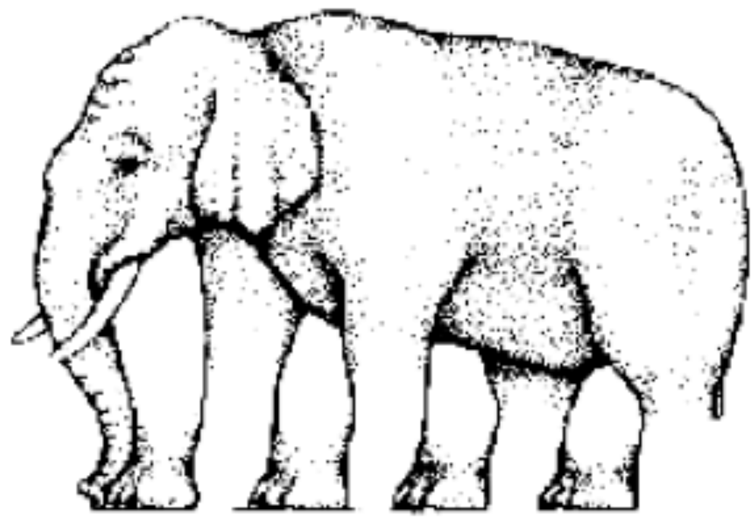


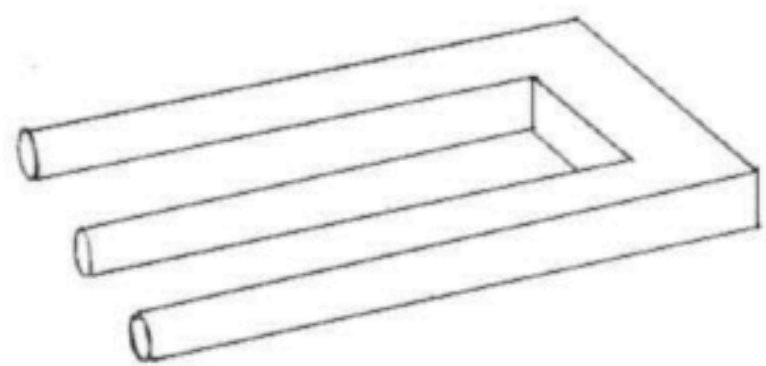
The four color theorem

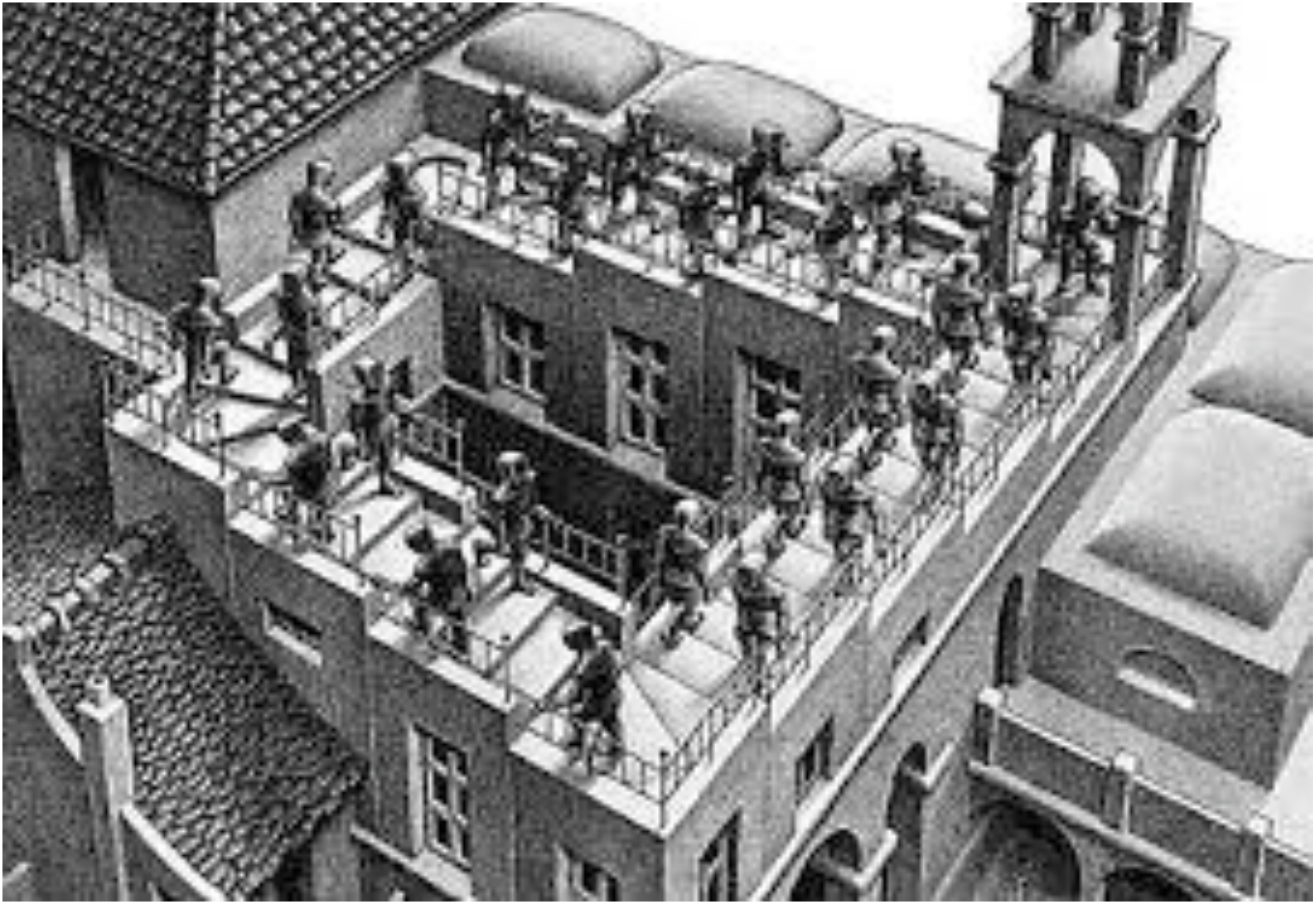
How many colors are required for painting a map so that neighboring countries will never have the same color? The problem was introduced by **Francis Guthrie, 1852**. **Alfred Kempe 1879** tried to show that 4-color map can be distorted to any map. His proof is wrong. Only in **1976 Kenneth Appel & Wolfgang Haken** used a computer to sort 1500 configurations painted in 4 colors that can be distorted to all maps.

The map can be transposed into a graph: Country=>vertex Border=>edge to another country

3D Topological shapes in 2D drawings:
False projection of three-dimensional bodies





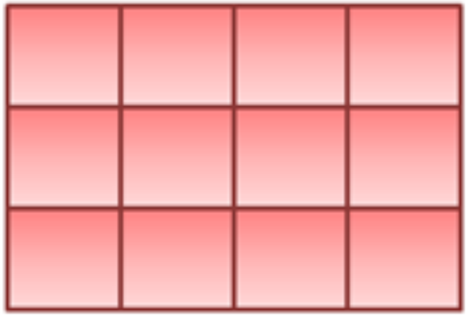


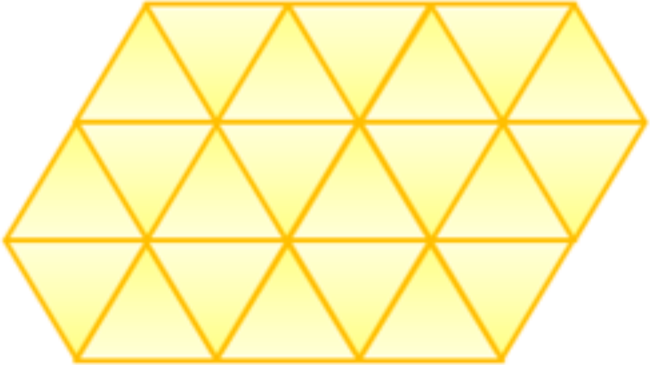
The stairs that only go up:
(**Maurits Cornelis Escher**, 1898-1972)

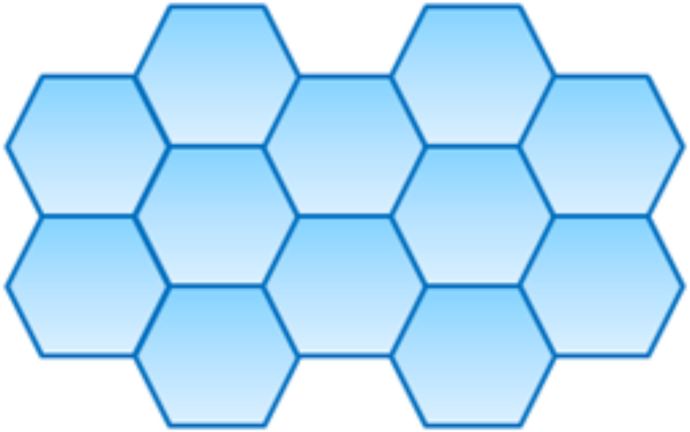
PACKING

Filling in a plane:

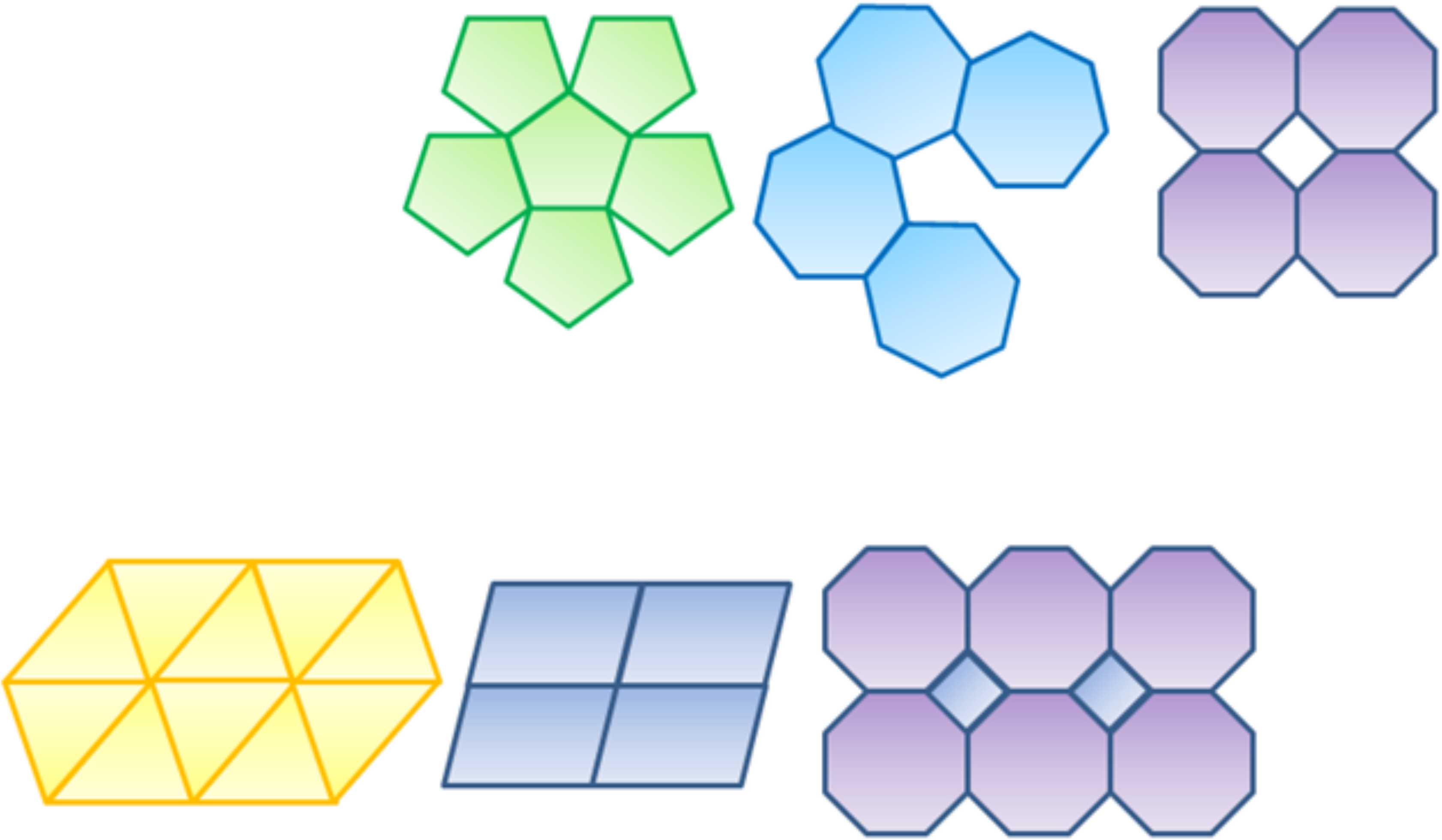
It is easy to understand that perfect filling in the plane can only be achieved for triangles, rectangles and hexagonals. However, if we want to pack these shapes in boxes, the "boundaries" will impose empty areas.







Packing of regular polygons other than 3,4,6 cannot fill the plane,
But we can do better with irregular polygons.



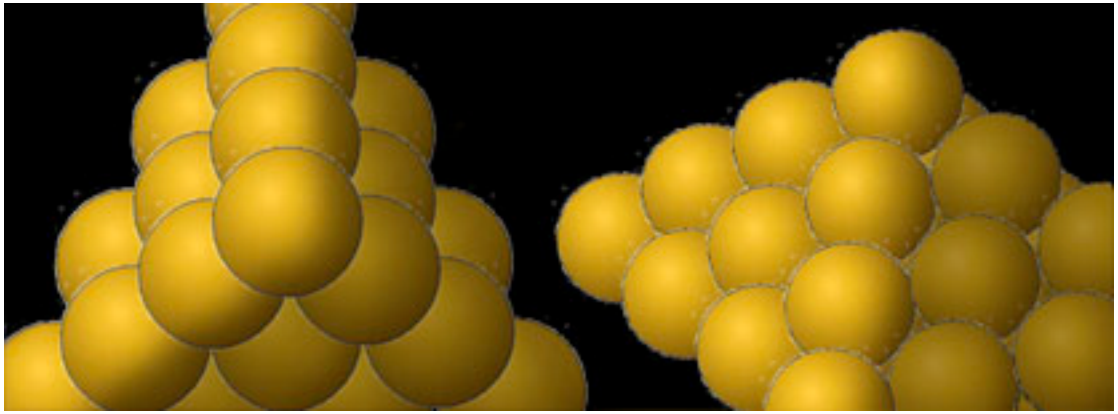
Filling space:

We can fill volumes with cubes, or rectangles.
Are there other non-regular space-filling shape? (hint: divide a cube into 4 Daltons)

Johanes Kepler 1606 studies how to best pack spheres.
Packing in a "cubic lattice" fills ~52% of the volume
Packing in "hexagonal lattice" fills 74% of the volume
Sir Walter Raleigh applied the problem for storing cannon balls in a ship.

How can we pack plums in a can? How to pack larger peaches ?
How is best to pack large watermelons in boxes ?

Computer simulation was carried by **Thomas Hales & Samuel P. Ferguson 1998** and considered boundary conditions: Best packing depends on the box …



FRACTALS

FRACTALS

Definition: a continuous function that has no derivative at any point.

First, simple behavior in "integer" dimensions:

One dimension: How many red sticks will match into one unit length?

Stick=1 N=1

Stick=1/2 N=2

Stick=1/3 N=3

Stick=ε N=ε^{-1}

Two dimensions: How many blue squares Will match into a unit square?

