Tutorial 13 – Feynman diagrams

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Contents

1 Lightning review of perturbation theory

We review the various steps that result in the perturbation series for time-ordered correlation functions in QFT.

We are given a theory with fields $\{\phi_i\}$ (these stand for all the fields in the theory) and Hamiltonian:

$$\mathcal{H} = \mathcal{H}_0 + \lambda \mathcal{H}_{\text{int}} \tag{1}$$

where $\mathcal{H}_0$ is the 'free' part, containing all the kinetic terms for the fields; and $\mathcal{H}_{\text{int}}$, containing all the interactions. The object that we're after is the time-ordered correlator $\langle \Omega | T\{\phi_i ...\} | \Omega \rangle$, where the brackets contain the fields of the theory at various different space-time points.

1. Some amount of algebra shows that:

$$\langle \Omega | T\{\phi_i ...\} | \Omega \rangle = \frac{\langle 0 | T\{\phi_i ... e^{-i\lambda \int \mathcal{H}_{\text{int}} \text{d}t}\} | 0 \rangle}{\langle 0 | T\{e^{-i\lambda \int \mathcal{H}_{\text{int}} \text{d}t}\} | 0 \rangle} \tag{2}$$

2. The philosophy behind perturbation theory is that there is a small parameter λ which, in some sense, captures the strength of the interactions; and that the observables of the interacting theory may be computed as a Taylor series in λ (sometimes there are multiple interactions and hence $\lambda \to \lambda_i$). In this spirit, the right-hand side is written as an expansion in λ.

$$\langle \Omega | T\{\phi_i ...\} | \Omega \rangle = f^{(0)} (\{x_i\}) + \lambda f^{(1)} (\{x_i\}) + \mathcal{O} \left(\lambda^2\right) \tag{3}$$

3. All the $f^{(i)}$'s may be evaluated via Wick's theorem (each theory comes with its own fields and hence its own set of contractions, but the statement of the theorem stays the same) which equates each correlator to a sum over contractions. To compute the correlator is to compute this sum for as many $f^{(i)}$'s as needed.

4. As is written in the lecture notes, each contraction may be represented by a diagram, and lots of contractions lead to the same diagram (and the same expression). So while there are thousands of contractions that lead to every $f^{(i)}$, there are relatively few diagrams. Hence, in practice, we would like to have some (Feynman) rules that let us go from diagrams to integrals, so that all we'll need to do is to draw the necessary diagrams at each stage, and sum up the corresponding expressions. This is the topic of this tutorial.

2 Feynman rules for a theory of real scalars

Let's look at the example of the ϕ^4 theory that was discussed in class:

$$\mathcal{L} = \frac{1}{2} (\partial \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4 \tag{4}$$

$$\lambda \mathcal{H}_{\text{int}} = \frac{\lambda}{4!} \int \text{d}^3 x \; \phi^4(x) \tag{5}$$

There is just one relevant contraction, as was discussed in class:

$$\overbrace{\phi(x_1)\phi}(x_2) = D_F (x_1 - x_2) \tag{6}$$

We consider the expansion of the correlator:

$$\langle \Omega | \phi(x_1) \phi(x_2) | \Omega \rangle \tag{7}$$

We focus on the numerator for now (we will discuss a method to account for the denominator in a bit). At leading order, this is just the free theory correlator:

$$f^{(0)}\left(x_1, x_2\right) = \langle 0|\phi(x)\phi(y)|0\rangle \tag{8}$$
$$= D_F\left(x_1 - x_2\right) \tag{9}$$

In this case, there are only two vertices in the problem (x and y), both external, and they are connected by the propagator – this may be represented by a diagram:

$$x_1 \text{ - - - - - - - - - } x_2 \quad = D_F\left(x_1 - x_2\right)$$

Let's go to $\mathcal{O}(\lambda)$:

$$\lambda f^{(1)}\left(x_1, x_2\right) = \langle 0|\phi(x_1)\phi(x_2)\left(-i\frac{\lambda}{4!}\int \mathrm{d}^4x\ \phi^4(x)\right)|0\rangle \tag{10}$$

There are two *types* (or *topologies*) of contractions that contribute at this level:

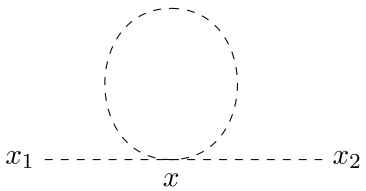
$$\langle 0|\phi(x_1)\phi(x_2)\left(-i\frac{\lambda}{4!}\int \mathrm{d}^4x\ \phi(x)\ \phi(x)\ \phi(x)\ \phi(x)\right)|0\rangle = -i\frac{\lambda}{4!}\int \mathrm{d}^4x\ D_F\left(x_1 - x_2\right) D_F\left(x - x\right)^2 \tag{11}$$

$$\langle 0|\phi(x_1)\phi(x_2)\left(-i\frac{\lambda}{4!}\int \mathrm{d}^4x\ \phi(x)\ \phi(x)\ \phi(x)\ \phi(x)\right)|0\rangle = -i\frac{\lambda}{4!}\int \mathrm{d}^4x\ D_F\left(x_1 - x\right) D_F\left(x_2 - x\right) D_F\left(x - x\right) \tag{12}$$

