ANCIENT AND CLASSIC WORLS
Until the 5-th Century

ALGEBRA

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Algebraic expressions serve the quantitative form of the physical laws of nature.

Although ALGEBRA, as we know it today, was only formulated during middle ages by the Arabs, Algebraic thinking and algorithm descriptions appears already at the Babylonians 3500 years ago, who wrote formulae for calculating area of rectangles by multiplying length by width, area of triangles from half the product of base and height, and for calculating products using tables of squares, $ab=[(a+b)^2-(a-b)^2]/4$, since tables of N squares were used for calculating all combinations of n^2 products.

We review shortly algebra here, and included problems from a Greek and Chinese scripts for you to be engaged in solving.

As we remember, integers were invented as a generalization to number of any object. Algebra was invented to express a general quantitative relation between variables. These variables, annotated by letters e.g. x,y,z, can be replaced by any numerical value to calculate the value of the algebraic expression (or formula).

Algebraic equations may contain unknown variables that can be resolved numerically by solving these equations. Algebra define rules for eliminating unknowns from equations.

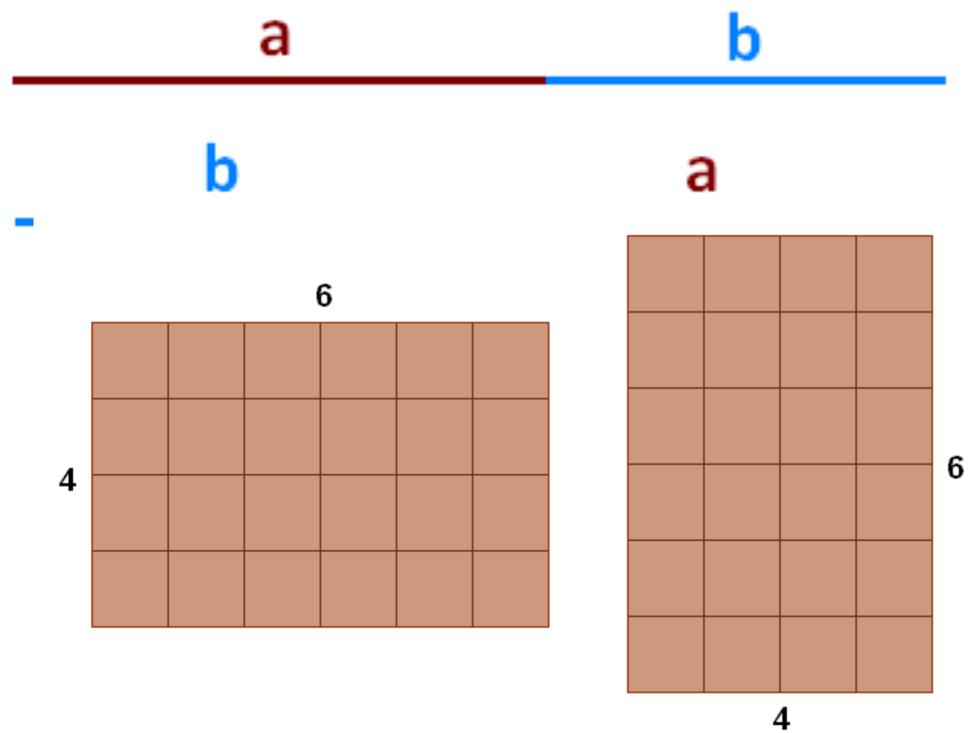
Functions (e.g. f(x), g(x,y) etc.) are further generalization of numbers and variables: they define a value for the function given numerical values to its variables. Functions are defined by algebraic expressions. Functions form a correspondence between members of two sets of numbers, e.g. x & y=f(x). If a function defines a one-to-one correspondence there exists the inverse function, e.g. $x=f^{-1}(y)$. If functions behave "nicely", they can be plotted by smooth curves, for example in x,y plane. But there are functions that are not smooth, e.g. f(x)=1/x around x=0.