Square=1 N=1

Square=1/2 N=4

Square=1/3 N=9

Square=ε N=ε^{-2}

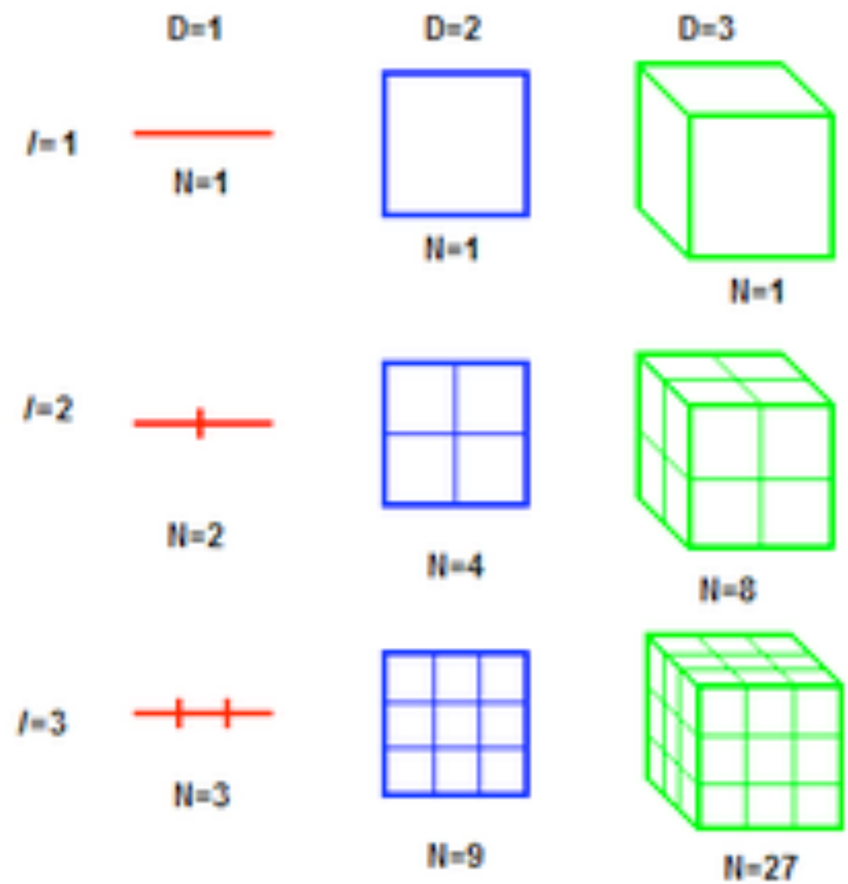
Three dimensions: How many green cubes will match into a unit cube ?

Cube=1 N=1

Cube=1/2 N=8

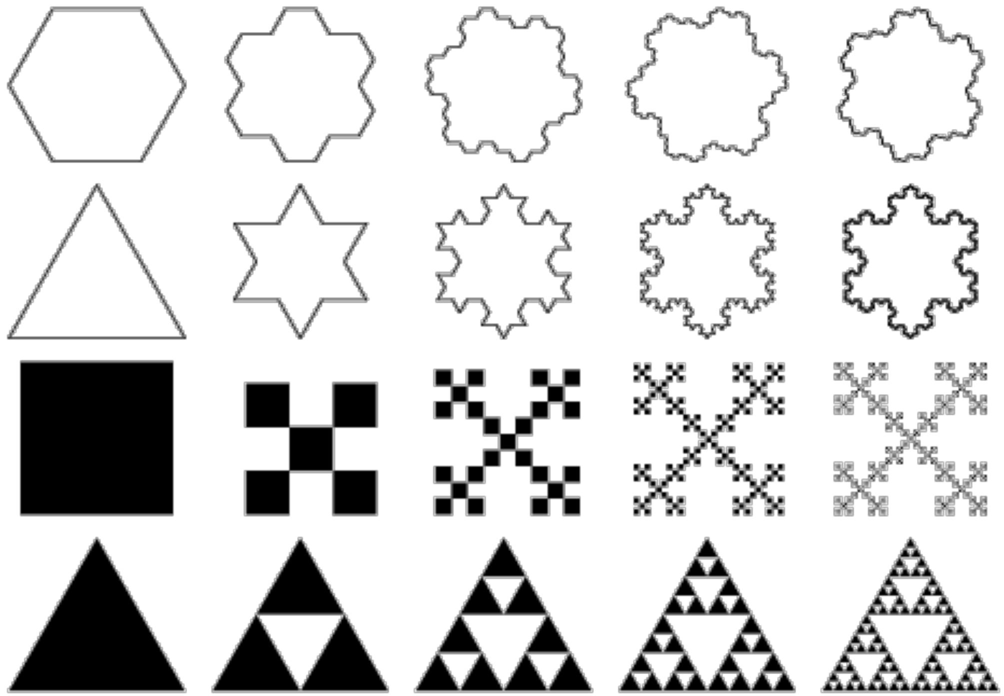
Cube=1/3 N=27

Cube=ε N=ε^{-3}



But we can create shapes with properties scaled by “fractional dimension”.

The example here is due to **Helge von Koch,** therefore called “Kock’s snow flakes”



What is the perimeter of the triangular snow flakes? $3*(4/3)^n$

And the area? $1/5*[8-3*(4/9)^n]$

Thus the perimeter of the snow flakes diverge, but the area converge to 8/5

What is the total black area of the square? $(5/9)^n$

And the triangle $(3/4)^n$

Thus for these cases the areas of both square and triangle diminish to zero.

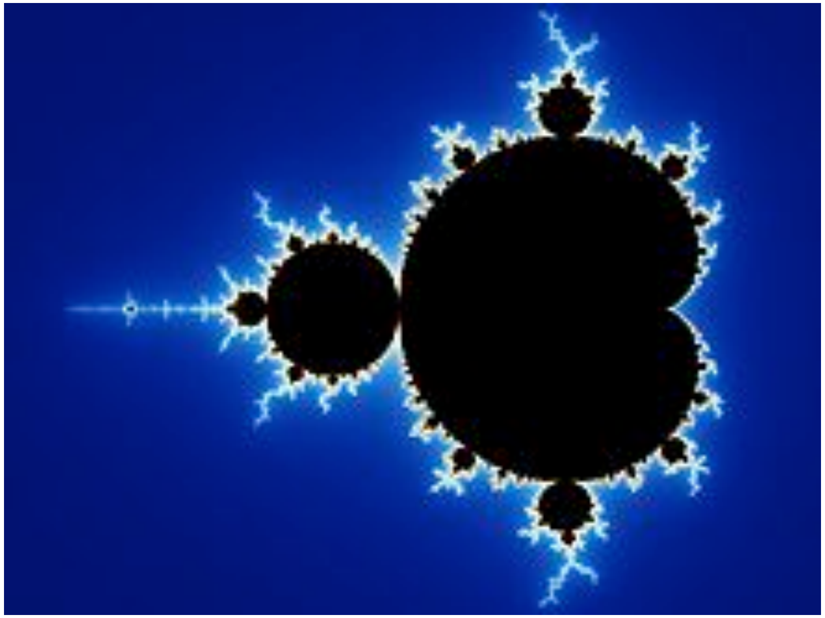
For “normal” shapes, area $\propto$ size^2 and perimeter $\propto$ size^1. For fractals they are proportional to a fractional power.

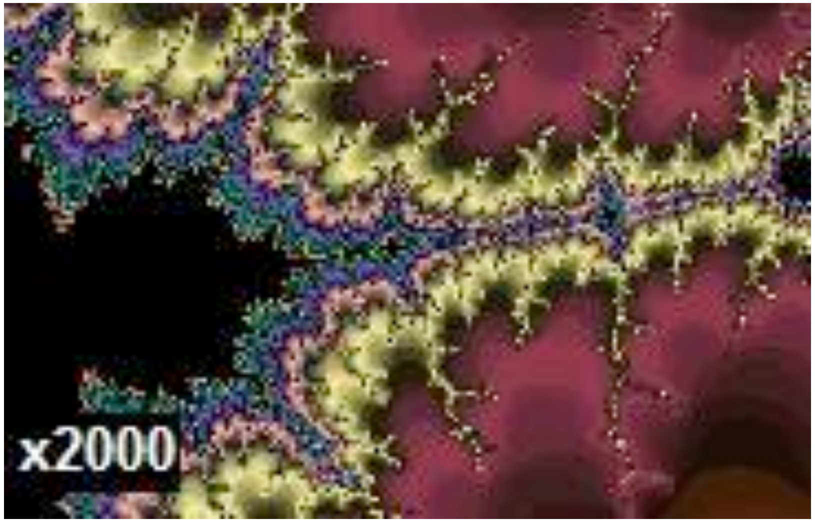
What is the surface area of a mountain?
It depends on our “ruler”: if we consider rocks, caves, stones and sand grains.

Similarly, the length of the coast line of England increase with our yardstick length:

11.5 x 200 = 2300 km   28 x 100 = 2800 km   70 x 50 = 3500 km

Benoît Mandelbrot created beautiful patterns by computers, based on fractal algorithms





APPENDIX

High School Geometry

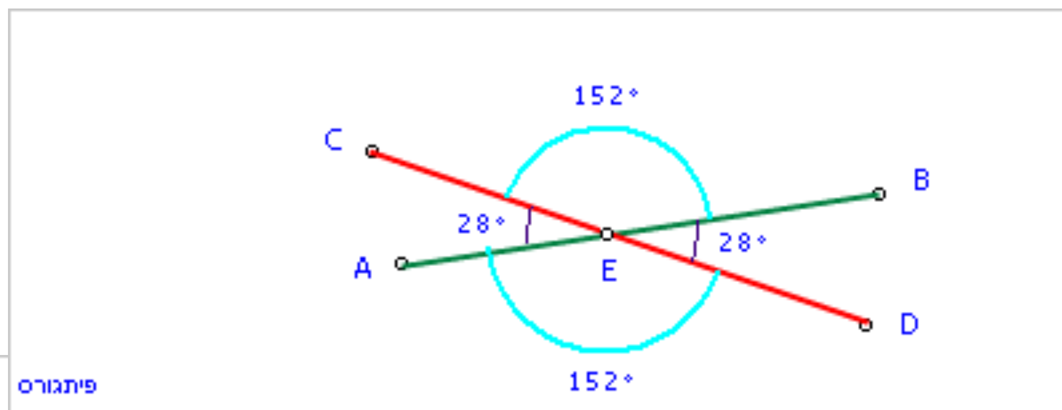
Brought here to get a flavor of the achievements of Greek geometers during a few hundred years

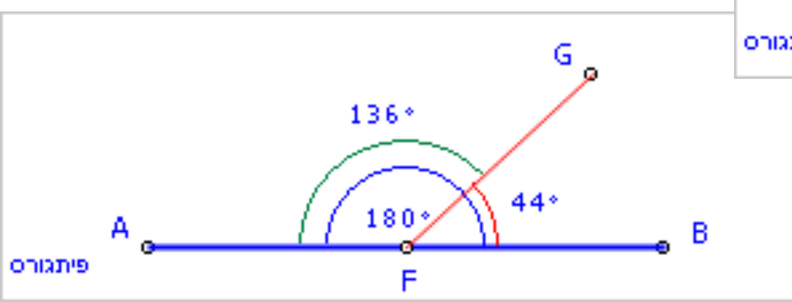
Summary: Geometry for high schools

Planar and three-dimensional geometry, Analytical geometry, Trigonometry

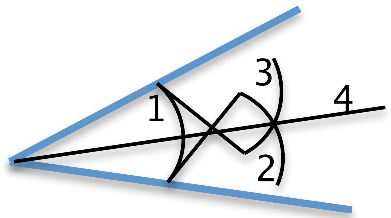
Lines and Angles

* Through two (non identical) points only one line can pass
- Two lines have one intersecting point at most (if parallel – no intersection)
- A point on a line divides it into two rays
- Two points on a line define a segment
- Definitions of addition and subtraction of segments
- Opposite angle between two lines with common vertex are equal
- Nearby angles complement each other to 180^o

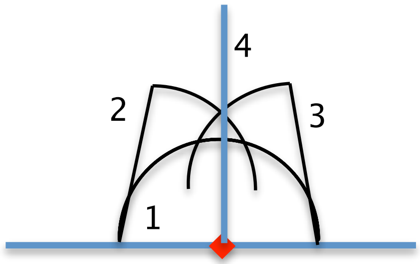




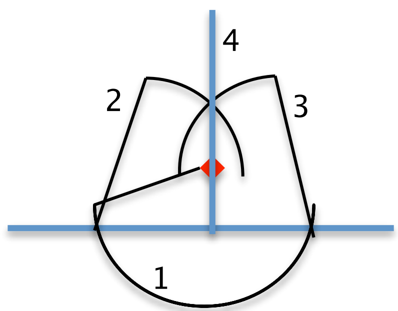
Constructions with ruler and compass



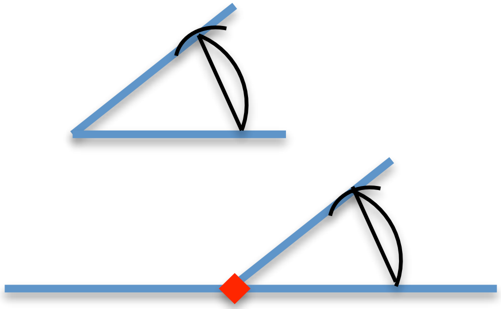
Bisect an angle,



Perpendicular to a line at a point on the line



Perpendicular to a line at a point off the line



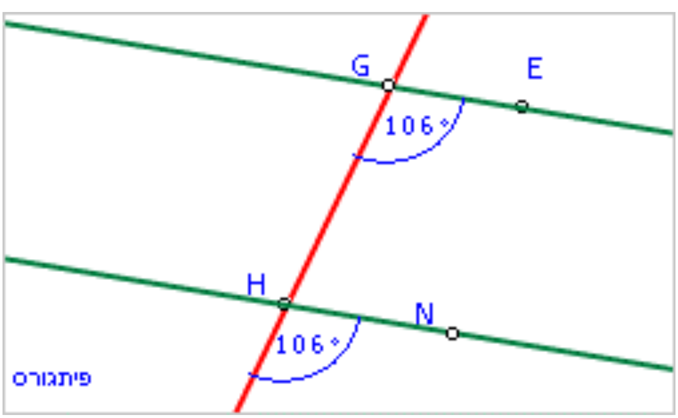
Copy an angle on a line

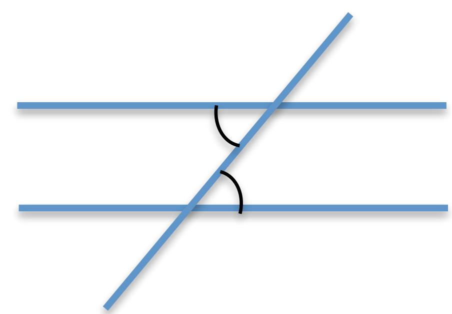
Parallels

Axiom: two parallel lines do not meet

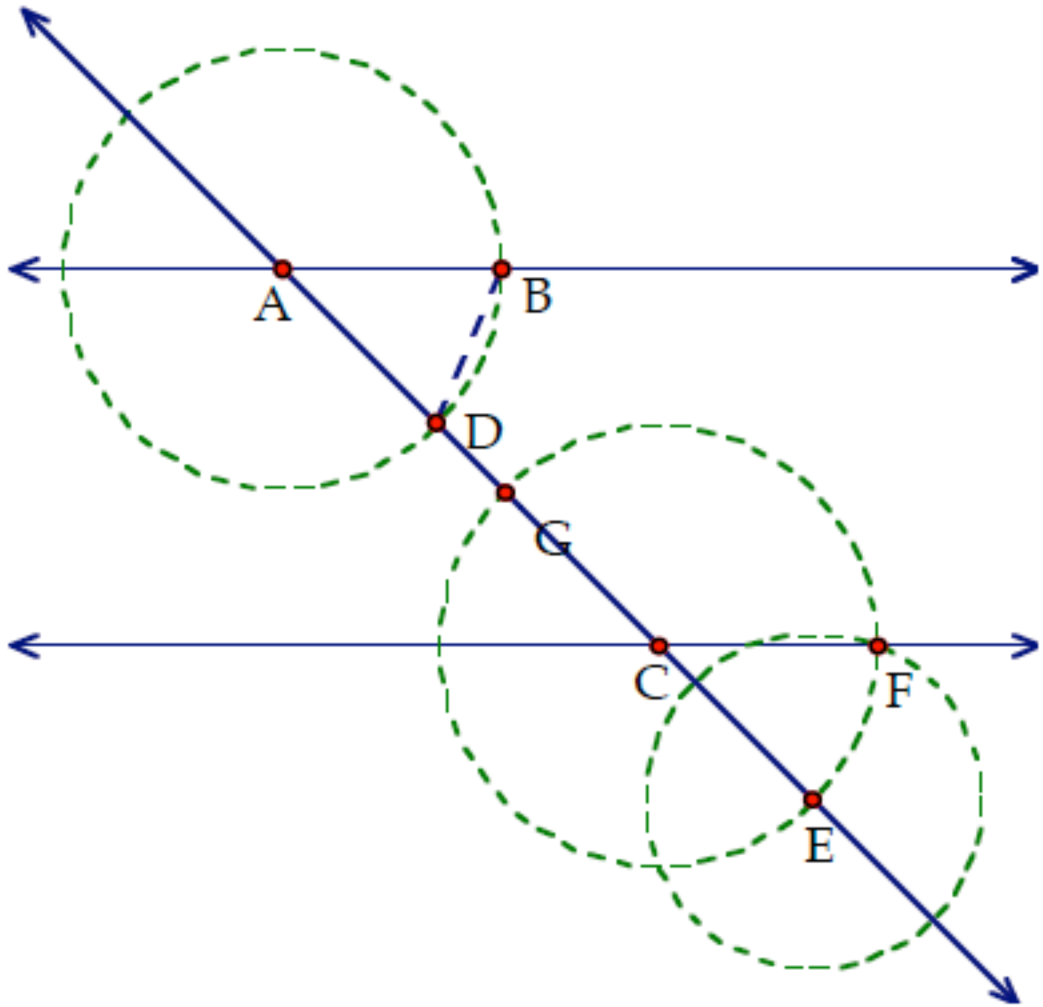
Axiom: only one parallel to a line pass through a point