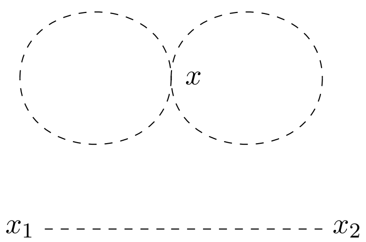
It is easy to see that any other contraction gives one of these two expressions. For example:

$$\langle 0|\phi(x_1)\phi(x_2)\left(-i\frac{\lambda}{4!}\int \mathrm{d}^4x\ \phi(x)\ \phi(x)\ \phi(x)\ \phi(x)\right)|0\rangle = -i\frac{\lambda}{4!}\int \mathrm{d}^4x\ D_F\left(x_1 - x\right) D_F\left(x_2 - x\right) D_F\left(x - x\right) \tag{13}$$

Hence, the expression that any contraction gives is only dependent on the topology of the contraction, and it is this feature that is captured by Feynman diagrams. Hence, each Feynman diagram will stand for one such expression, and is the sum of many individual contractions. Let's write down the two types of contractions we have in diagrammatic form:



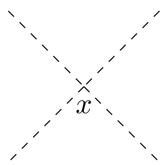
$$\propto -i\lambda\int \mathrm{d}^4x\ D_F\left(x_1 - x\right) D_F\left(x_2 - x\right) D_F\left(x - x\right) \tag{14}$$



$$\propto -i\lambda\int \mathrm{d}^4x\ D_F\left(x_1 - x_2\right) D_F\left(x - x\right)^2 \tag{15}$$

There are three vertices – x_1 and x_2 are external, and x is internal (i.e., integrated over). A $D_F\left(x_i - x_j\right)$ means that the vertices x_i and x_j are to be connected, $D_F\left(x_i - x_j\right)^2$ means that they are to be connected twice, and so on. Each internal vertex should come with $-i\lambda\int \mathrm{d}^4x$, and hence all internal vertices must be integrated over. External vertices don't seem to contribute anything.

To summarise (as seen in class):

x_1 - - - - - - - - - x_2	$D_F\left(x_1 - x_2\right)$		
x_1 - - - - - - - - - -	1		$-i\lambda \int \mathrm{d}^4 x$ (16)
- - - - - - - - - - x_2	1		

It is straightforward to convince oneself that these rules work at higher orders.

Of course, to get the full expression that each diagram contributes to the relevant $f^{(i)}$, one must sum over all contractions that have the same topology as the diagram, and divide by the 4! that came with the interaction term. This means that the actual expression that each Feynman diagram contributes is the expression given by the rules we just wrote down times a combinatorial factor that we'll call $1/S$ (so that S is an integer), where S is the symmetry factor of the diagram. As far as this course is concerned, you will not need to know how to compute symmetry factors, and you may move on (you may skip the next subsection).

2.1 Optional: Symmetry factors

This subsection is only for the curious. Let's sum over *all* contractions, and write down the full expression for $\lambda f^{(1)}\left(x, y\right)$:

$$\lambda f^{(1)}\left(x_1, x_2\right) = \langle 0|\phi(x_1)\phi(x_2)\left(-i\frac{\lambda}{4!}\int \mathrm{d}^4x\ \phi^4(x)\right)|0\rangle \tag{17}$$

$$= 12\left(-i\frac{\lambda}{4!}\int \mathrm{d}^4x\ D_F\left(x_1 - x\right) D_F\left(x_2 - x\right) D_F\left(x - x\right)\right)$$
$$+3\left(-i\frac{\lambda}{4!} D_F\left(x_1 - x_2\right)\int \mathrm{d}^4x\ D_F\left(x - x\right)^2\right) \tag{18}$$

It is easy to count the number of contractions in this case: to create the first term, one needs to contract both x_1 and x_2 with an x. To do this, one needs to select an x for x_1 (4 choices) and *then* select an x for x_2 (3 choices). The remaining contractions are then fixed, and we have 12 contractions. This cancels with the 4! from the denominator to give a 1/2 (i.e., $S = 2$ for this diagram). For the second term, x_1 goes with x_2, and all that we need to do is pick the first $\phi(x)$ inside the integral and contract it with another $\phi(x)$ (3 choices); this fixes the remaining contraction, and hence we have 3 contractions that lead to this diagram. This factor cancels with the 4! to give a symmetry factor of $S = 8$.

