Cosmology 2021 - Tutorial 9

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1 Big Bang Nucleosynthesis (BBN) and particle physics

BBN is a powerful observational probe. It is the earliest direct probe of the early universe, when it was in a temperature of $\sim 1\text{ MeV}$, in the radiation-dominated era. BBN is a strong evidence in favour of the big-bang paradigm, and it sets strong constraints on many modifications to the Standard Model of particle physics and gravity. In the following we explore such modifications.

1.1 $Y_p$

We derived in class an expression for the mass fraction of $^4\text{He}$, let us refresh our memory. In the radiation domination era the Hubble expansion rate is given by

$$H \approx \frac{g_*^{1/2} G_N^{1/2} T^2}{\hbar^3 c^3},$$

(1)

where $g_* = \Sigma_{\text{bosons}} g_i + \frac{7}{8} \Sigma_{\text{fermions}} g_i$ (Why $7/8$? Fermi-dirac vs. Bose-Einstein). At high temperatures the weak interaction is dominating the following processes: $n + \nu_e \leftrightarrow p + e$, $n + e^+ \leftrightarrow p + \bar{\nu}$, $n \leftrightarrow p + e + \bar{\nu}$ that dictates thermal equilibrium between neutrons and protons. The weak interaction rate is given by (assuming interacting particles are relativistic)

$$\Gamma_{\text{wk}} \approx \frac{G_N^2 T^5}{\hbar}.$$  

(2)

As the temperature decrease to $T_{\text{fo}} \approx 8 \times 10^9[\text{K}]$ the weak interactions becomes too slow and the ratio between neutrons and protons freezes out.

$$\frac{X_n}{X_p} \approx \exp(-Q/T_{\text{fo}}) \approx \frac{1}{6},$$

(3)

with $Q = m_n - m_p = 1.29\text{MeV}$. Until deuterium can be created appreciably ($T_D = 7.5 \times 10^8[\text{K}]$), thus allowing for heavier elements to form, the neutrons can ($\beta$-) decay with lifetime of $\tau_n = 880.3 \pm 1$ s. 

$$\frac{X_n}{X_p} \approx \exp(-Q/T_{\text{fo}}) \exp(-t/\tau_n).$$

(4)

Assuming all of the neutrons end up as Helium ($p + n \leftrightarrow d + \gamma$, $d + d \rightarrow p + t$, $d + d \rightarrow ^3\text{He} + n$, $d + t \rightarrow ^4\text{He} + p$) the mass fraction is

$$Y_p = \frac{4(X_n/2)}{X_p + X_n} = \frac{2(X_n/X_p)}{1 + X_n/X_p} = \frac{2 \exp(-Q/T_{\text{fo}}) \exp(-t/\tau_n)}{1 + \exp(-Q/T_{\text{fo}}) \exp(-t/\tau_n)} = \frac{2}{1 + \exp(Q/T_{\text{fo}}) \exp(t/\tau_n)}.$$  

(5)

We employ this relation to have some control over the parametric dependence of $Y_p$. 

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1.2 BBN vs. modified gravity

Today $G_N = (6.67408 \pm 0.00031) \times 10^{-8} \text{cm}^3 \text{g}^{-1} \text{s}^{-2}$. Let us probe possible deviations of $G_N$ from its current value during the epoch of BBN, based on the $^4\text{He}$ abundance measurement $Y_p = 0.2449 \pm 0.0040$.\(^1\) We estimated freeze-out temperature as (assuming interacting particles are relativistic)

$$H \sim \Gamma_{wk} \rightarrow g_*^{1/2} G_N^{1/2} T_{fo}^2 \frac{h^3 c^5}{\hbar^3} \sim \frac{G_N^2 T_{fo}^5}{\hbar},$$

thus

$$T_{fo} \propto (G_N)^{1/6}.$$ \hspace{1cm} (7)

It follows then that

$$\frac{\delta Y_p}{Y_p} \approx \frac{Y_p}{2} \left( \frac{2}{Y_p} - 1 \right) \frac{Q}{T_{fo}} \frac{\delta T_{fo}}{T_{fo}} = \frac{1}{6} \left( 1 - \frac{Y_p}{2} \right) \frac{Q}{T_{fo}} \frac{\delta G_N}{G_N},$$

which yields $\delta G_N/G_N \approx 10\%$, therefore only a mild change is allowed. Notice that we neglected any variation in $t$. We can do that since $t$ is quite smaller than $\tau_n$, $\exp(-t/\tau_n)$ does not change appreciably compared the variation from $T_{fo}$.

1.3 Additional neutrino generations

At a temperature of $T \sim 1\text{MeV} \sim 10^{10}\text{K}$ the relativistic degrees of freedom are $e^\pm$, $\nu$, $\bar{\nu}$ and $\gamma$, yielding

$$g_* (T \sim 1 \text{MeV}) = 2 + \frac{7}{8} (2 \times 2 + N_\nu \times 2) = 10.75 + 1.75 \Delta N_\nu,$$

where $\Delta N_\nu \equiv N_\nu - 3$ ($N_\nu$ being the number of relativistic neutrinos during BBN). After electrons become non-relativistic ($T \sim 500\text{keV}$), we have

$$g_* (T \sim 0.1 \text{MeV}) = 2 + \frac{7}{8} \times N_\nu \times 2 \times \left( \frac{4}{11} \right)^{4/3} = 3.36 + 0.45 \Delta N_\nu.$$ \hspace{1cm} (10)

Recall again that for fermions we multiply by $7/8$ compared to bosons. Recall also that the factor $(4/11)^{4/3}$ appears because this is after neutrino decoupling (which is before $e^\pm$ become non-relativistic), so that neutrino temperature $T_\nu$ is lower compared to photon temperature $T_\gamma$ by a factor of $(4/11)^{1/3}$ (entropy conservation).