Alternating angles of a line crossing two parallels are equal,
and vice versa:
If a line intersect two other lines at equal angles – the two line are parallel





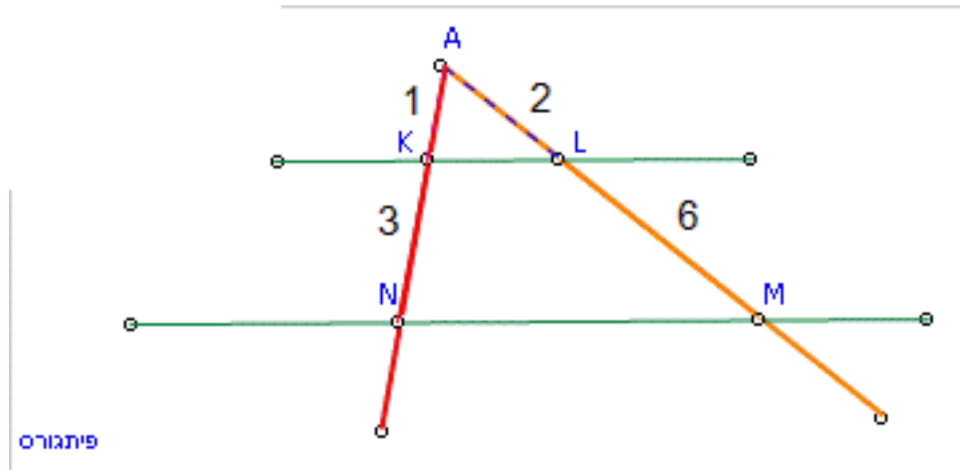
Construct a parallel to a given line that pass through a given point



In a non-equilateral triangle, the longer edge lies opposite the larger angle, and vice versa.

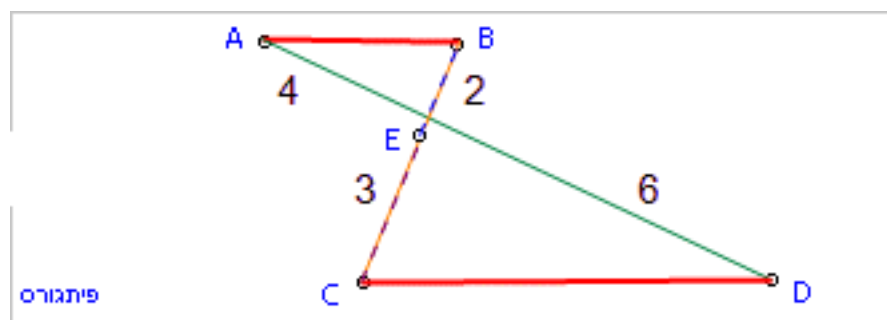
Thales theorem

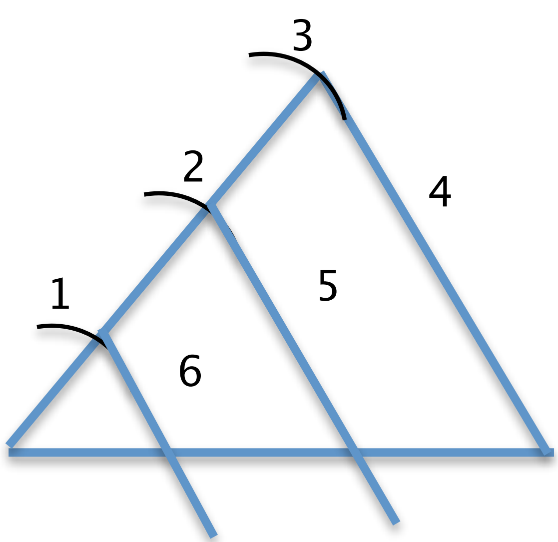
Two parallel line cutting the edges of an angle create proportional segments, And vice versa.



Extended Thales theorem:

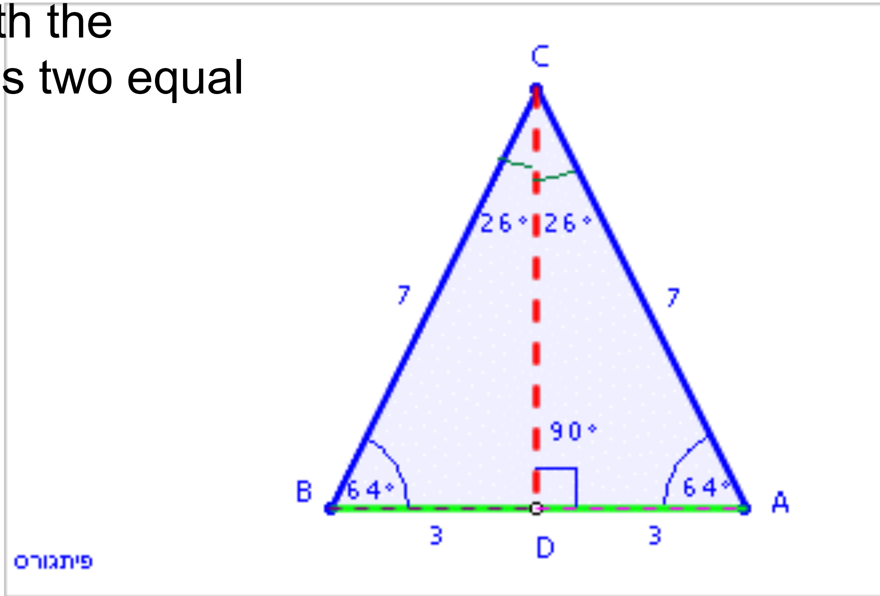
A line parallel to one of a triangle edges Cut the two other edges (or their continuation) in proportional segments





In a triangle, edges opposite equal angles have equal length,
And vice versa.

In triangle with two equal edges, the head angle bisector, the middle to the edge, and perpendicular to the base center coincide.
And vice versa: If the angle bisector coincide with the perpendicular to the base center, the triangle has two equal edges.
Same if bisector and middle edge coincide,
Or if middle edge and perpendicular coincide.

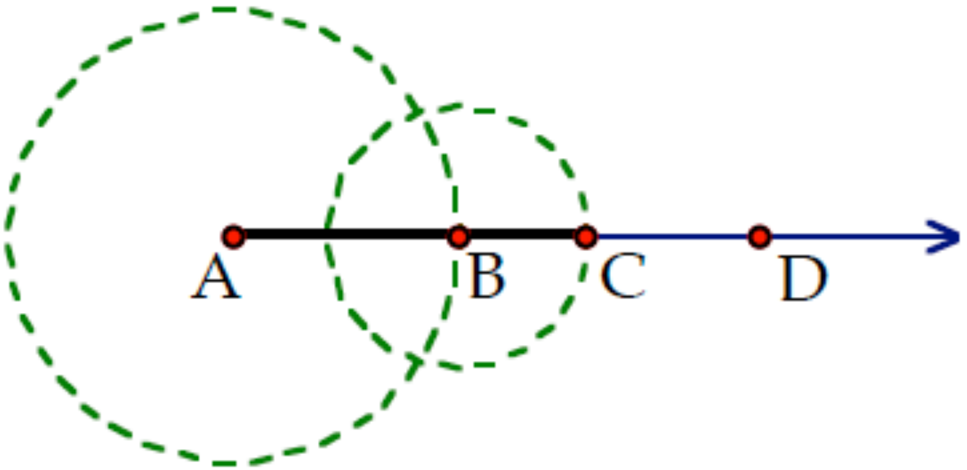


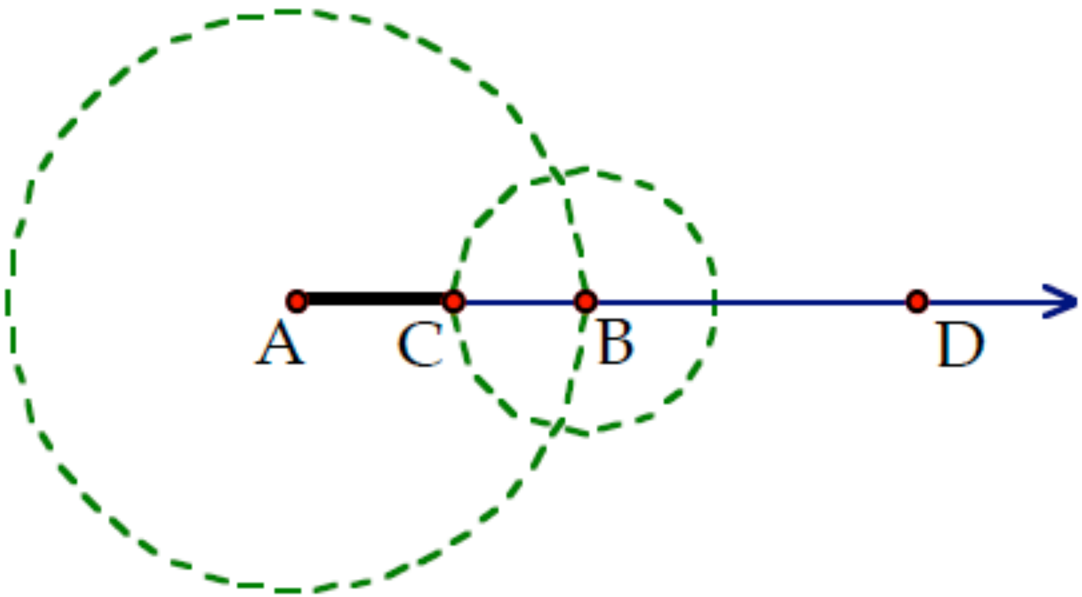
The sum of two edges in a triangle is larger then the third edge

Field's Parallae ruler
(named after Captain William Andrew Field 1790/1 – 1870)

Another kind of tool (in addition to ruler, compass and "rotating ruler" or Neusis) for geometrical constructions, providing solutions to problems that cannot be solved by compass and ruler alone.
Was used for locating ships in navigation.

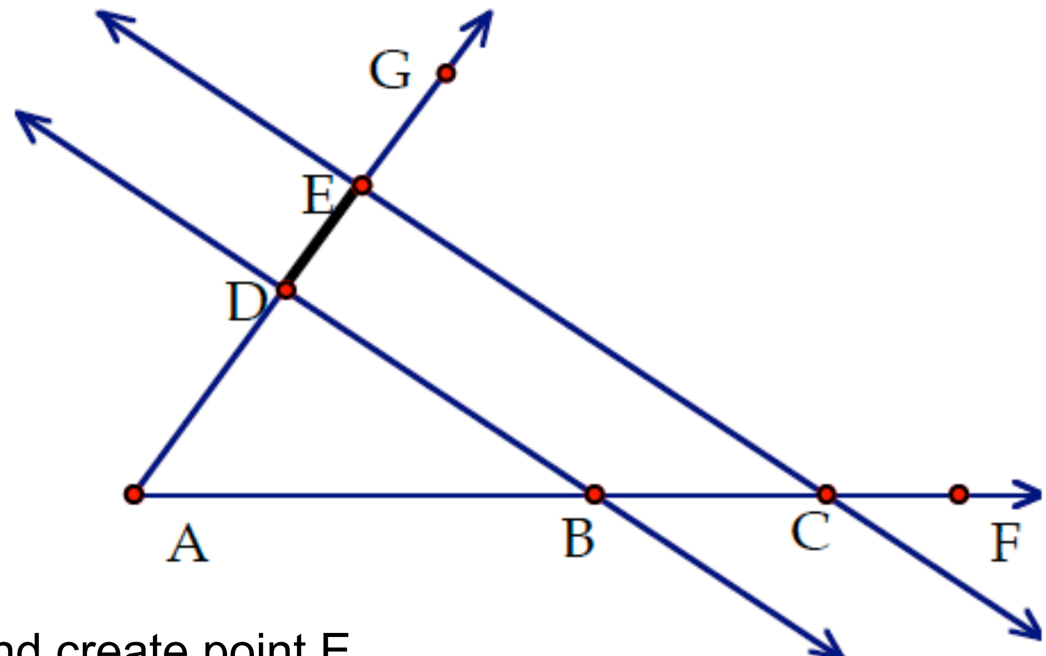
Adding and Subtracting





Multiplication, XY, given unit length

Note: Addition and Subtraction are independent on the unit length, but multiplication depends on it.