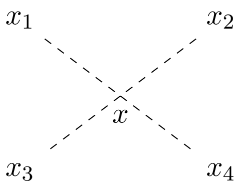
Hence, the symmetry factor is the residual factor left over from the cancellation of the number of contractions with the 4! that comes with the coupling (and later an additional $N!$ that comes from expanding the interaction exponential $e^{-i\frac{\lambda}{4!}\int \mathrm{d}^4x\ \phi^4(x)} \to \left(-i\frac{\lambda}{4!}\int \mathrm{d}^4x\ \phi^4(x)\right)^N/N!$). One may wonder why we attached a 4! to the coupling to begin with. The answer is: the 4! (times the N! from the exponent) is the actual number of contractions that correspond to a generic diagram! Let's look at the following correlator at first order in $\mathcal{O}\left(\lambda\right)$:

$$\langle\Omega|\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|\Omega\rangle \sim \langle 0|\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)e^{-i\lambda\int \mathcal{H}_{\mathrm{int}}\mathrm{d}t}|0\rangle \tag{19}$$

We consider the following contraction at first order in λ:

$$\langle 0|\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\left(-i\frac{\lambda}{4!}\int \mathrm{d}^4x\ \phi(x)\ \phi(x)\ \phi(x)\ \phi(x)\right)|0\rangle \tag{20}$$

which gives the following diagram:



$$\propto -i\lambda \int \mathrm{d}^4x \; D_F\left(x_1 - x\right) D_F\left(x_2 - x\right) D_F\left(x_3 - x\right) D_F\left(x_4 - x\right) \quad (21)$$

Let's count all the possible contractions: the external field $\phi(x_1)$ is paired with some internal $\phi(x)$ (4 choices), and the same is done in the case of $\phi(x_2)$ (3 choices), $\phi(x_3)$ (2 choices), and $\phi(x_4)$ (1 choice). This gives exactly a factor of 4!, which means that the symmetry factor is 1. The point here is that in a generic diagram that isn't "symmetric" in any way (here, there are four independent external legs connected to the vertex, so the diagram isn't symmetric), the 4! exactly cancels the number of contractions and the "naive" rules give the right result. And in the cases we considered earlier (Eqn. 14 and 15), it is the diagram's fault that there were fewer contractions than expected (recall that certain contractions were forced upon us once we made a few choices – this comes from the symmetry of the diagram).

It is not very straightforward (at least as far as we know) to learn how to deduce the symmetry factor of a diagram just by looking at it. For those who want some practice, the textbook by Peskin has a few examples, and the textbook ("Quantum Field theory") by Mark Srednicki (available online for free) has more.

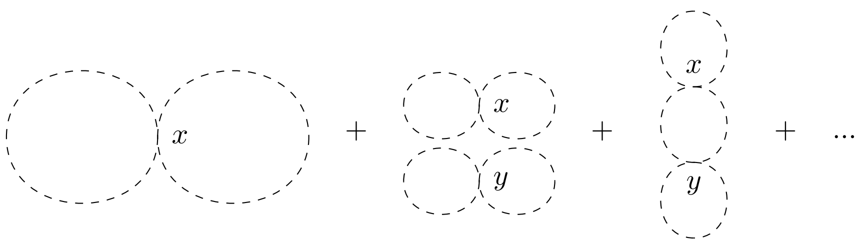
2.2 Dealing with the denominator

So far we've been focussed on the numerator. But how does one account for the terms that come from down under? In this subsection, we will see (but not prove, since this will involve computing symmetry factors) that these contributions systematically cancel, and the end result is that to properly account for the denominator, one need just ignore the "disconnected" diagrams that contribute to the numerator.

Let's start by looking at what kind of diagrams make up the denominator in ϕ^4 theory:

$$\langle 0|e^{-i\lambda \int \mathrm{d}^4x \; \phi^4(x)}|0\rangle \quad (22)$$

It is easy to see that what contributes to this correlator are diagrams without any external legs ("sources"). Such diagrams are called "bubble diagrams". Let's draw a few of these:

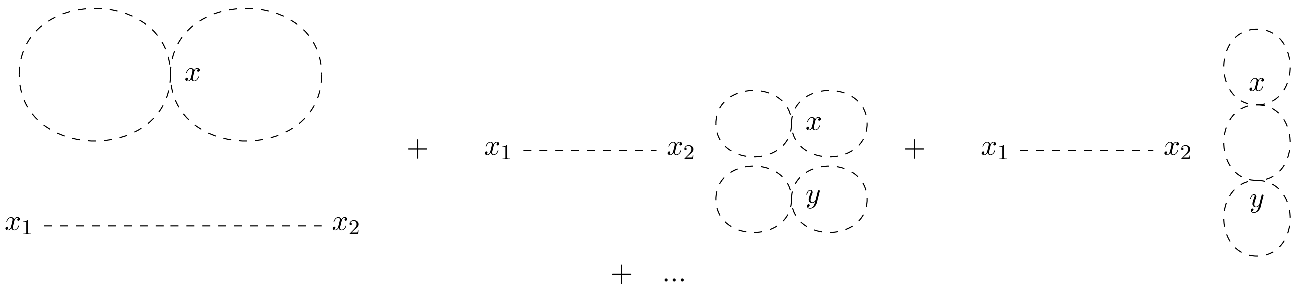


(23)