Algebraic expressions are bound to “laws of order of operations”:
For example, without laws the value of the expression 3+2*5 may be 3+10 or 5*5
The order laws define the order of carrying arithmetic calculations:

- Highest are brackets () : first evaluate the expression inside the brackets
- Next powers:
- Next multiplications and divisions (commutative between them)
- Last additions and subtractions (also commutative between them)

The laws of simplifying algebraic expressions :
commutative, distributive, associative.

Geometric demonstration of
Commutative in addition
And multiplication:
a+b = b+a
x*y = y*x



Associative between multiplications/divisions and between additions/subtractions

$$a*(b*c) = (a*b)*c \qquad (a+b)+c = a+(b+c)$$

Distributive of multiplication over addition/subtraction:

$$a(b+c) = ab + ac \qquad a(b-c) = ab - ac$$

Elimination of unknown variables from algebraic equation is achieved by successive <u>identical operations on both sides of the equation that help elimination of the unknown</u>:

If a+b=c then a=c-b (subtracting b from both sides of the equation)

if a*b=c then a=c/b (dividing both sides of the equation by b)

if $a^2=c$ then $a=c^{½}$ or $a=\sqrt{c}$ (square root of both sides of the equation)

If I am three years older than my sister, you can surely tell me my age if you know my sister's age. Using my age as x and my sisters as y we can write:

$$x=y+3$$

And eliminating my sisters age from above:

$$y=x-3$$

Let us examine a similar problem with a catch:

I bought a racket with a ball at $33, the racket costs 30$ more than the ball, what is the price of the ball?

Let us use algebra to avoid the intuitive but wrong answer (ball cost $3…):

Ball cost x, racket x+30, therefore 2x+30=33

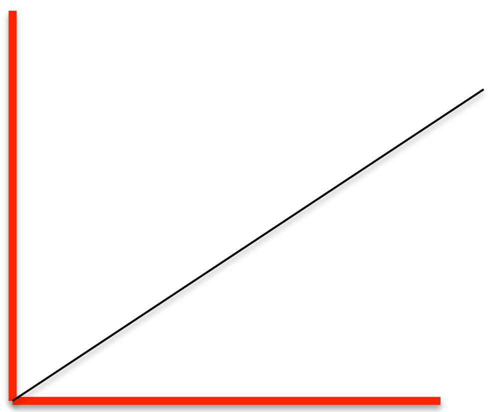
$$x=\$1.50$$

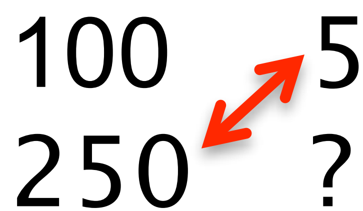
Proportionality: many common variables in every-day life are proportional to each other, e.g. weight and price of cheese. The graphic presentation of price vs. weight is a straight line starting at the origin, (0,0).

Question: if 100gr of the cheese cost $5 how much 250gr costs?

We write the weights as first column, and costs as the second, then apply the algorithm: multiply the diagonal and divide the number opposite the unknown:

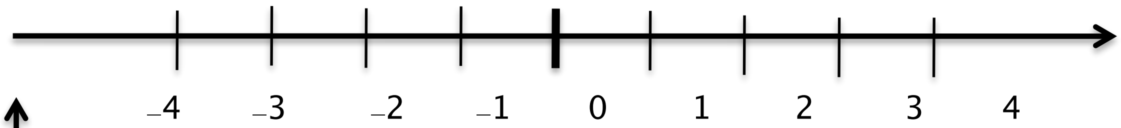
Cost = 250*5/100



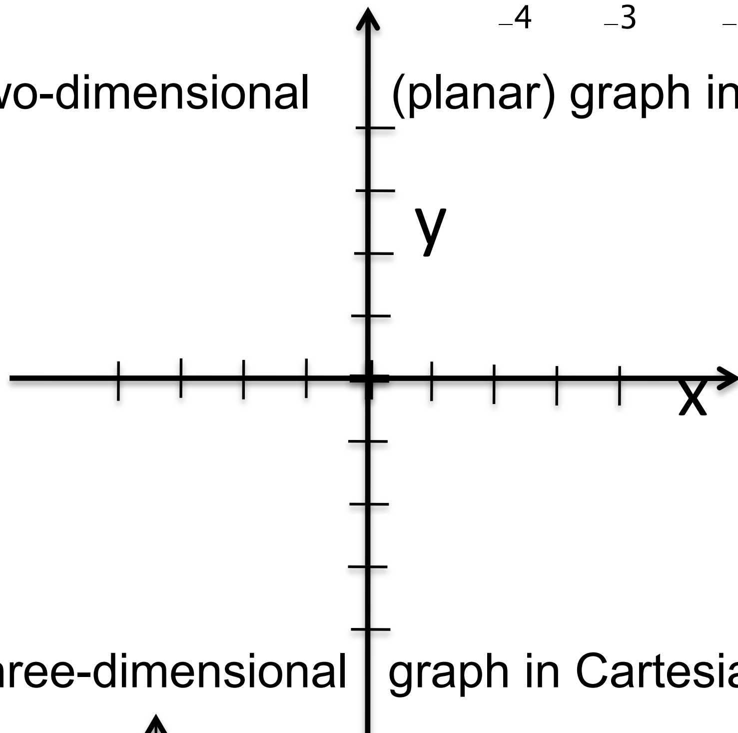


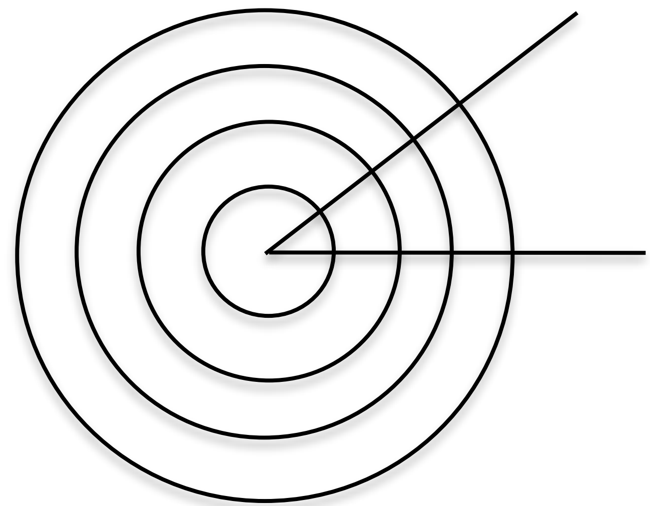
Algebra and Graphs

One-dimensional (linear) graph

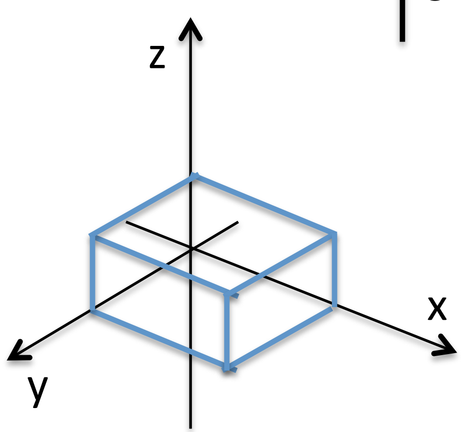


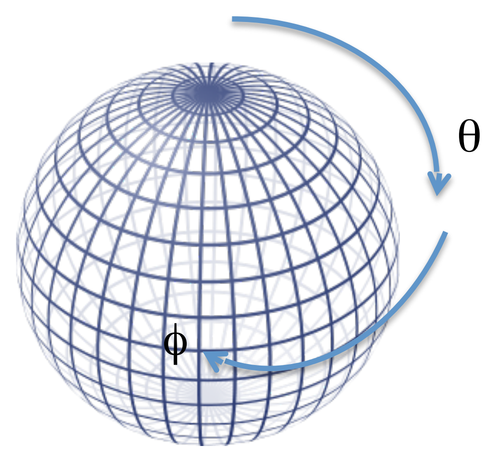
Two-dimensional (planar) graph in Cartesian (x,y) and polar (r,ø) coordinates