Build an angle FAG

Build AD=y BC=x AB==1

Build parallel to BD that pass C, and create point E

xy=DE

Proof: Triangles ACE and ABD are similar, therefore: AE/AC=AD/AB

(y + z)/ (1 + x) = y/1

z=xy

Division

Build AE=y AB=x AC=1

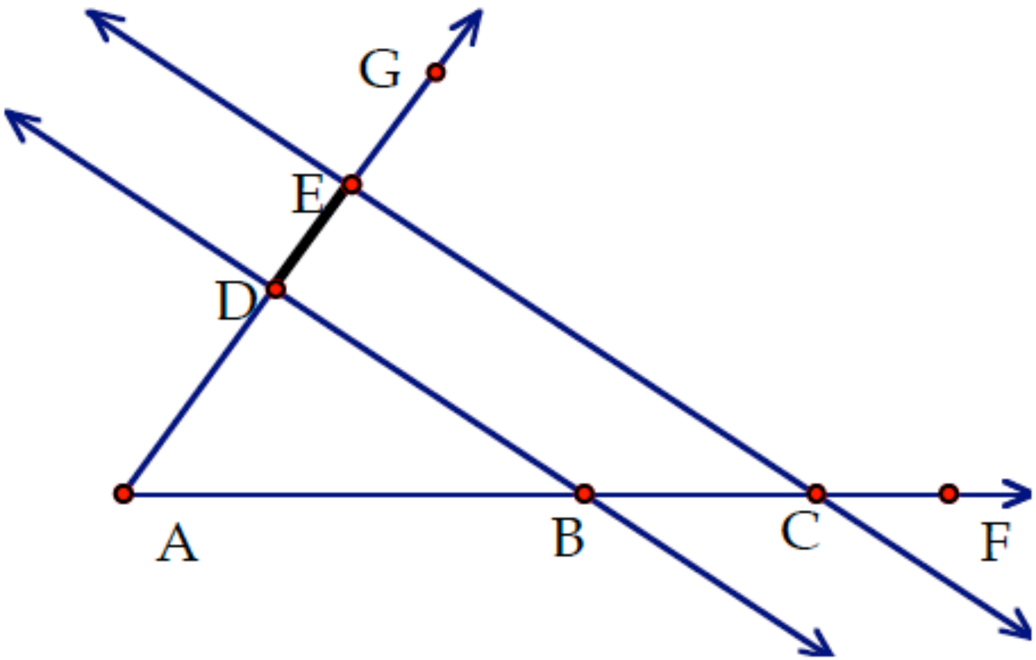
Build the parallel to CE that pass through B creating point D

AD=y/x

Proof: again since similar triangles: : AE/AC =AD/AB

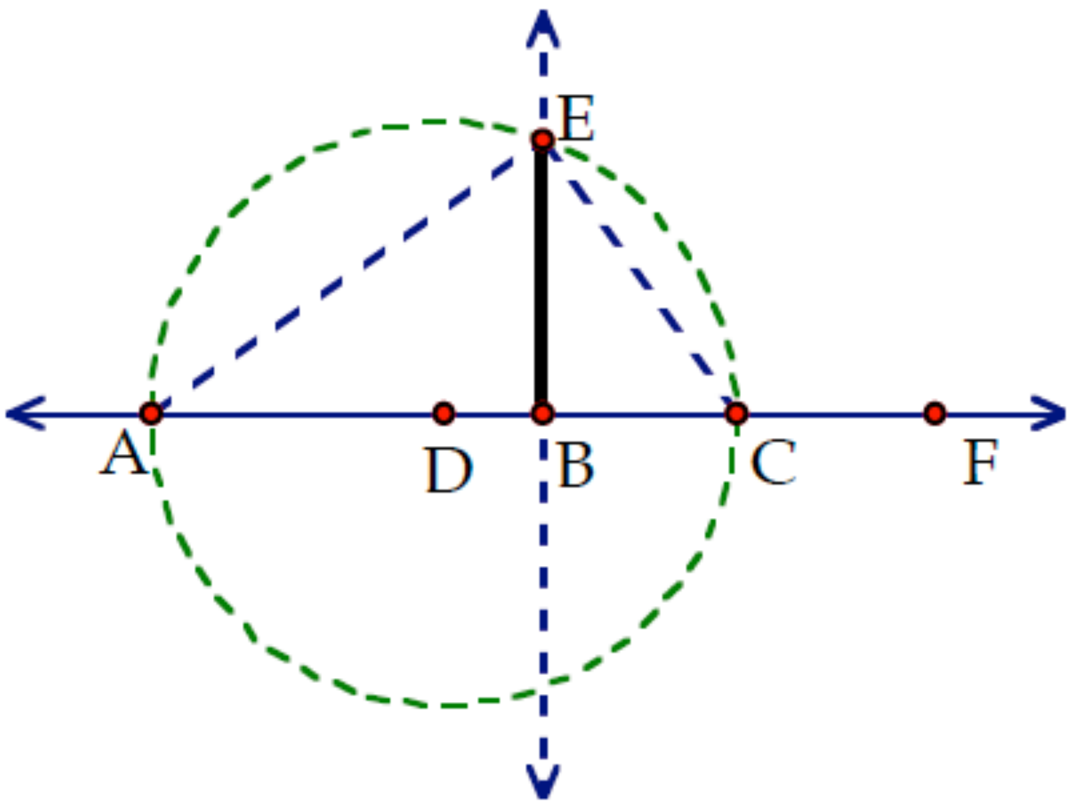
y/1 =z/x

z=y/x



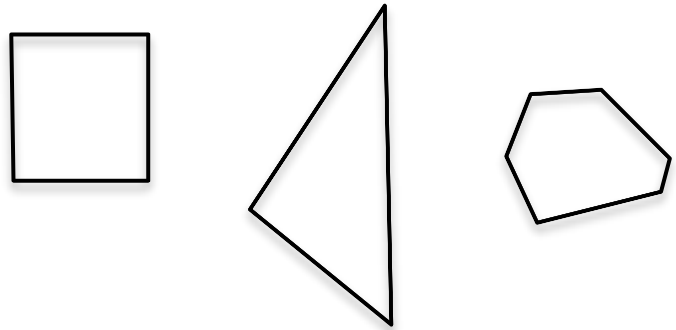
Square root

Build a circle center D radius AD=DC
The perpendicular line to AC that pass through B cuts the circle at E
BE= $\sqrt{x}$
Proof: EBC and ABE are Similar triangles, therefore $EB/BA = CB/BE$
AB*BC=BE^2



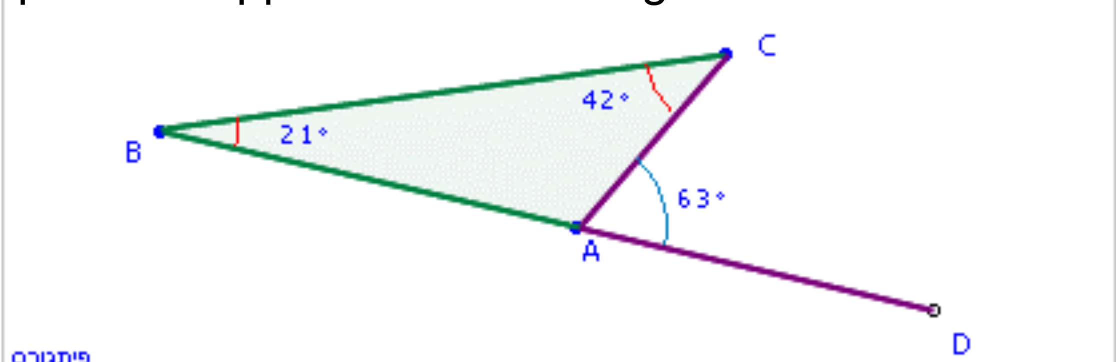
Other problems:
* Dividing an angle into three.
* Building Inscribed and
Circumscribed circles to a triangle

Polygons



Sum of internal and external angles for a polygon.

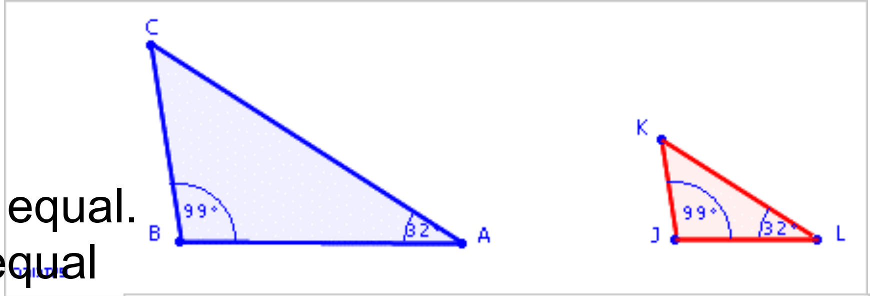
In a triangle, sum of two angles equals the opposite external angle.



Triangle with two equal edges has equal base angles.
Proof: divide into two triangles.

In equilateral triangle – all angles are equal.

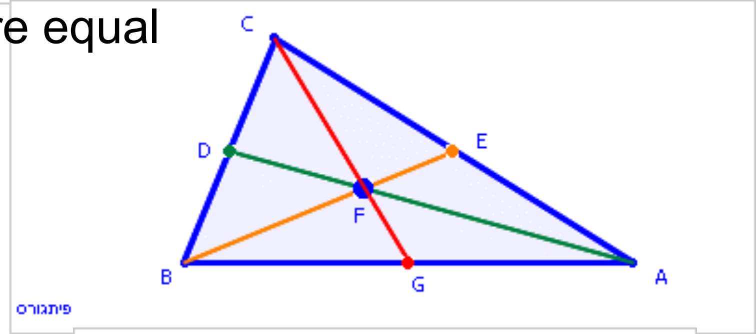
- Similar triangles – all angles equal
- Ratio of areas equals square of edge ratio



<u>Overlapping triangles rules</u>:

- If two angles and the edge in between are equal.
- Two edges and the angle in between are equal
- Two edges and the angle opposite the longer are equal
- Three edges are equal

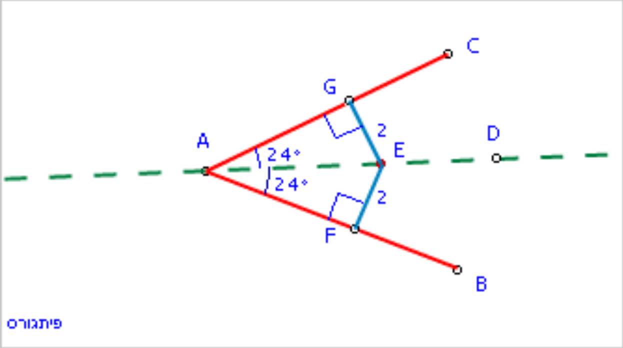
Proofs: by construction



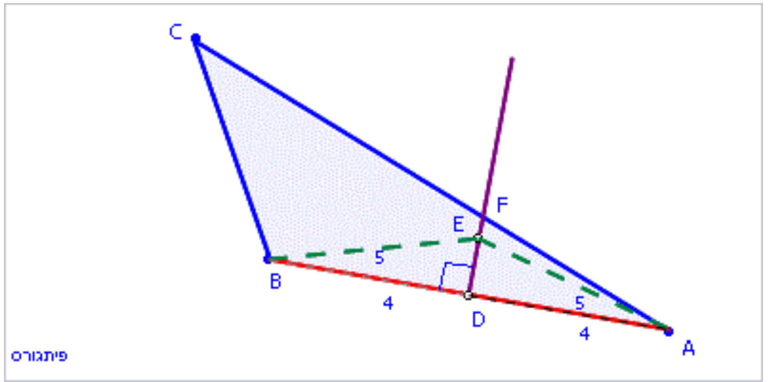
Vertex to Middle edge lines meet at 2/3 their length (also the center of mass)

All three meet at one point

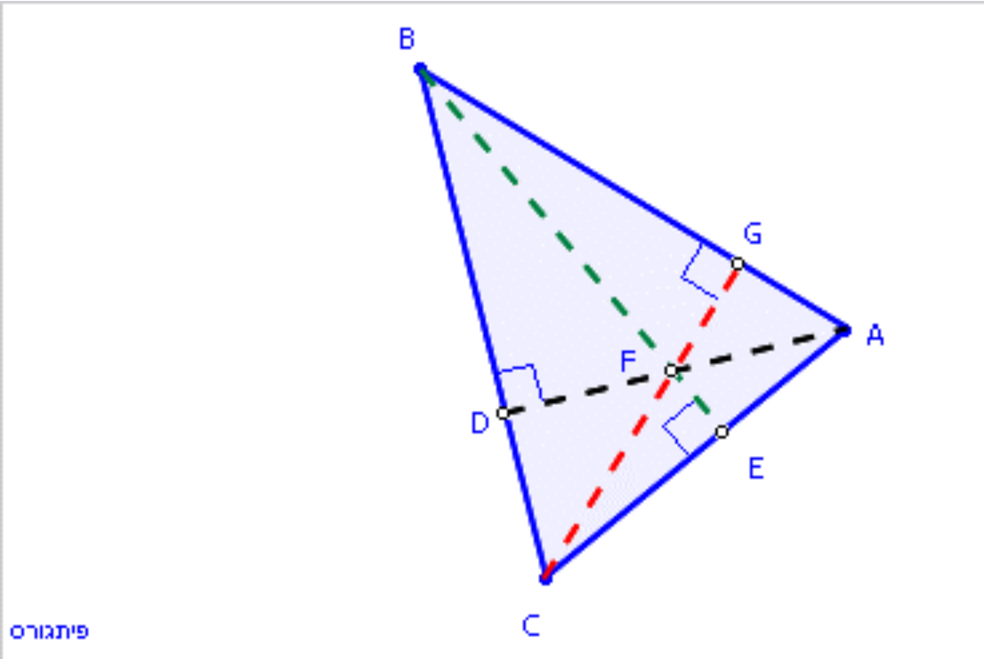
Also three angle bisectors meet at one point, and is at equal distances to all edges.



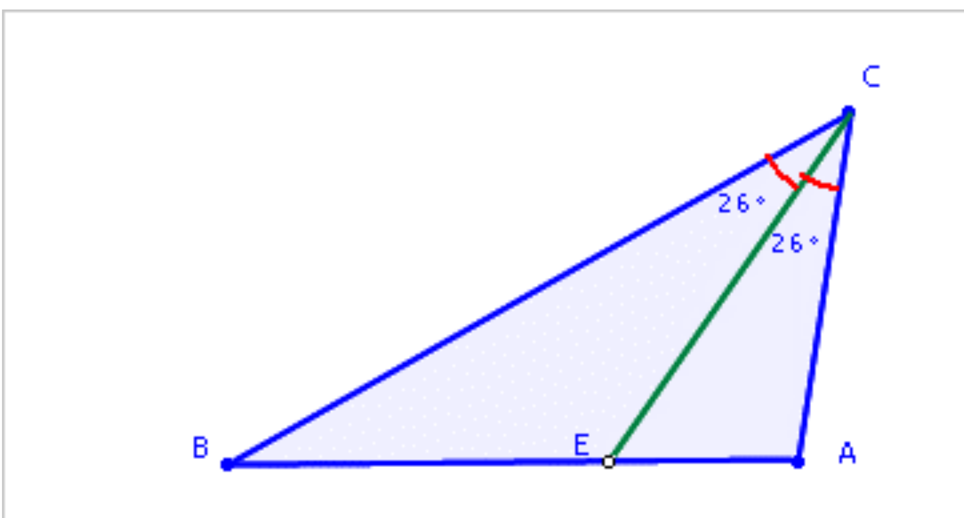
Every point on center-edge-perpendicular is at equal distances to the vertices of this edge



Three heights (perpendicular from a vertex to opposite edge) meet at one point



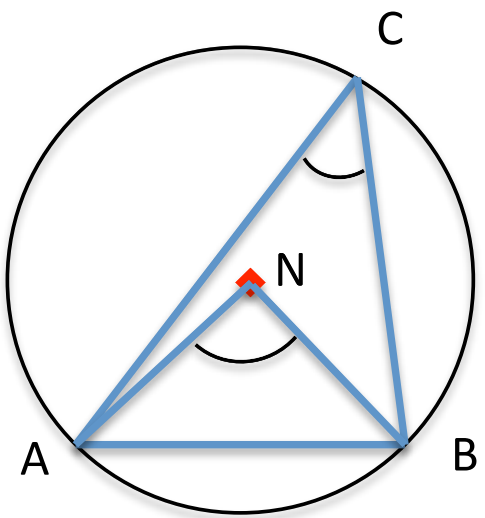
Angle bisectors cut the opposite edge into two sectors with ratio equals the ratio of the edges enclosing the angle, and vice versa.



The Circle

All points at equal distance (radius) from a point (center).
Only one circle pass through any three points that are not on one line
Any two points on a circle define an arc and a chord

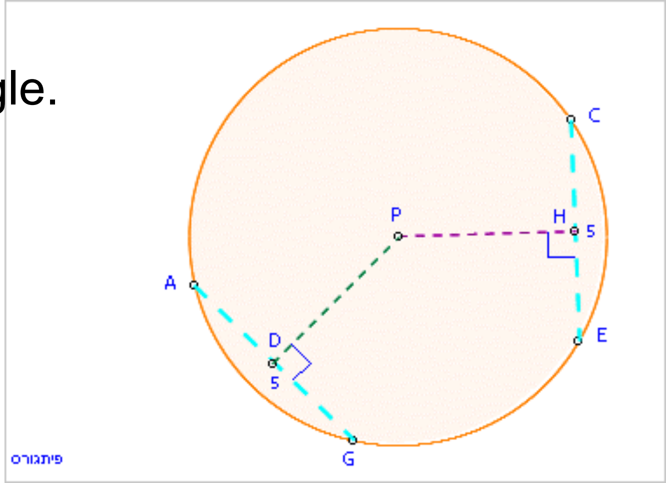
Angles in a circle



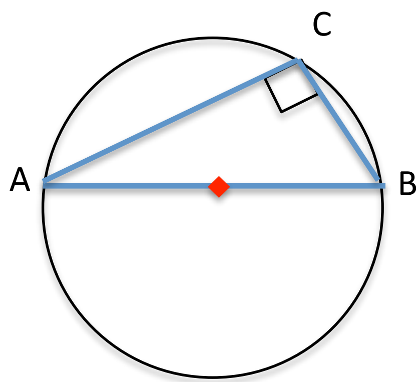
Equal arcs belong to equal peripheral and central angles, where the peripheral angle equals half of the central angle.