(we don't care about orders in λ for this particular argument). Now let's look at the correlator $\langle 0|\phi(x_1)\phi(x_2)e^{-i\lambda \int \mathrm{d}^4x \; \phi^4(x)}|0\rangle$, and consider the diagram at $\mathcal{O}\left(\lambda^0\right)$:

$$x_1 \text{ - - - - - - - - - } x_2 \quad (24)$$

It is now easy to see that the following are possible diagrams too (that occur at higher orders in λ):



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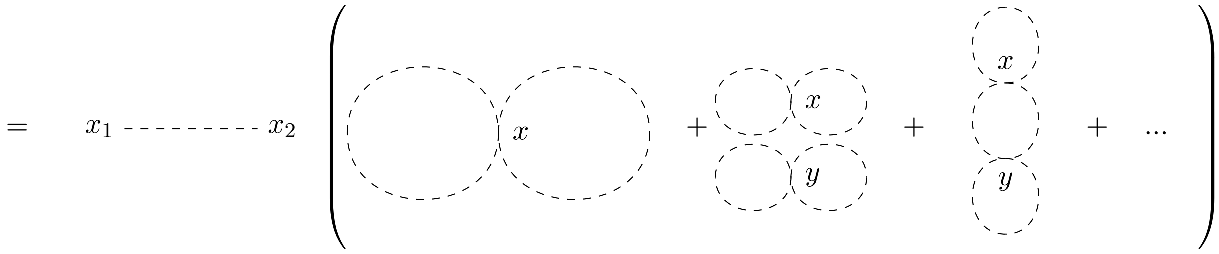


It may help to write down a couple of these to see that they do indeed factor. It is easy to see that this holds for every diagram that contributes to the numerator. More specifically, the claim is that the numerator factors into: (sum of diagrams without bubble subdiagrams) $\times$ (sum of all bubble diagrams), when we consider terms of all orders in perturbation theory. And since the denominator is just the sum of all the bubble diagrams, both these factors neatly cancel, leaving behind only those numerator diagrams that do not have bubble diagrams. The difficult part of this proof is to show that the various symmetry factors that accompany every term allow for factorisation and cancellation. We do not do this here.

But the takeaway is that to account for the numerator, one needs to sum only those numerator diagrams that do not have bubble subdiagrams. These are called 'connected diagrams'. Alternatively, these may be defined as those diagrams where any given vertex is connected to at least one source (external vertex).

2.3 Momentum space Feynman rules

The rules that were stated earlier (Eqn. 16) are called the position-space Feynman rules. For various applications, we find it easier to do things in momentum space. In this subsection, we will motivate the momentum space version of the rules by Fourier transforming the diagrams we were looking at earlier (Eqn. 14 and 15). We recall the following:

$$D_F(x-y) = \int \bar{d}^4p \; e^{-ip.(x-y)} \frac{i}{p^2 - m^2 + i\epsilon} \tag{26}$$

The diagram in Eqn. 14 is:

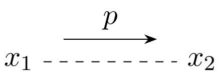
$$A1 = \frac{1}{S}\left(-i\lambda \int \mathrm{d}^4x \; D_F(x_1 - x)\, D_F(x_2 - x)\, D_F(x - x)\right) \tag{27}$$

$$= \frac{1}{S}\left(-i\lambda \int \mathrm{d}^4x \; \left(\int \bar{d}^4p_1 \; e^{-ip_1.(x_1-x)} \frac{i}{p_1^2 - m^2 + i\epsilon}\right)\left(\int \bar{d}^4p_2 \; e^{-ip_2.(x_2-x)} \frac{i}{p_2^2 - m^2 + i\epsilon}\right)\left(\int \bar{d}^4p_3 \; e^{-ip_3.(x-x)} \frac{i}{p_3^2 - m^2 + i\epsilon}\right)\right) \tag{28}$$

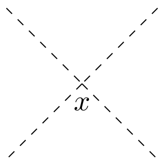
$$= \frac{1}{S}\int \bar{d}^4p_1 e^{-ip_1x_1} \bar{d}^4p_2 e^{-ip_2x_2} \bar{d}^4p_3 \,(-i\lambda) \int \mathrm{d}^4x \; e^{i(p_1+p_2+p_3-p_3).x} \; \frac{i}{p_1^2 - m^2 + i\epsilon}\frac{i}{p_2^2 - m^2 + i\epsilon}\frac{i}{p_3^2 - m^2 + i\epsilon} \tag{29}$$

$$= \frac{-i\lambda}{S}\int \bar{d}^4p_1 e^{-ip_1x_1} \bar{d}^4p_2 e^{-ip_2x_2} \bar{d}^4p_3 \bar{\delta}(p_1 + p_2 + p_3 - p_3) \; \frac{i}{p_1^2 - m^2 + i\epsilon}\frac{i}{p_2^2 - m^2 + i\epsilon}\frac{i}{p_3^2 - m^2 + i\epsilon} \tag{30}$$