The simplest constraint we can derive is again by asking how the freeze-out temperature would be modified. The exercise is analogous to the exercise in Eqs. 6-11, such that

$$\frac{\delta Y_p}{Y_p} \approx \frac{Y_p}{2} \left( \frac{2}{Y_p} - 1 \right) \frac{Q}{T_{fo}} \frac{\delta T_{fo}}{T_{fo}} = \frac{1}{6} \left( 1 - \frac{Y_p}{2} \right) \frac{Q}{T_{fo}} \frac{\delta g_*}{g_*},$$

such that $\delta g_*/g_* \sim 10\%$, disfavouring additional generations of neutrinos (c.f. Eq. 9, $1.75/10.75 \approx 16\%$).

\(^1\)Reported in arXiv:1503.08146.
2 Dark matter annihilations vs. ionization

In the widespread picture of WIMPs, we typically think of some (Dirac) fermion $\chi$ with mass $m_\chi$ in an equal abundance of particles and anti-particles $n_\chi = n_{\bar{\chi}}$, such that the total energy density today is $\rho = 2m_\chi n_\chi c^2$ (under the assumption that it is non-relativistic today). We assume that this particle has a velocity-weighted pair annihilation cross section $\langle \sigma v \rangle$ which does not depend on temperature, i.e. $\langle \sigma v \rangle = \langle \sigma v \rangle_0$ is constant. Let us further assume that an annihilation releases all the energy into the SM plasma.

1. What is the energy injection rate per unit volume due to dark matter annihilation?

Assuming a given volume $V$ has $N$ particles of $n_\chi$,

$$Q(z) = \frac{1}{V} N \times \epsilon \times \frac{P_{\text{annihilation}}}{\Delta t} = n_\chi \times 2m_\chi c^2 \times \frac{n_{\bar{\chi}} \langle \sigma v \rangle_0}{\Delta t} = \frac{\rho_\chi^2 c^2 \langle \sigma v \rangle_0}{2m} \propto (1 + z)^6. \quad (12)$$

2. Assume that a fraction $(1 - X)/3$ of the energy released per annihilation event goes into ionizing neutral hydrogen. In the homework you compute the standard ionization history by integrating $dX/dT = f(T, X)$ with appropriate cosmological parameters. If we cast it as

$$\frac{dX}{dz} = I_{\text{standard}} - I_\chi, \quad (13)$$

what is $I_\chi$, the term due to dark matter annihilation?

We can write

$$\delta X \approx \frac{\delta n_e}{n_H} \propto \frac{1}{n_H} \frac{1}{1 - X} \times \frac{1}{3} \times \frac{1}{\epsilon_H} (Q \Delta t), \quad (14)$$

where $\epsilon_H = 13.6 \text{ eV}$. We neglected changes in $n_H$ and also neglected the effect of excited states and of primordial $^4H\text{e}$. Translating this into redshift evolution is straightforward. Since $a = 1/(1 + z)$, $z = 1/a - 1$ and $dz/da = -1/a^2$, such that

$$\frac{dz}{dt} = \frac{dz}{da} \frac{da}{dt} = \frac{1}{a^2} \times H a = -H(1 + z), \quad (15)$$

and

$$\frac{dx}{dz} = \frac{dx}{dt} \frac{dt}{dz} = -\frac{1}{H(z)(1 + z)} \frac{dx}{dt}. \quad (16)$$

We conclude

$$\delta X = -\frac{Q(z)}{H(z)(1 + z)} \frac{1 - X}{3n_H \epsilon_H} \delta z, \quad (17)$$

thus

$$I_\chi = \frac{Q(z)}{H(z)(1 + z)} \frac{1 - X}{3n_H \epsilon_H}. \quad (18)$$
3. At the homework, you integrated until \( z \sim 1000 \) and found \( X \ll 1 \) where photons decoupled, leaving most Hydrogen non-ionized. How much can dark matter change that?

Let us integrate

\[
\frac{dX}{dz} \approx -\frac{Q(z)}{H(z)(1 + z)} \frac{1 - X}{3n_H \epsilon_H},
\]

from \( z_0 = 1000 \) up to \( 1 \ll z < 1000 \), setting \( X = 0 \) on RHS of Eq. 19, with

\[
H = H_0 \sqrt{\Omega_\Lambda + \Omega_{m,0}(1 + z)^3 + \Omega_{r,0}(1 + z)^4} \approx H_0 \sqrt{\Omega_{m,0}(1 + z)^3}, \quad n_H = n_{H,0}(1 + z)^3,
\]

such that

\[
\Delta X(z) = \int_{z}^{z_0} I_{\chi}(z') dz' \approx \int_{z}^{z_0} \frac{Q(z')}{H(z')(1 + z')} \frac{dz'}{3n_H \epsilon_H(z')}
\]

\[
\approx \int_{z}^{z_0} \frac{Q_0 (1 + z')^6}{H_0 \sqrt{\Omega_{m,0}(1 + z')^3(1 + z')^3}} \frac{dz'}{3n_H n_{H,0}(1 + z')^3}
\]

\[
\approx \frac{Q_0}{3H_0 \epsilon_H n_{H,0}} \int_{z}^{z_0} \frac{(1 + z')^2 dz'}{\sqrt{\Omega_{m,0}(1 + z')^3}}
\]

\[
\approx \frac{Q_0}{3H_0 \epsilon_H n_{H,0} \sqrt{\Omega_{m,0}}} \frac{2}{3} (1 + z_0)^3/2 \approx \frac{2}{3} I_{\chi}(z_0)(1 + z_0).
\]