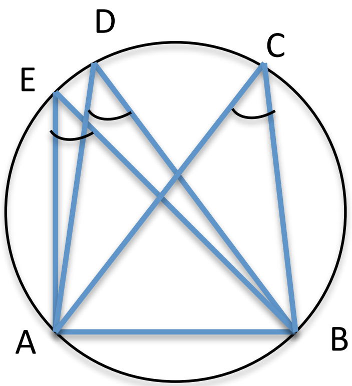
ACB = 1/2 ANB

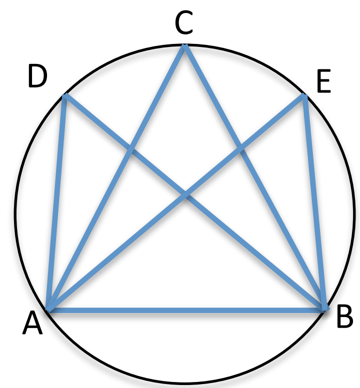
Chords at equal distances from the circle center have equal length.
The perpendicular line from circle center to a chord cuts it to half, and bisect to half the corresponding arc angle.



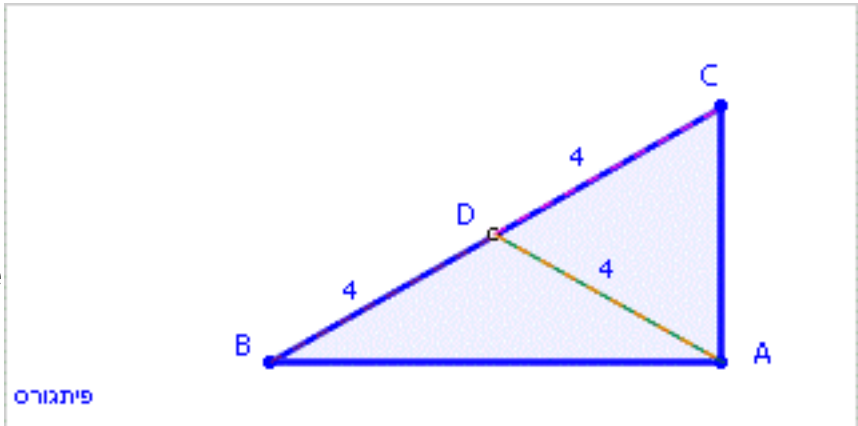
Chords of equal length are at equal distances from the circle center
There is no line passing through three points on a circle
Equal peripheral angles lie on equal arcs, and unequal angles on unequal arcs.







On a chord that is a diameter lies a right angle
Pythagoras theorem, and its inverse
In a right-angle triangle the length of the middle-edge from the right angle to the edge opposite the right angle is half the length of the edge, and vice versa.

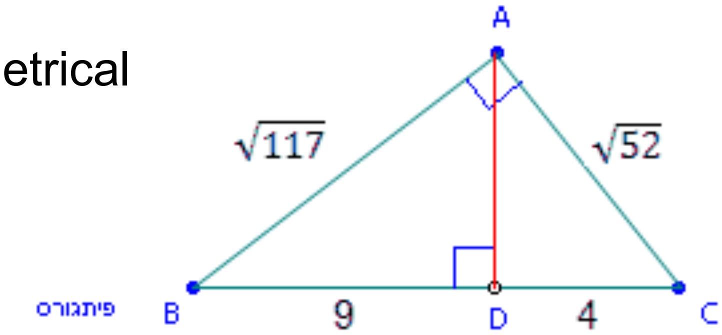


All the peripheral angles with vertices on a circle that lie on the same arcs are equal (e.g. ADB=ACB=AEB) and vice versa: the geometrical place of all equal angles lying on the same arc is a circle containing this arc.

In a right-angle triangle the height AD is the geometrical average of the two segments it cuts from the base,

AD = sqrt[BD*DC]

(e.g. sqrt[9*4]=6)



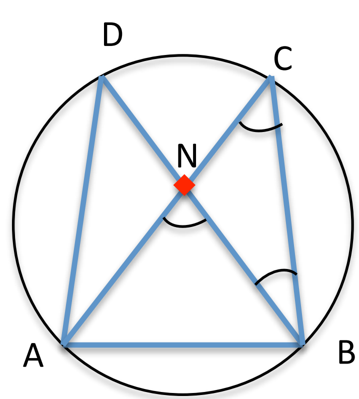
An angle lying on a chord AB with vertex inside a circle (e.g. ANB), equals the sum of the two peripheral angles lying on the arcs cut by the continuation of the lines AB and CD (or half the central angles on the same arcs)

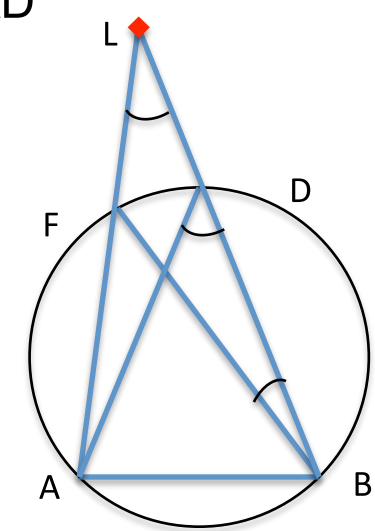
Proof: Angle ANB is external to triangle CNB

An angle lying on a chord AB with a vertex out of the circle (e.g. ALB) equals the difference between the peripheral angles lying on the arcs cut by the lines FD and AB

Proof: angle ADB is external to triangle ADL therefore ADB=ALD+LAD

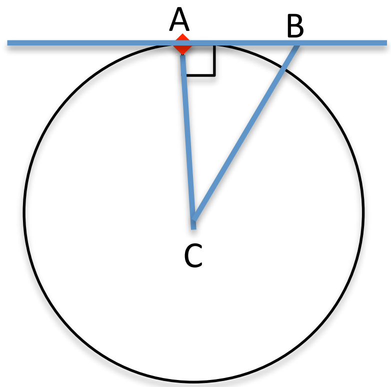
therefore: ALD==ALB=ADB–LAD





Tangential

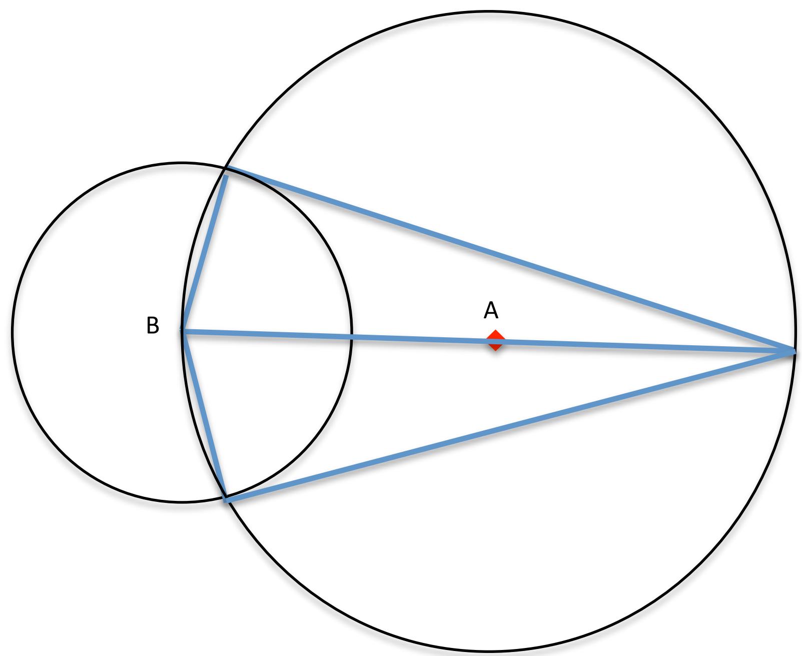
The tangent to a circle is a line that meets the circle only once.
Tangent to a circle is perpendicular to the radius from center to the tangential point.
And vice versa: the perpendicular to a radius at the point of meeting the circle is a tangent.
Proof: any point (e.g. B) other than A on the perpendicular has distance to the center BC larger than the radius, therefore is outside the circle.
Hence, two tangential passing through the same point are the same.



Construction problems

Building a tangent to a circle from a point on the circle:

Build a perpendicular line to the radius to this point.

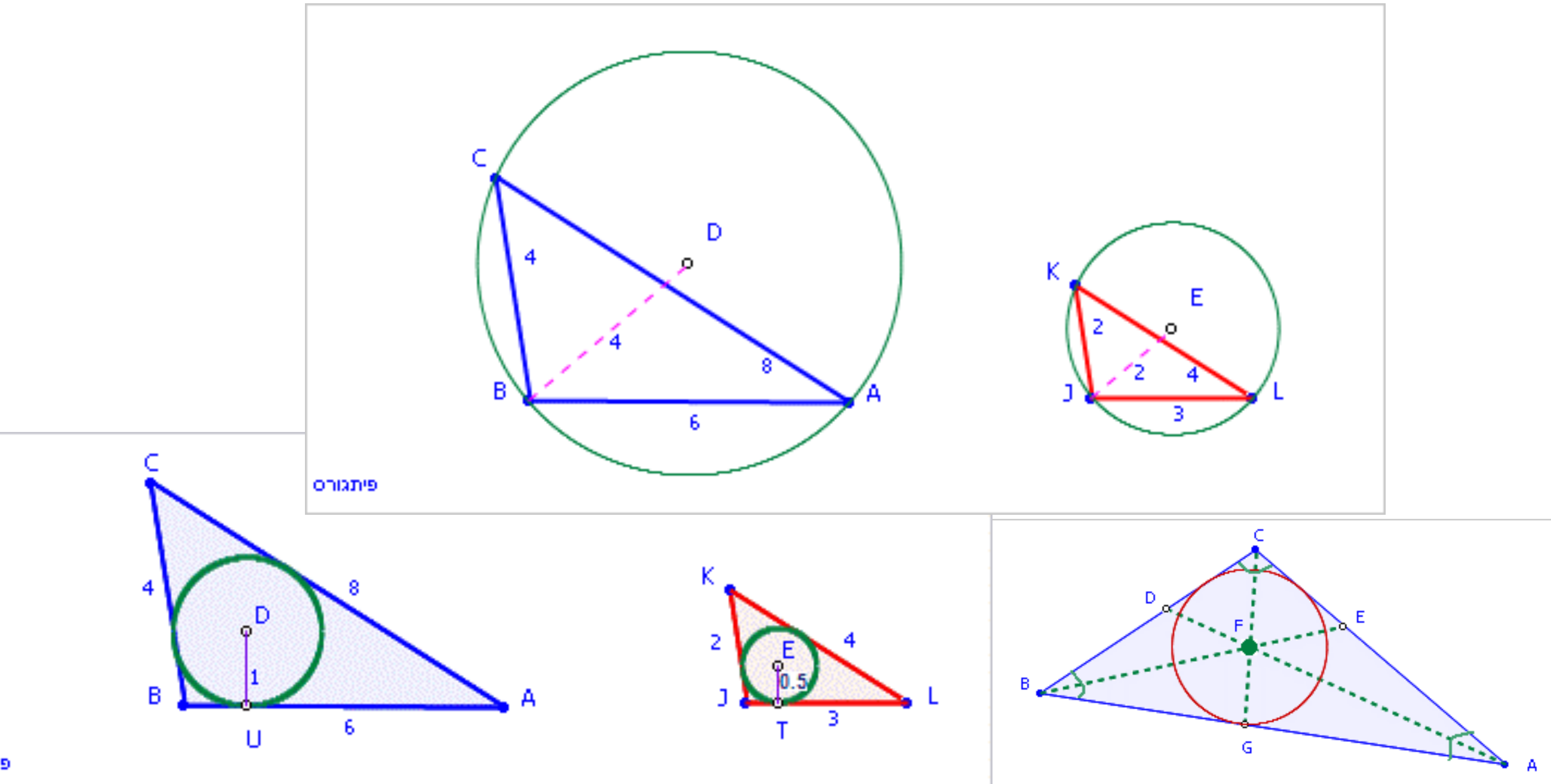


Building a tangent to a circle from a point out of the circle:

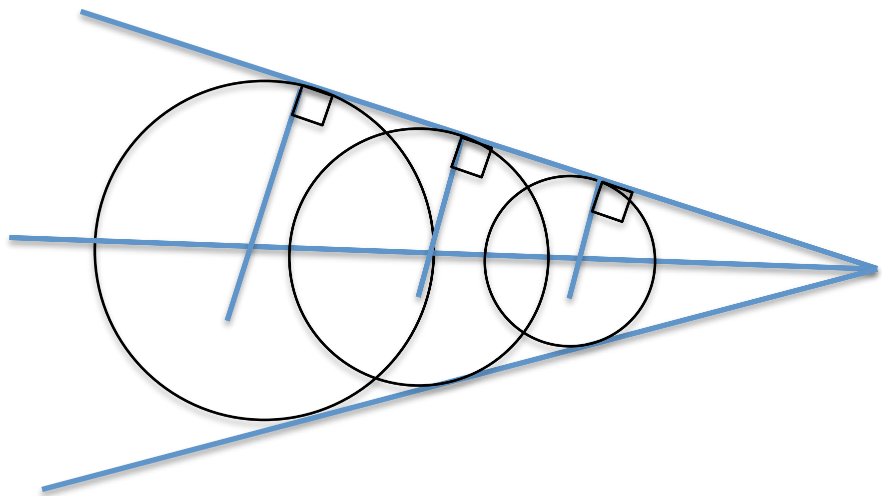
Build a circle with radius AB and center at A. It cuts the circle at the two points we need. Prove it !

Inscribed and circumscribing circles to a triangle:

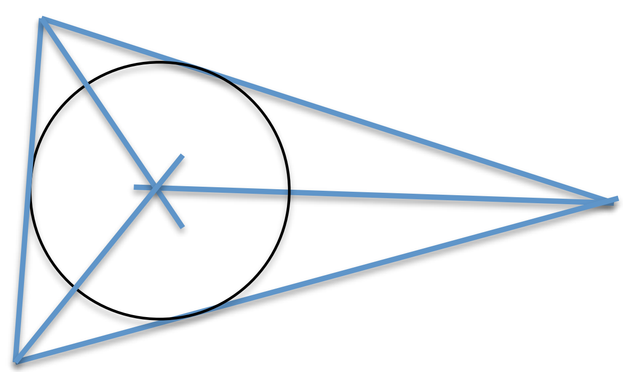
The three angle bisectors cur at one point – the center of the inscribed circle.
The three middle-edge perpendiculars meet at one point – the center of the circumscribing circle. Prove it !



Every center of circles tangential to the two lines making an angle lies on the angle bisector. Any line perpendicular to the lines cross the angle bisector at a circle center with the radius equal the segment of the perpendicular cut by one edge and the angle bisector.



There is a single inscribed circle for each triangle, with center at the meeting point of his angles bisectors.

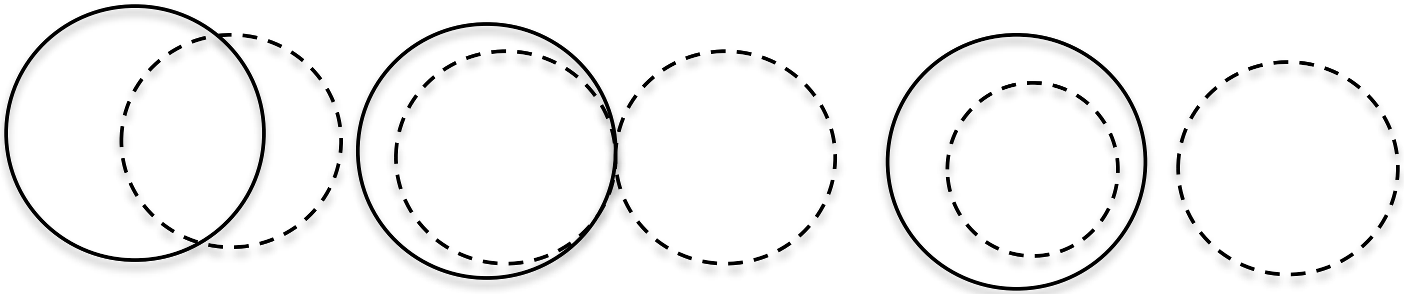


Two circles

They can meet at two, one point or not meet at all.