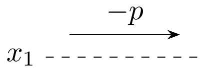
It is straightforward to see what happens: each propagator gets a $\frac{i}{p^2-m^2+i\epsilon}$, along with a $e^{ip.(x_i-x_f)}$ (x_i and x_f are the points the propagator starts and ends at; the respective p may be thought of as the momentum of the propagator). The integral over any internal vertex then collects all the $e^{ip.(x_i-x_f)}$'s that have a leg in the vertex, and turn this into a delta function (of course, there is still an $-i\lambda$ for every vertex) that imposes momentum conservation at each vertex. Since all the momenta are integrated over, and there are delta functions at every vertex, the only momentum integrals that survive are the ones which are left undetermined by momentum conservation (i.e., the ones which aren't killed by the delta functions). This leads to the following rules:



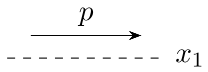
$$\frac{i}{p^2-m^2+i\epsilon}$$



$$-i\lambda \tag{31}$$



$$e^{-ip.x_1}$$



$$e^{-ip.x_1}$$

(Note the direction of the external momenta). Furthermore, impose momentum conservation at each vertex and integrate over all undetermined momenta (more precisely, add 2π times a momentum-conserving delta function at each vertex and integrate over *all* momenta). Then, divide by the symmetry factor.

Note on what was discussed in the tutorial The rules as listed above still describe the computation of $\langle 0|T\left[\phi(x_1)...e^{-i\frac{\lambda}{4!}\int d^4x\ \phi(x)^4}\right]|0\rangle$. The rules discussed in the tutorial did not include the external factors of e^{-ipx}, and had an extra momentum-conserving δ function in all correlators. This is because they describe the computation of the Fourier-transformed quantity:

$$\int d^4x_1...e^{i(p_1.x_1+...)}\langle 0|T\left[\phi(x_1)...e^{-i\frac{\lambda}{4!}\int d^4x\ \phi(x)^4}\right]|0\rangle \tag{32}$$

Let's now apply the new rules to a different diagram (see Eqn. 21)

$$A2 = -i\lambda\int d^4x\ D_F(x_1-x)\,D_F(x_2-x)\,D_F(x_3-x)\,D_F(x_3-x) \tag{33}$$

$$\begin{aligned} &= -i\lambda\int d^4x\ \int đ^4p_1\frac{i}{p_1^2-m^2+i\epsilon}e^{-ip_1.(x_1-x)}\int đ^4p_2\frac{i}{p_2^2-m^2+i\epsilon}e^{-ip_2.(x_1-x)} \\ &\qquad \int đ^4p_3\frac{i}{p_3^2-m^2+i\epsilon}e^{-ip_3.(x_3-x)}\int đ^4p_4\frac{i}{p_4^2-m^2+i\epsilon}e^{-ip_4.(x_4-x)} \end{aligned} \tag{34}$$

$$\begin{aligned} &= -i\lambda\int đ^4p_1đ^4p_2đ^4p_3đ^4p_4\ e^{-i(p_1x_1+...+p_4x_4)}\frac{i}{p_1^2-m^2+i\epsilon}\frac{i}{p_2^2-m^2+i\epsilon}\frac{i}{p_3^2-m^2+i\epsilon}\frac{i}{p_4^2-m^2+i\epsilon} \\ &\qquad \bar{\delta}(p_1+...+p_4) \end{aligned} \tag{35}$$

which is exactly the expression given by the rules (the symmetry factor is 1 in this case).

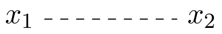
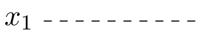
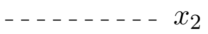
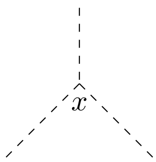
2.4 A different kind of interaction – ϕ^3 theory

Let's now switch out the ϕ^4 interaction for a ϕ^3 one. More precisely,

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{3!}\phi^3 \tag{36}$$

$$\lambda\mathcal{H}_{\text{int}} = \frac{\lambda}{3!}\int d^3x\ \phi^3(x) \tag{37}$$

Surveying the various steps involved in perturbation theory, we notice that since the free Hamiltonian is the same, all the steps stay essentially the same, except that we need to use a ϕ^3 interaction in place of the ϕ^4 one. It is straightforward to work out a couple of examples and arrive at the following position space rules:

	$D_F(x_1 - x_2)$
	1
	1
	$-i\lambda \int \mathrm{d}^4x$ (38)

Dividing by the symmetry factor then gives the full answer. Ignoring the disconnected diagrams while doing computations accounts for the denominator.

This is an instance of the general pattern that the kinetic terms control the external leg factors and the propagator, while the interaction terms control the vertex factors (this is entirely reasonable).