Three-dimensional graph in Cartesian (x,y,z) and spherical (r,θ,ø) coordinates





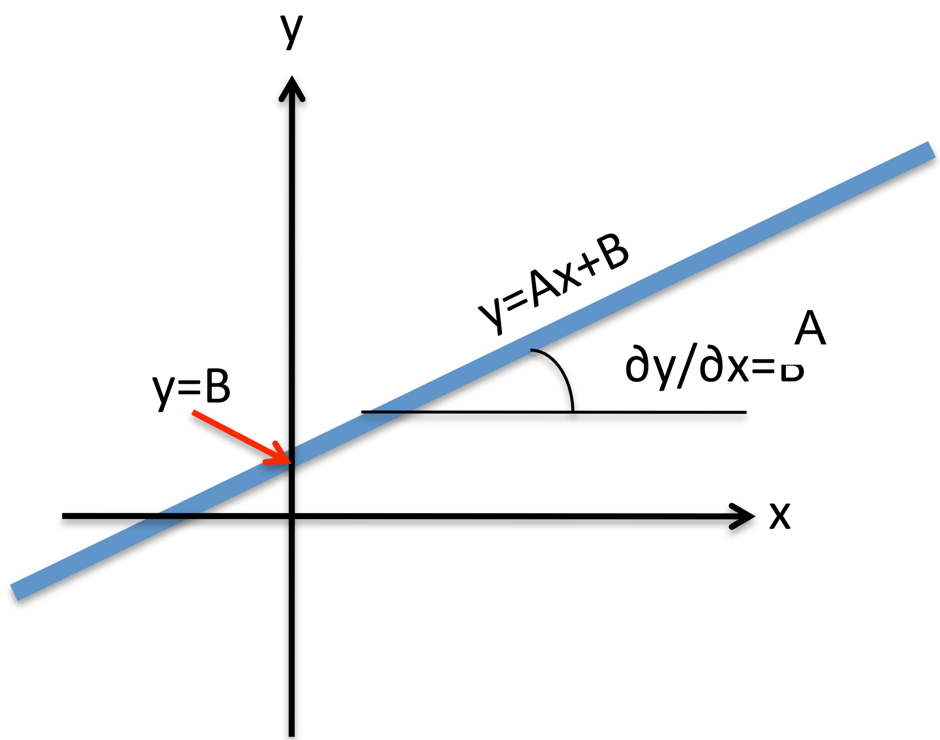
A linear equation: $y=Ax+B$ or $ax+by+c=0$

Is plotted by a line in xy plane.

What is the relation between A,B and abc ?

We need only two parameter to define a line. But what happens if $b=0$?

Therefore the “canonical” form $ax+by+c=0$ is often used, with a,b,c defined up to a factor.