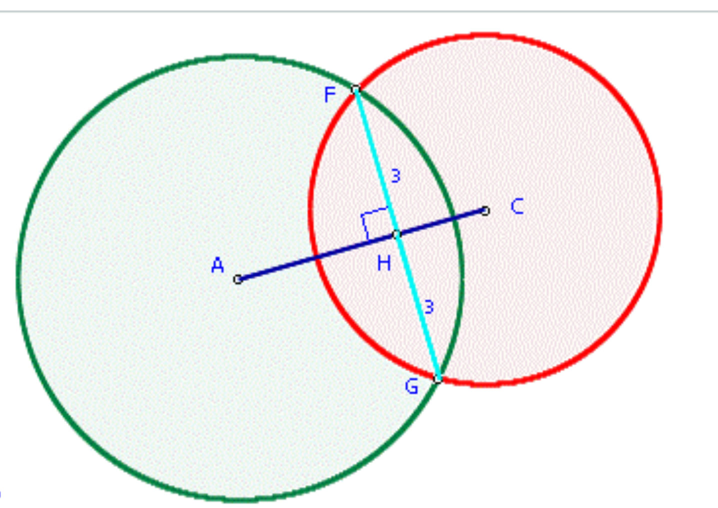
If they meet at one point, this point is on the line connecting the two centers.

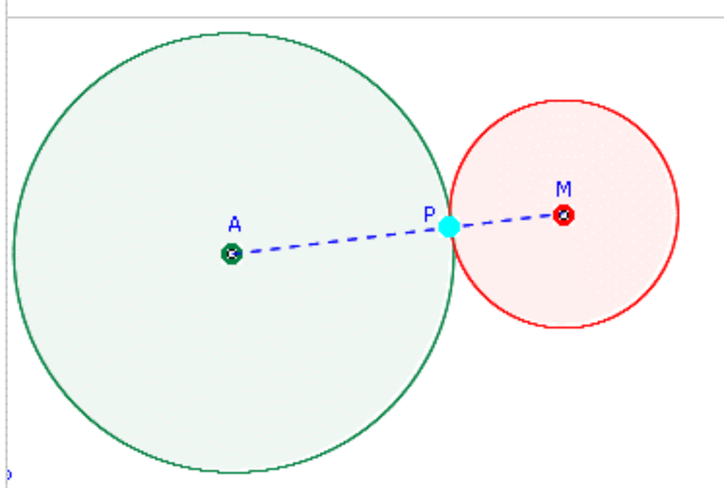
Proof: by falsifying the opposite assumption.

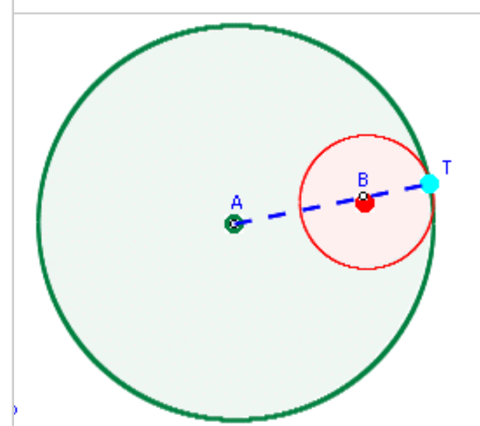


The line connecting the centers of two intersecting circles is perpendicular to the line between their two meeting points.

The point of touch of two tangential circles lies on the line between their centers or its continuation





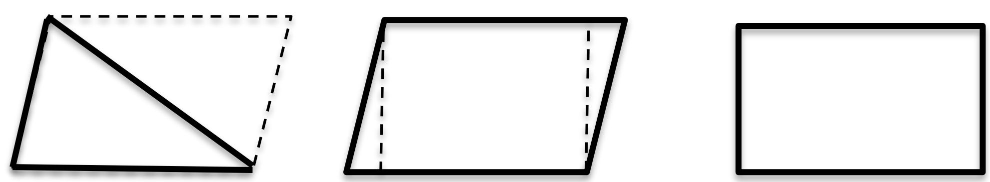


Symmetry

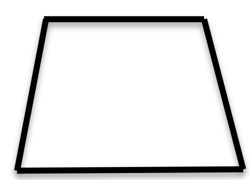
Two points are called symmetrical to a line if the line joining them is perpendicular to the symmetry line, and are at equal distance from it.
A circle is symmetric to itself for every diameter.

Area

The area of a rectangle = width * height
The area of a parallelogram = base * height
The area of a triangle = ½ base * height
also: = ½ perimeter * radius of circumscribing circle

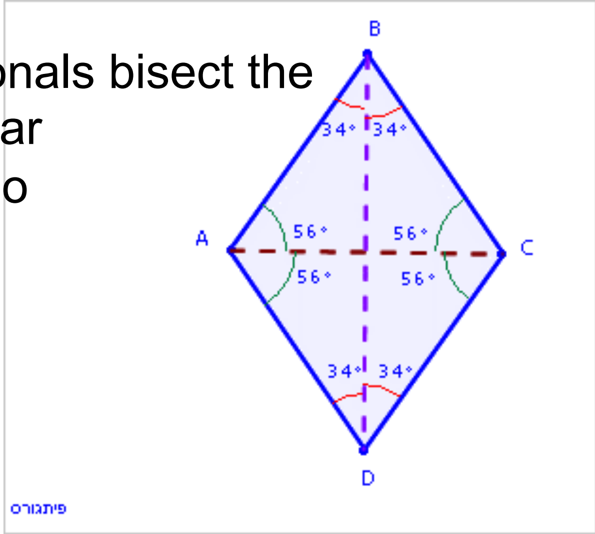


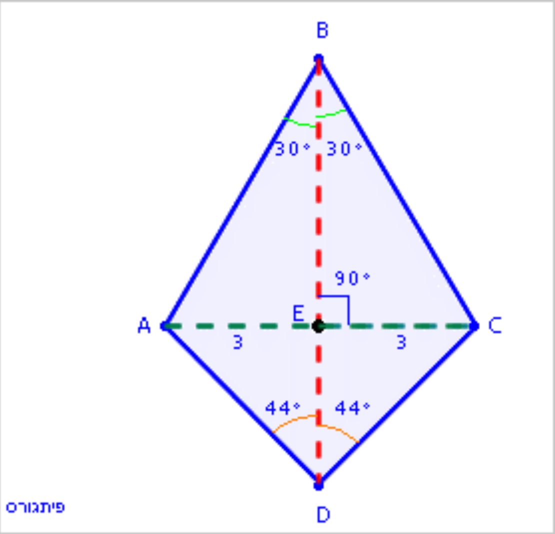
The area of a trapeze = average of its two bases * height



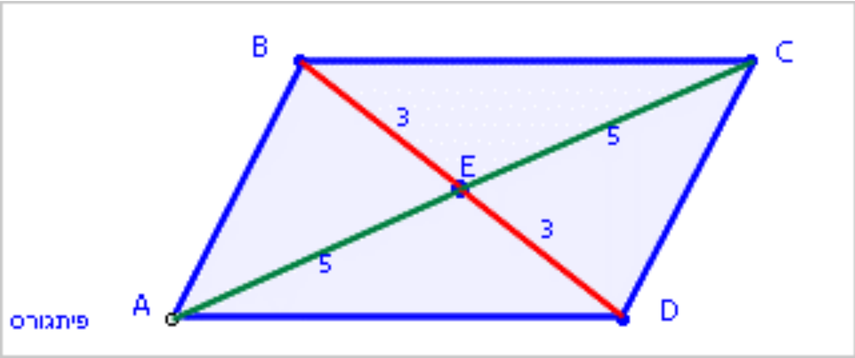
The diagonals of a Dalton bisects its head angles, bisects the other diagonal, and is perpendicular to it.

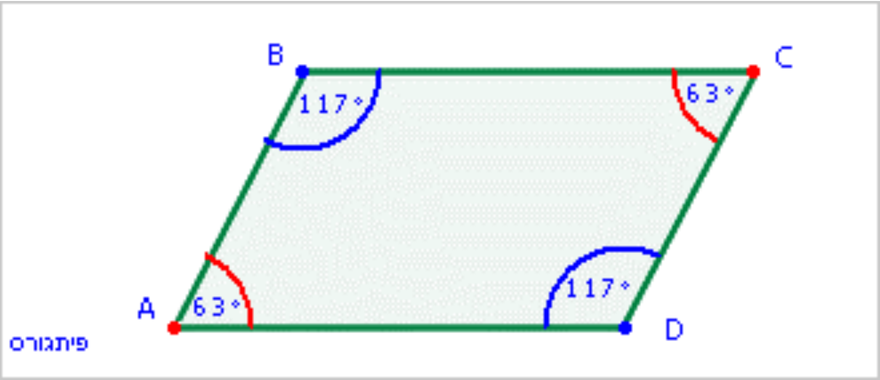
In a diamond (rhombus) the diagonals bisect the head angles, and are perpendicular to each other, and vice versa: if so This is a diamond



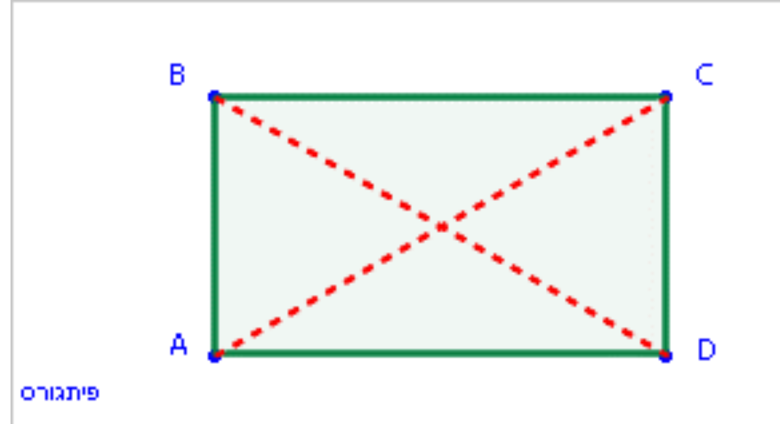


In parallelograms opposite angles are equal, and opposite edges are equal
The diagonals bisect each other,
and vice versa: if bisect – a parallelogram
A rectangle with equal opposite angles is a parallelogram
A quadrate with equal opposite edges is a parallelogram.

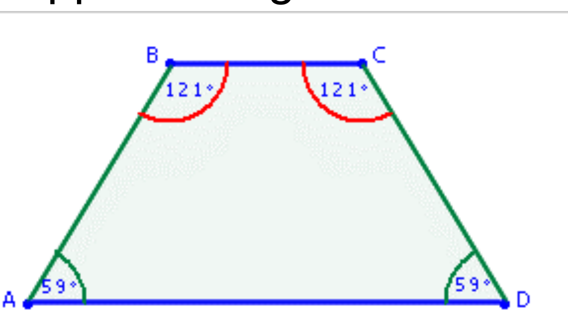


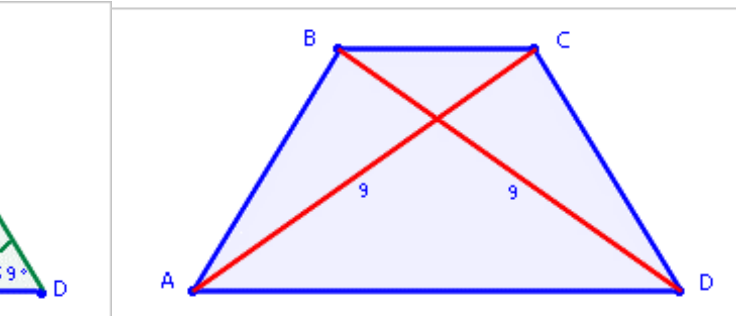


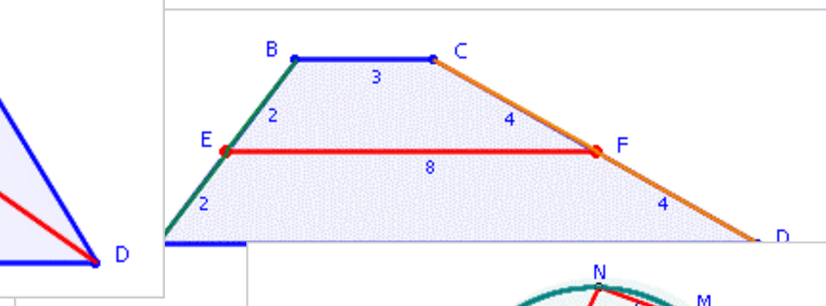
The diagonals of rectangles are equal

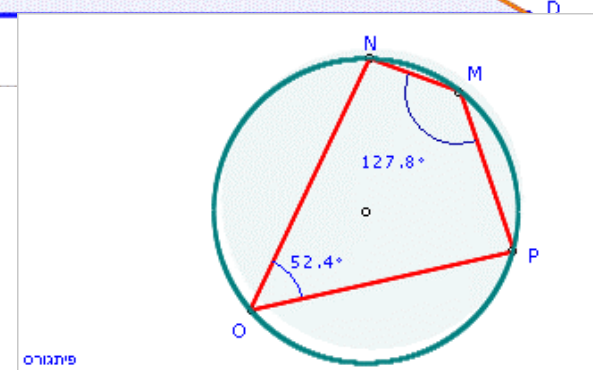


In equal edges trapeze the angles next to the same base are equal, and vice versa.
In equal edges trapeze the diagonals are equal, and vice versa.
The middle line in a trapeze is parallel to the bases and its length is their average.
In a symmetric trapeze, the line bisecting one edge and parallel to the base bisects the opposite edge



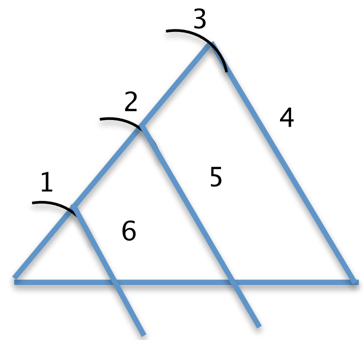






A quadrate can be circumscribed in a circle only if the sum of opposite angles equals 180°

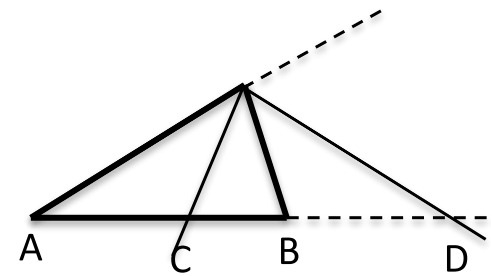
Proportional Segments



Similar Triangles

Harmonic division:

D from outside AB and C inside, so that CA/CB=DA/DB

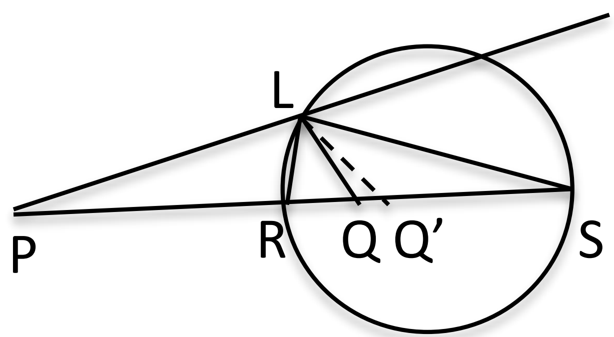


Inner angle bisector cuts AB at C, and outer angle bisector cuts at D then CA/CB=DA/DB

Apollonius’ theorem:

The circle (diameter RS) is the geometric place of points L with distances from two points PQ has a given proportion.

Proof: if we divide PQ harmonically (inside and outside points RS with the given ratio.