3 Feynman rules for a theory of complex scalars

We now look at an instance where the kinetic part of the Lagrangian is different. Specifically,

$$\mathcal{L} = \partial^\mu \varphi^* \partial_\mu \varphi - m^2 \varphi^* \varphi - \frac{\lambda}{4} (\varphi^* \varphi)^2 \tag{39}$$

$$\lambda \mathcal{H}_{\text{int}} = \frac{\lambda}{4} \int \mathrm{d}^3x \ \ (\varphi^* \varphi)^2 \tag{40}$$

This time, we have more fields, and hence correlation functions generally contain both φ and φ^*. Then everything is the same until we get to Wick's theorem. As was seen in previous tutorials, there are two different kinds of contractions in this theory:

$$\varphi^*(x)\varphi(y) = \varphi(x)\varphi^*(y) \tag{41}$$

$$= D_F(x - y) \tag{42}$$

where D_F is the same function as in the real scalar case. The first distinct feature we note in the complex scalar case is that one cannot contract φ or φ^* with itself. One has to be contracted with the other, and we need to keep track of this while drawing diagrams (this wasn't a problem in the previous example).

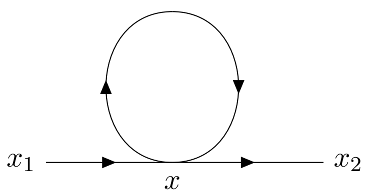
Let's look at a concrete example. For simplicity, consider:

$$\langle 0 | T \left\{ \varphi(x_1) \varphi^*(x_2) e^{-i\frac{\lambda}{4}\int \mathrm{d}^4x \ \ (\varphi^*(x)\varphi(x))^2} \right\} |0\rangle \tag{43}$$

Let's try to draw a diagram at $\mathcal{O}(\lambda)$. We have two different kinds of external legs – φ and φ^*; this also happens at the vertex. To keep track of these, we introduce *charge arrows*, with the convention that if the arrow points away from the vertex, it's a φ and if the arrow points toward the vertex, it's a it's a φ^*. So for the contraction:

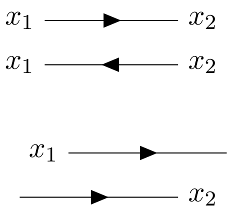
$$\langle 0 | \varphi(x_1) \varphi^*(x_2) \left(-i\frac{\lambda}{4} \int \mathrm{d}^4x \ \varphi^*(x)\varphi(x)\varphi^*(x)\varphi(x) \right) |0\rangle \tag{44}$$

we have the diagram:



(45)

It is easy to see that the position space Feynman rules are the same for real and complex scalars, except for the fact that one needs to keep track of the charge arrows.

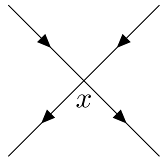


$D_F(x_1 - x_2)$

$D_F(x_1 - x_2)$

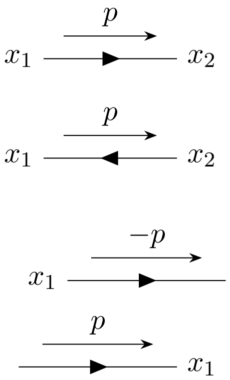
1

1



$$-i\lambda \int \mathrm{d}^4 x \tag{46}$$

The remaining steps are the same as in the real scalar – one needs to divide by the symmetry factor and include only connected diagrams. A little bit of work (of the same kind as was done in the previous section) then shows that the momentum space Feynman rules also stay the same:

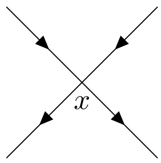


$\dfrac{i}{p^2 - m^2 + i\epsilon}$

$\dfrac{i}{p^2 - m^2 + i\epsilon}$

$e^{-ip.x_1}$

$e^{-ip.x_1}$



$$-i\lambda \tag{47}$$

Note that the only thing the charge arrows do is keep track of the contractions; the expressions only depend on the direction of the momentum arrow[1].

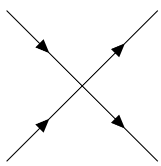
3.1 Scattering amplitudes and correlation functions

The reason we're after time-ordered correlators is that ultimately, these may be used to compute the amplitudes for various physical processes that may be observed directly – the most important of which are scattering amplitudes and decay rates. All that we'll say here is that there are QFT tools that allow for computation of specific amplitudes from specific correlators (one correlator allows for computation of various related amplitudes). In this subsection, we'll answer the question: if we're after a specific scattering amplitude, which correlator must we look at?

Let's look at the complex scalar theory that we were just discussing. The process that we can study is scattering between various φ particles and φ^* particles. Let's we want to compute the scattering amplitude of the process: $2\varphi \to 2\varphi$. Then, we need to feed in the correlator:

$$\langle \Omega | T \left\{ \varphi(x_1)\varphi(x_2)\varphi^*(x_3)\varphi^*(x_4) \right\} | \Omega \rangle \tag{48}$$

This is very intuitive from the way the diagrams look. For instance, one of the diagrams that contribute to the above correlator is:



(49)

[1]This is no longer true for fermions. However, we won't worry about writing down expressions in the fermionic case.

and it is easy to view this as two φ particles coming in from the left and two φ particles exiting to the right (with 'time' flowing from left to right). And this is indeed one of the diagrams that contribute to this process. In general,

$$\begin{aligned} \text{number of } \phi\text{'s in the correlator } &= \text{number of incoming } \phi \text{ particles} + \text{number of outgoing } \phi^* \text{ particles} \\ \text{number of } \phi^*\text{'s in the correlator } &= \text{number of outgoing } \phi \text{ particles} + \text{number of incoming } \phi^* \text{ particles} \end{aligned} \tag{50}$$

Note that in the real scalar theory, $\phi^* = \phi$ and hence one just needs to insert as many ϕ's as the number of external particles.

3.2 Fermions

For the purposes of this course, we won't be writing down expressions for diagrams involving fermions. All that we'll care about is drawing these diagrams, and then it turns out that fermions behave exactly like complex scalars. Let's discuss a couple of examples.

Firstly, the "Gross-Neveu model":

$$\mathcal{L} = i\bar{\psi}\partial\!\!\!/\psi - m\bar{\psi}\psi - \frac{\lambda}{4}\left(\bar{\psi}\psi\right)^2 \tag{51}$$

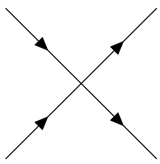
The correlators of the theory take the form:

$$\langle\Omega| T\left\{\psi, \bar{\psi}\right\}|\Omega\rangle \tag{52}$$

(i.e., there are many insertions of ψ and $\bar{\psi}$). As far as diagrams are concerned, this theory looks exactly like the complex scalar model we discussed in the previous subsection. For example, the correlator:

$$\langle\Omega| T\left\{\psi(x_1)\psi(x_2)\bar{\psi}(x_3)\bar{\psi}(x_4)\right\}|\Omega\rangle \tag{53}$$

(this may be used to calculate the scattering amplitude $2\psi \to 2\psi$, among other things). This behaves exactly like the corresponding scalar correlator, and all of the diagrams are the same (the expressions are vastly different, but we won't worry about those). One of the diagrams at $\mathcal{O}(\lambda)$ is:



(54)

exactly as before.

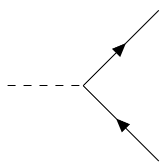
The second example is a theory with a "Yukawa" interaction.

$$\mathcal{L} = \frac{1}{2}\left(\partial\phi\right)^2 - \frac{1}{2}M^2\phi^2 + i\bar{\psi}\partial\!\!\!/\psi - m\bar{\psi}\psi - \lambda\phi\bar{\psi}\psi \tag{55}$$

The correlators of this theory will include both ψ and ψ insertions. For example,

$$\langle\Omega| T\left\{\phi(x_1)\psi(x_2)\bar{\psi}(x_3)\right\}|\Omega\rangle \tag{56}$$

is the correlator that may be used to compute the amplitude of the process: $\phi \to \bar{\psi}\psi$ (this may be interpreted as a decay process with a ϕ particle decaying into a ψ particle-antiparticle pair). One of the diagrams at $\mathcal{O}(\lambda)$ is:

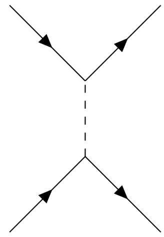


(57)

Meanwhile, the $2\psi \to 2\psi$ scattering process is given by the correlator:

$$\langle\Omega|T\left\{\psi(x_1)\psi(x_2)\bar{\psi}(x_3)\bar{\psi}(x_4)\right\}|\Omega\rangle \tag{58}$$

And one of the diagrams at $\mathcal{O}\left(\lambda^2\right)$ is:



(59)

4 Selection rules in perturbation theory

The fact that (continuous) symmetries lead to conserved currents should be a familiar one. These conserved currents may then be integrated to charges, which then lead to constraints on physical processes. This section deals with seeing how this manifests in our perturbation theory formalism. To be precise, we will see how conserved charges give constraints on correlation functions (which will lead to constraints on scattering amplitudes via QFT machinery).