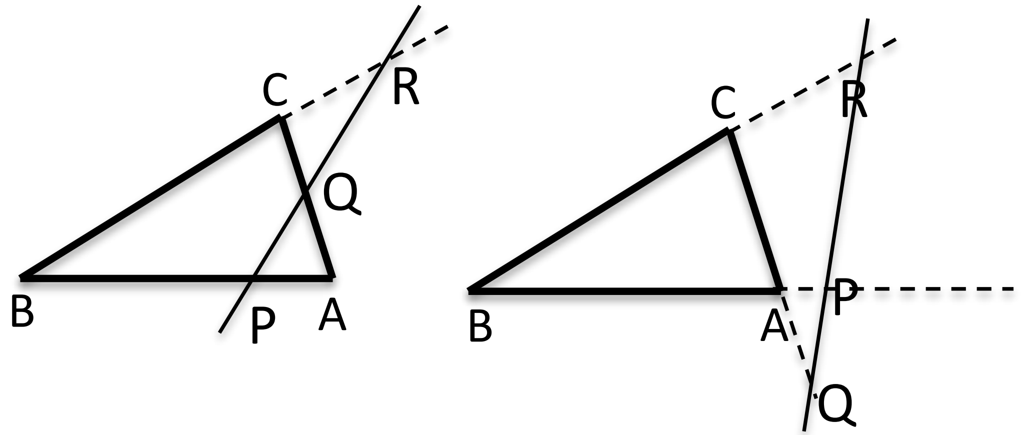


Menelaus' theorem:

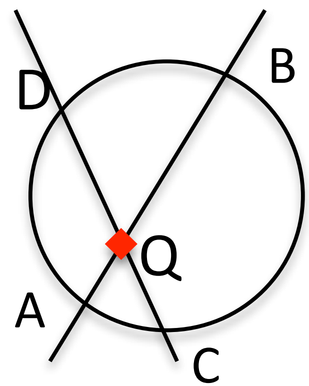
If a line cuts the edges of a triangle or their continuations, then:

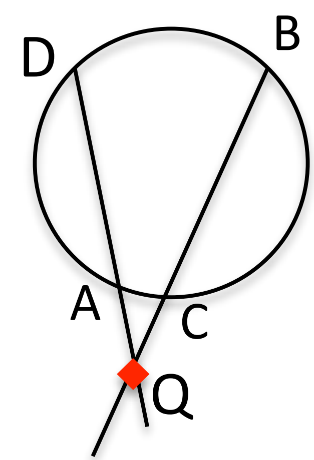
$$AP/BP \cdot BR/CR \cdot CQ/QA = 1$$

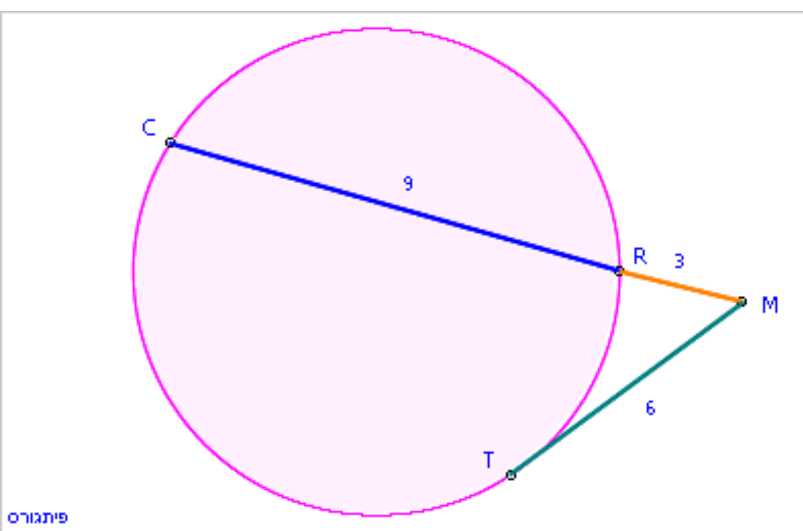
And vice versa: if the ratio holds, the points PQR are on one line



If two chords in a circle cut at point Q, the product of the two segments of one chord equal the product of the other chord segments: QA*QB=QC*QD
This is true for Q inside or outside the circle.
Therefore the product also equals the square of the length of a tangential to the circle from Q that is outside the circle.

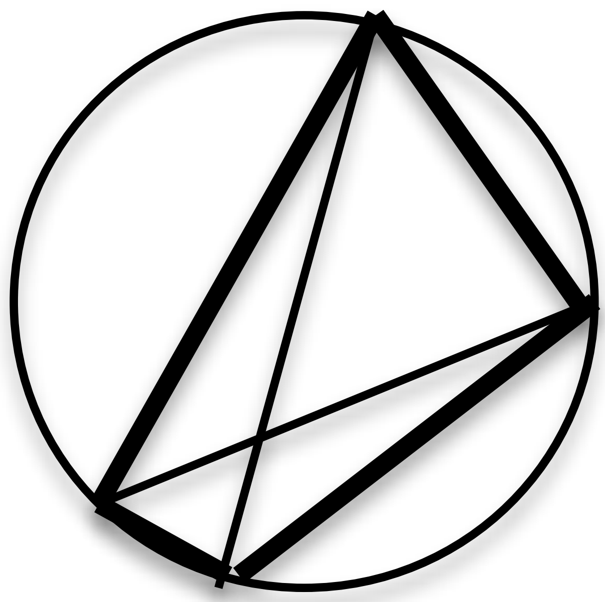






Ptolemy's theorem

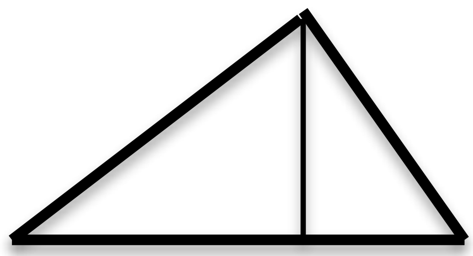
For a rectangle circumscribed in a circle, the product of its diagonals equals sum of products of opposite edges.



Similarity of triangles and polygons

Metric properties of triangles

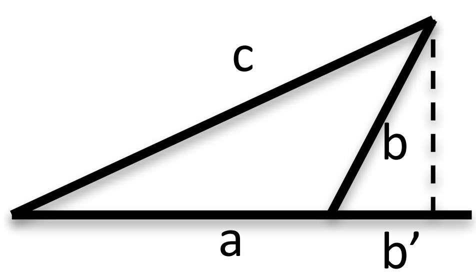
Right angle triangle divided by the height into two triangles similar to itself



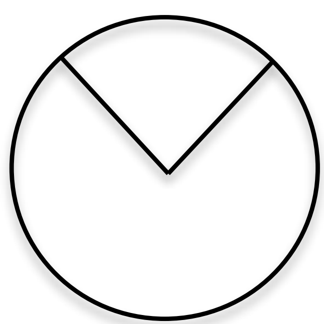
in a triangle the square of the edge opposite a sharp angle (<90°) equals the sum of squares of the other edges, minus twice their product times the cosine of the angle between them: $c^2=a^2+b^2-2ab\cos(ACB)$

Heron's equation: the area of a triangle in terms of its three edges:

$$A=\text{sqrt}[p(p-a)(p-b)(p-c)] \quad p=(a+b+c)/2$$



Area of a parallelogram = edge * height from this edge
Area of a triangle = ½ edge * height from this edge
Area of diamond = ½ product of its diagonals
Area of a trapeze = height * average of bases
Area of a circle = $\pi * R^2$
Area of circle sector = ½ radius * arc length
Area of a circle sector = ½ radius^2 * arc angle in radians



Perfect and other polygons

Every perfect polygon (equal edges and angles) can be circumscribed in a circle.

The sum of inner angles of convex polygon is $(180^0)*(n-2)$

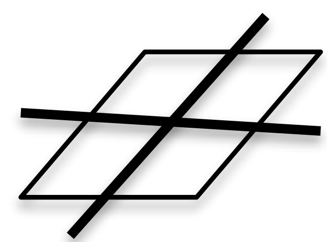
Three-dimensional geometry

Definitions:

1. Space, plane, how to define a plane is space:
   a. Two dissecting lines
   b. two parallel lines
   c. 3 points not on the same line
   d. a line and a point not on it.

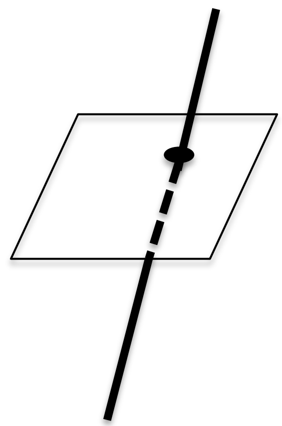
2. Two mutual configurations of two planes:
   a. intersect. They intersect in a line
   b. parallel. No intersecting line.

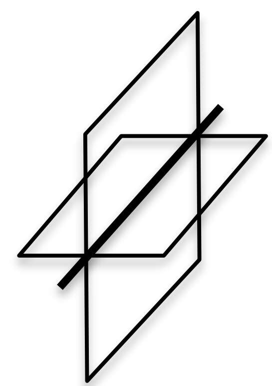
3. Three mutual configurations of two lines:
   a. intersecting
   b. Crossing
   c. Parallel



4. A line is in a plane if two points on it are in the plane
5. Through a line and a point that is not on it pass one and single plane
6. A line that is not in a plane and is parallel to one line in the plane is parallel to the plane

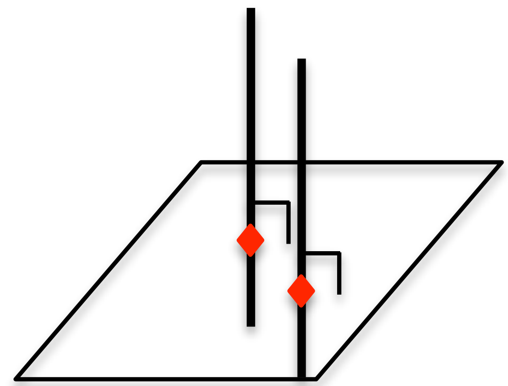
The intersect of a line with a plane (not parallel to it) is called the **Heel**



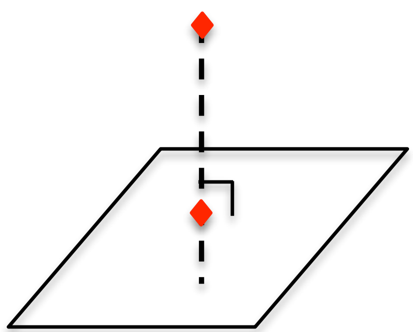


If two planes have a common point they have a common (intersecting) line.
Two lines that are not in the same plane are called crossed lines.
Two line that are parallel to a third line are parallel to each other.
A line is perpendicular to a plane if it is perpendicular to two lines in this plane.
All lines that meet a line at a point and are perpendicular to it are in the same plane
From every point in space there is only one perpendicular line to a plane.
From every point in a plane there is a single line perpendicular to the plane

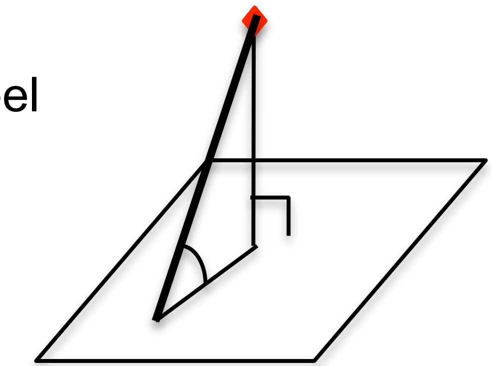
The distance between a point and the point a perpendicular line cut a plane is the distance of a point to a plane. It is the minimal distance to any point in the plane.



Two perpendicular line to a plane are parallel.
Two planes perpendicular to a line are parallel
A plane intersects two parallel planes in parallel lines
All points at equal distance from a plane are on a parallel plane
Two angles created by parallel edges are equal, or sum to 180^o
Projection of a point on a plane is the intersect with the perpendicular line to the plane that pass through the point.



A tilted line is a line that is not perpendicular to a plane.
The **<u>projection of a tilted line</u>** on a plane pass through its heel
And the projection of any point on the tilted line
(other than the heel itself)



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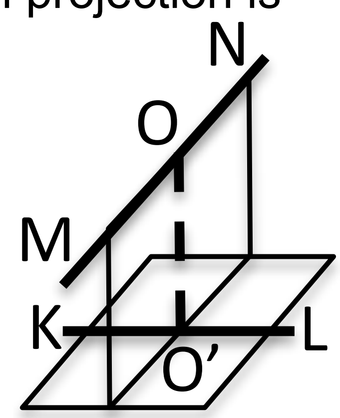
The angle between a tilted line and its projection
is the minimal angle of all lines in the plane passing through
its Heel.

A line in a plane that is perpendicular to a tilted line is also perpendicular to its projection.

A line perpendicular to a plane, all intersecting planes are also perpendicular.

The distance between two crossing lines KL and MN is OO' built as follows:
Build a plane containing KL and parallel to MN (thus every line with one point on KL is parallel to MN). The projection of MN on this plane will cut KL at point O'. We draw from it a perpendicular line to the plane that cut MN at point O (which projection is O').



The angle of a **corner** built by two dissecting planes is the minimal angle between any two lines in these two planes that pass through a common point on the line of intersection between the two planes.

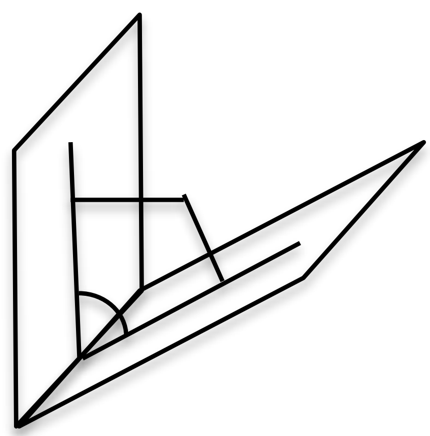
Construction: from a point on the intersecting line build two perpendicular lines in the two planes. The angle between them is the corner angle.

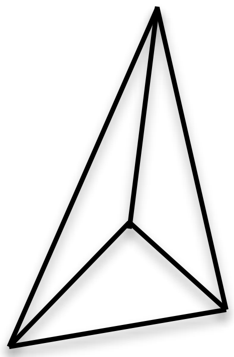
Two parallel planes have equal corner angles with any other plane dissecting them (this is the three-dimensional equivalent of the exchanging angles of a line crossing two parallel lines).

A plane passing through a line that is perpendicular to another plane is perpendicular to this plane.

A line in a plane that is perpendicular to its intersection with a second plane is perpendicular also to it.

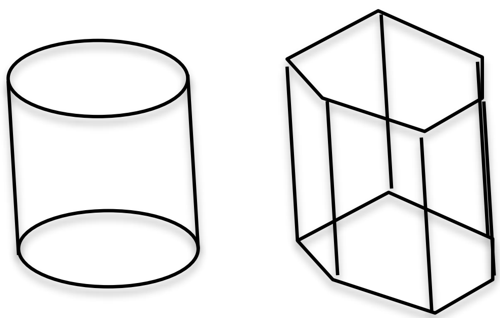
The line of intersect of two planes that are both perpendicular to a third plane, is also perpendicular to it.





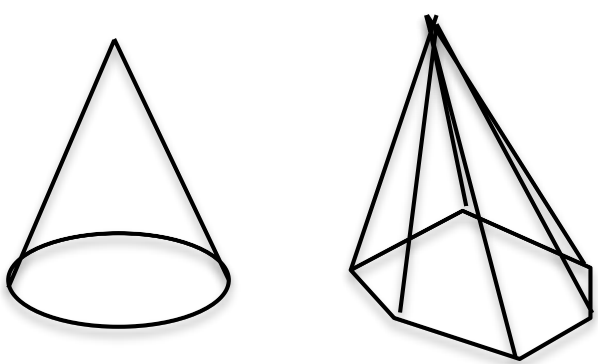
Triangular corner created at a Dalton’s vertex

A prism and a cylinder – Surface area and volume
Volume = base area * height



Pyramids and cones – Surface area and volume
Volume = 1/3 Base area * height

A chopped cone
Volume = 1/3 Average of bases * height



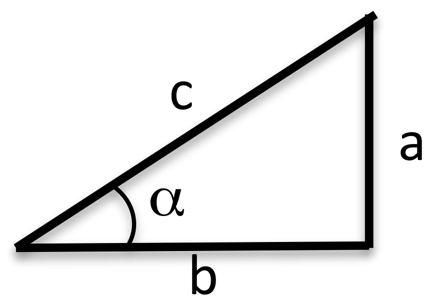
Volumes and surface areas of rotational bodies: see Guldin’s law

Volume of sphere = $4/3 \pi R^3$
Surface area of a sphere = $4\pi R^2$
Intersection of a sphere by a plane = circle
Intersection of a plane passing through sphere center = **large circle**
Plane tangential to a sphere
Only one sphere can pass through 4 points not on the same plane.
Spherical triangle is created by three large circles.
Loon sickle is build from two large circles.