We start with a free theory with fields ϕ and some continuous symmetry with a conserved current j^μ. A conserved charge may be constructed by integrating the current $Q = \int \mathrm{d}^3x\; j^0$. The conserved charge generates the symmetry transformations on the field via the relation:

$$\phi'(x') = e^{i\alpha Q}\phi(x)e^{-i\alpha Q} =: U\phi(x)U^\dagger \tag{60}$$

(recall that Q is also an operator). In addition, if the interactions (this means that the transformations remain a symmetry of the interacting theory) and vacuum are invariant under the symmetry group[2],

$$\begin{aligned}\chi &:= \langle 0|\phi\phi...\phi e^{-i\int\lambda\mathcal{H}_{\text{int}}\mathrm{d}t}|0\rangle && (61)\\ &= \langle 0|U^\dagger U\phi U^\dagger U\phi U^\dagger U...U^\dagger U\phi U^\dagger U e^{-i\int\lambda\mathcal{H}_{\text{int}}\mathrm{d}t}U^\dagger U|0\rangle && (62)\\ &= \langle 0|\phi'\phi'...\phi' e^{-i\int\lambda\mathcal{H}_{\text{int}}\mathrm{d}t}|0\rangle && (63)\end{aligned}$$

Let's go through the steps above: the first equality is a definition, the second is inserting $UU^\dagger = \mathbb{1}$ throughout the expression (recall that the conserved charge is a Hermitian operator and hence U is unitary), and in the third, we've used the fact that the vacuum and interactions are invariant. In the following, we will use this equation to derive some of the consequences of specific symmetries on correlators.

We will deal with two examples: $\mathbb{Z}_2$, and $U(1)$; but it should be clear how to generalise this to other symmetry groups.

4.1 $U(1)$

It is easy to apply Eqn. **??** directly to the case of a $U(1)$ symmetry acting on, say the theory of complex scalars given by Eqn. 39:

$$\begin{aligned}U\varphi(x)U^\dagger &= e^{i\alpha}\varphi(x) && (64)\\ U\varphi^*(x)U^\dagger &= e^{-i\alpha}\varphi^*(x) && (65)\end{aligned}$$

and hence:

$$\begin{aligned}\chi &:= \langle 0|\phi\phi^*...\phi e^{-i\int\lambda\mathcal{H}_{\text{int}}\mathrm{d}t}|0\rangle && (66)\\ &= \langle 0|\phi'(\phi^*)'...\phi' e^{-i\int\lambda\mathcal{H}_{\text{int}}\mathrm{d}t}|0\rangle && (67)\\ &= e^{i(\#(\phi)-\#(\phi^*))}\langle 0|\phi\phi^*...\phi e^{-i\int\lambda\mathcal{H}_{\text{int}}\mathrm{d}t}|0\rangle && (68)\\ &= e^{i(\#(\phi)-\#(\phi^*))}\chi && (69)\end{aligned}$$

[2] Situations where the second assumption doesn't hold are where you have spontaneous symmetry breaking.

This shows that any correlator that doesn't have the same number of ϕ's and ϕ^*'s vanishes. Looking at the corresponding scattering amplitude (Eqn. 50), this means that:

$$\#\,(\text{incoming } \phi\text{'s}) - \#\,(\text{incoming } \phi^*\text{'s}) = \#\,(\text{outgoing } \phi\text{'s}) - \#\,(\text{outgoing } \phi^*\text{'s}) \tag{70}$$

This is the constraint on physical processes, and is exactly charge conservation in the usual sense if ϕ, and ϕ^* particles are interpreted as particle-antiparticle pairs (you are encouraged to see for yourselves how this works out in terms of charge arrows). Since the transformation is a symmetry of the Hamiltonian, this works out diagram by diagram, and not just at the level of the full correlator.

4.2 $\mathbb{Z}_2$

It was discussed in an earlier tutorial that if a theory described by a Lagrangian has a symmetry under which the action stays invariant, Noether's theorem may be used to construct a conserved current and subsequently a conserved charge. But even discrete symmetries can have conserved charges (not to be confused with conserved currents). We will merely motivate this by looking at the actual constraints on the correlators. The symmetry transformation is:

$$\phi(x) \to \phi'(x') = -\phi(x) \tag{71}$$

(the construction of the unitary operator that implements this transformation in the sense of Eqn. 60 was what was discussed in the tutorial, but it's not very important). Repeating the same steps as before:

$$\begin{aligned}
\chi &:= \langle 0|\phi\phi...\phi e^{-i\int \lambda \mathcal{H}_{\text{int}}\text{d}t}|0\rangle && (72)\\
&= \langle 0|\phi'\phi'...\phi' e^{-i\int \lambda \mathcal{H}_{\text{int}}\text{d}t}|0\rangle && (73)\\
&= (-1)^{\#(\phi)}\,\langle 0|\phi\phi...\phi e^{-i\int \lambda \mathcal{H}_{\text{int}}\text{d}t}|0\rangle && (74)\\
&= (-1)^{\#(\phi)}\,\chi && (75)
\end{aligned}$$

The final result is that any correlator (and indeed any diagram) that doesn't have an even number of ϕ's vanishes. And at the level of amplitudes, any process where the number of ϕ's going in doesn't match the number of ϕ's coming out *modulo 2* vanishes. This may be given the interpretation of a charge conservation rule: if each ϕ particle is given a charge 1, then charge is conserved modulo 2.[3]

The same steps may be used to derive constraints for bigger symmetries.

[3] The charge operator is given by:

$$Q = (-1)^N \tag{76}$$

where N is the number operator. There is no conserved current. Please don't worry about this statement if you don't understand it.