Perspective (Three- to Two-dimensions projection) and Projection of world maps (a spherical surface onto a plane) See above.

Trigonometry

In similar triangles the edges are proportional.
In right-angle triangles the ratios between edges depend only on the angle.
These ratios define the trigonometric functions. Their tables provide lengths of all edges, given one angle and one edge.



The definitions are: sinus: $\sin\alpha = a/c$

tangence: $tg\alpha = a/b$

cosinus: $\cos\alpha = b/c$

Example: the height of a tree, without climbing to its top:

Measure the top angle of view from a point, α, and its distance to the tree, d. The tree height = $d*tg\alpha$

We see in “Astronomy” how the radius of earth and the distance to the moon were measured even before trigonometric functions were defined.

A few relations between the trigonometric functions:

$\cos\alpha = \sin(90^0 - \alpha)$

$tg\alpha = \sin\alpha / \cos\alpha$

$\sin^2\alpha + \cos^2\alpha = 1$ (use Pythagoras law)

A few specific values to trigonometric functions:

$\sin 0^0 = 0$ $\cos 0^0 = 1$

$\sin 30^0 = 0.5$ $\cos 60^0 = 0.5$

$\sin 45^0 = \cos 45^0 = \sqrt{2}/2$ $tg 45^0 = 1$

A few additional trigonometric functions

Cotangence: $ctg\alpha = 1 / tg\alpha$ קוטנגנס

Secance: $\sec\alpha = 1 / \cos\alpha$ סקנס

Cosecance: $cosec\alpha = 1 / \sin\alpha$ קוסקנס

Therefore:

$1 + ctg^2\alpha = 1/\cos^2\alpha = \sec^2\alpha$

$1 + \tan^2\alpha = 1/\cos^2\alpha = \sec^2\alpha$

How were these function calculated in ancient times?

Experimental measurements on triangles – limited accuracy

Computations – based on half angle and sum and subtract angles equations

$\sin(\alpha/2)=\pm\sqrt{[(1-\cos\alpha)/2]}$

$\cos(\alpha/2)=\pm\sqrt{[(1+\cos\alpha)/2]}$

$tg(\alpha/2)=\pm\sqrt{[(1-\cos\alpha)/(1+\cos\alpha)]}$

$\sin 2\alpha=2\sin\alpha\cos\alpha$

$\cos 2\alpha=\cos^2\alpha-\sin^2\alpha$

$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$

$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

$tg(\alpha+\beta)=(tg\alpha+tg\beta)/(1+tg\alpha\, tg\beta)$

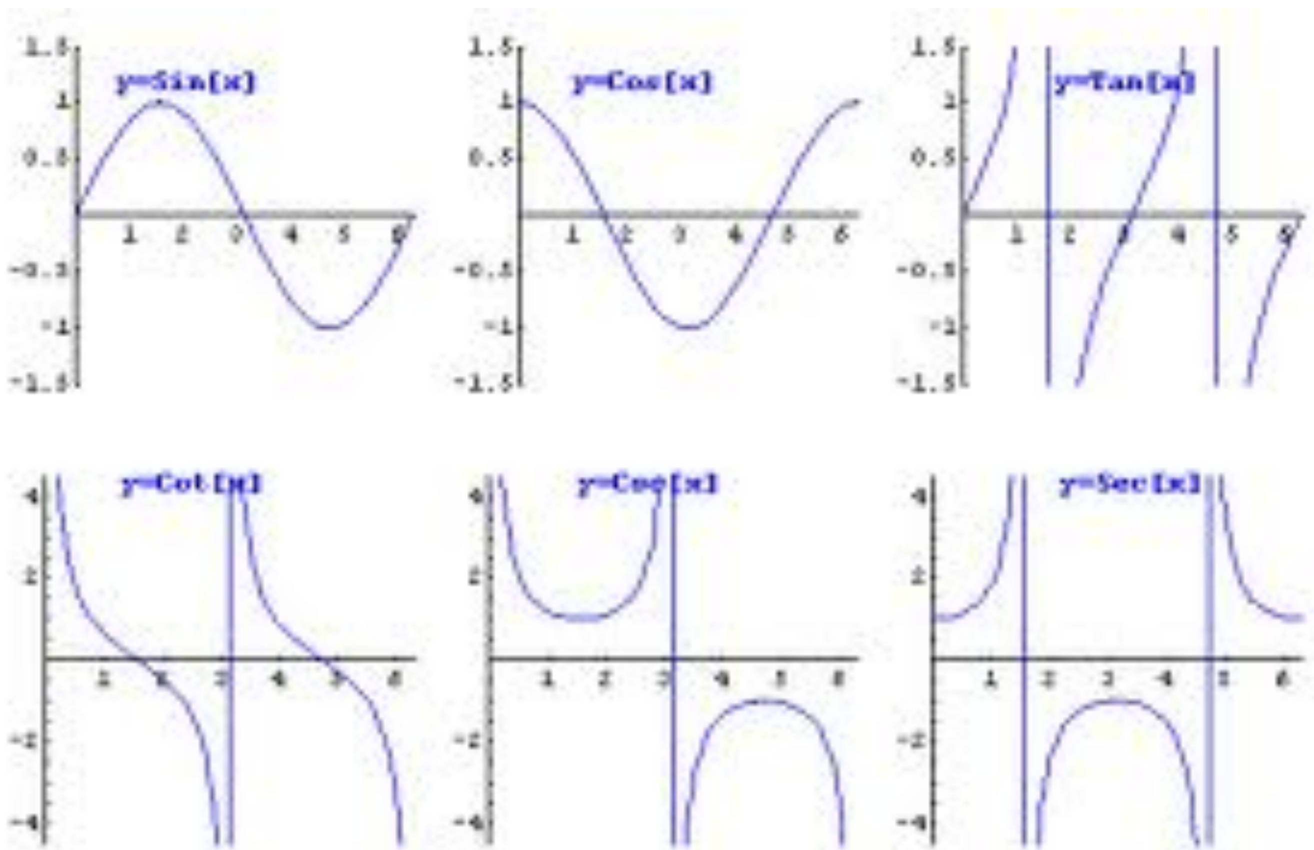
$\sin\alpha+\sin\beta=2\sin[(\alpha+\beta)/2]\cos[(\alpha-\beta)/2]$

$\cos\alpha+\cos\beta=2\cos[(\alpha+\beta)/2]\cos[(\alpha-\beta)/2]$

And also:

$tg(-\alpha)=-tg\alpha$ $\quad\sin(-\alpha)=-\sin\alpha$ $\quad\cos(-\alpha)=\cos\alpha$

The graphical plots of the trigonometric functions:



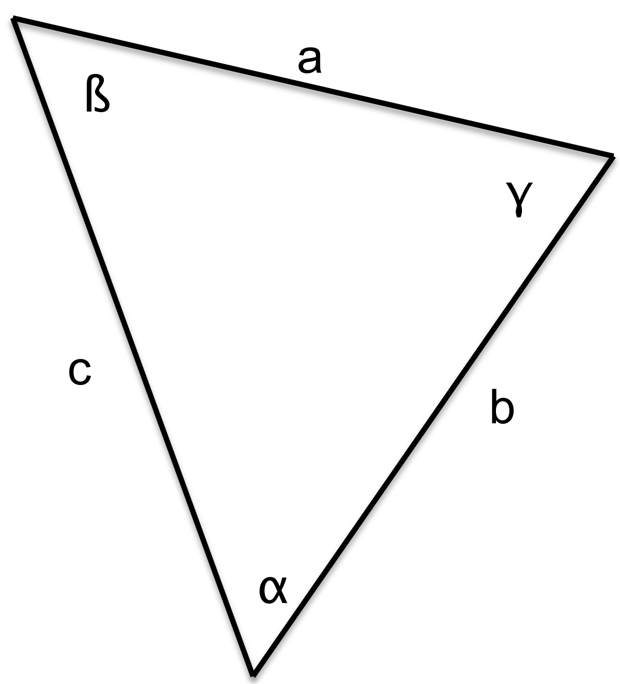
For triangles with edges a,b,c and corresponding angles opposite to the edges α,ß,γ

The sine law:

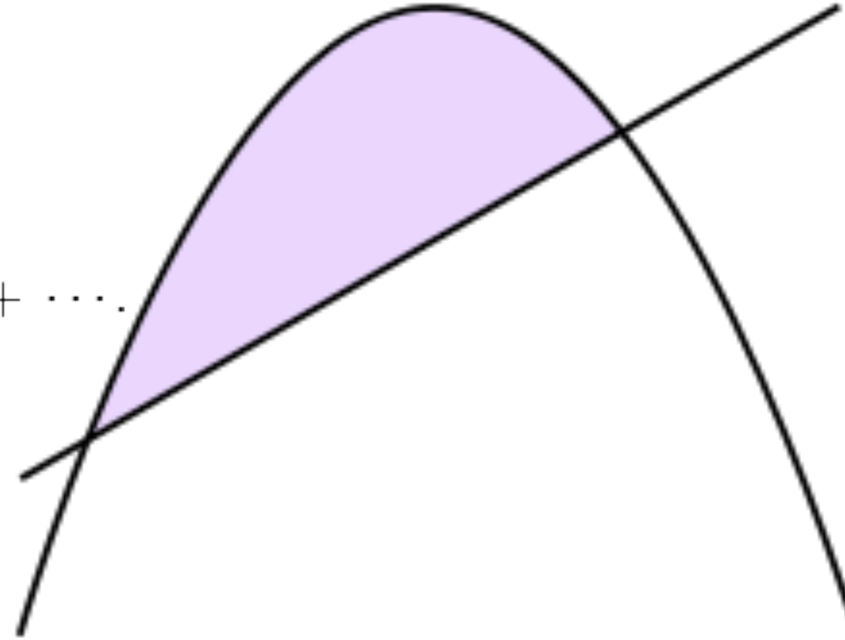
$a/\sin\alpha = b/\sin\beta = c/\sin\gamma$

The cosine law: (extending Pythagoras law for any angle:

$c^2 = a^2 + b^2 - 2ab\cos\gamma$

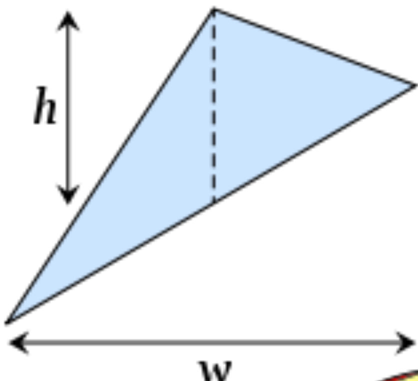


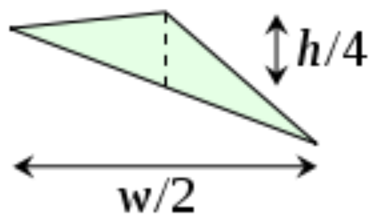
Archimedes wrote a theorem he proved using infinitesimal calculation: The area between a parabola and a line is 4/3 times the area of the light blue triangle. Proof:

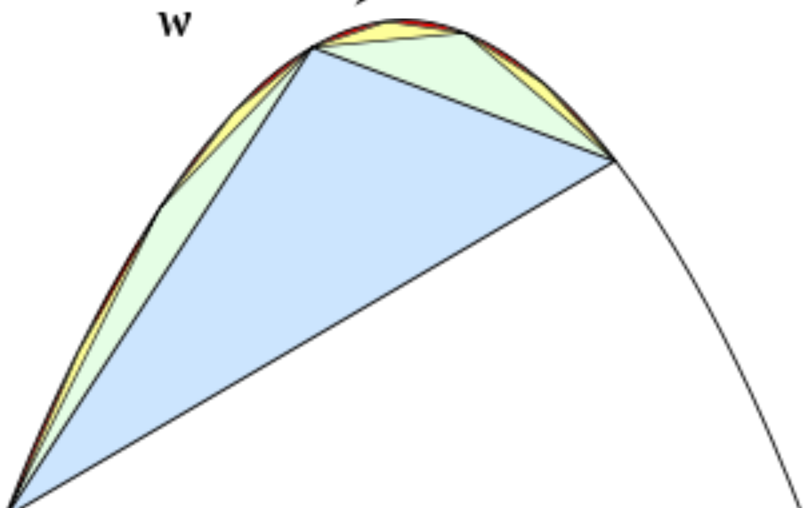


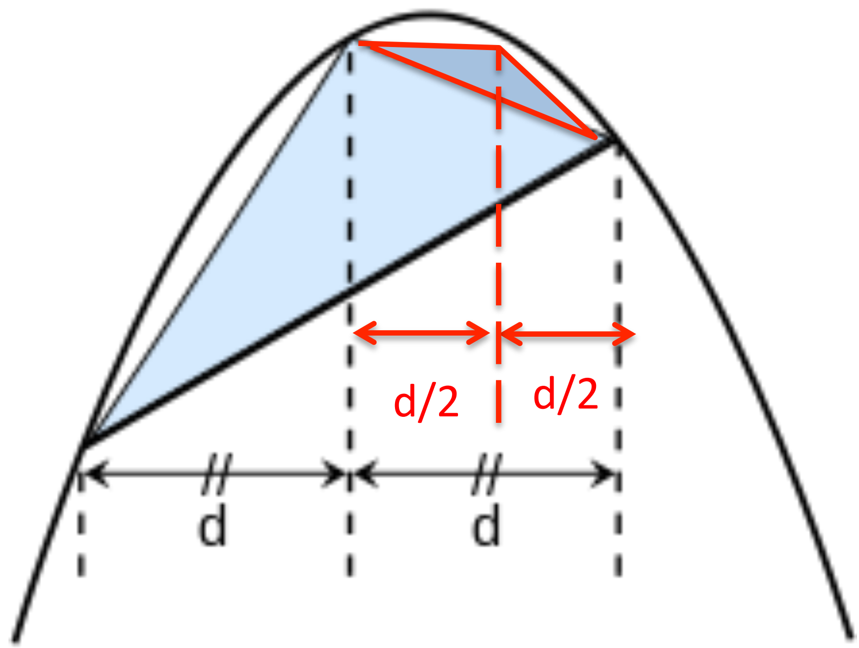
$$\text{Area} = T + 2\left(\frac{T}{8}\right) + 4\left(\frac{T}{8^2}\right) + 8\left(\frac{T}{8^3}\right) + \cdots.$$

$$\sum_{n=0}^{\infty} 4^{-n} = 1 + 4^{-1} + 4^{-2} + 4^{-3} + \cdots = \frac{4}{3}.$$









Greek philosophers that were not studying natural sciences:

Pyrrho of Elis (ca. 360–270 B.C.) – Skepticism
Epicurus (342–270 BC) – pleasures of life
Zeno of Citium (336–264 BC) – Stoicism
Diogenes of Synope (c. 412–323 BC) – cynicism

Diogenes is a contemporary of Plato and Aristo, and a student of Antisthenes. Cynicism is not what we refer to today: he preached integrity and personal honesty, in response to its loss in the public life in Athens. He wondered in the streets with a lantern, “searching for an honest man”. He moved to live in a barrel, and dumped all his belongings, including his last one: a cup, after seeing a farmer drinking from the paws of his hand.

When Alexander the Great came asking what he could do for him, he said “please move away from the sun that lights on me”.

A similar story is associated with Archimedes’ slaughter by a Roman soldier…

Greek historians

Polybius (ca. 200–118 BC) moved from Macedonia to Rome
Plutarch – (46 – 120 AD)
Cicero (Marcus Tullius Cicero) (106–43 BC)
Vergil [or Virgil, Publius Vergilius Maro**] (70-19 BC)** "the Roman Homer… **"**
Philo (c. 15 BC to 40 AD) or Philon from Alexandria – a Jewish historian
Diogenes Laertius (3rd cent. AD)