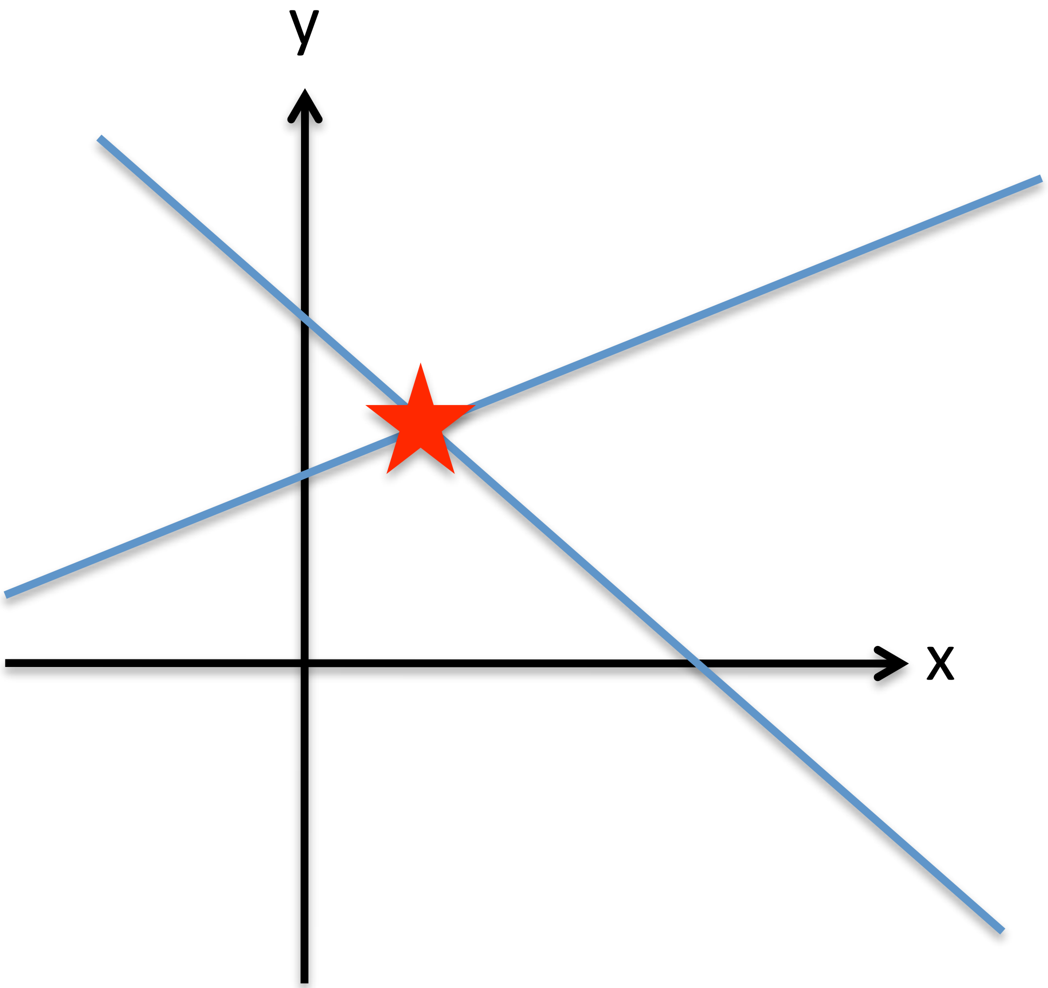


A line can be defined by intersect, B, with y-axis ($x=0$) and its slope, A.

Or from two points that it crosses: $y_2=Ax_2+B$ $y_1=Ax_1+B$

Solving two linear equations graphically:

$$a_1 x + b_1 y + c_1 = 0$$
$$a_2 x + b_2 y + c_2 = 0$$



Analytical solution: Eliminate y from 1st equation and substitute into the 2nd :

$$y = -(c_1 + a_1 x)/b_1$$

$$a_2 x - (c_1 + a_1 x)(b_2/b_1) + c_2 = 0$$

$$(a_2 b_1 - a_1 b_2) x = c_1 b_2 - c_2 b_1$$

$$x = (c_1 b_2 - c_2 b_1)/(a_2 b_1 - a_1 b_2)$$

This can be written symbolically:

$$x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

And for y:

$$y = \begin{vmatrix} a_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

This way of writing defines a “determinant” that can be extended to n equations with n unknowns. This organization of the variables motivated the development of **Vectors and matrix algebra** a field of seminal importance to physics for the equation of motion of bodies and wave mechanics. The same way we assign symbols x,y,… to number variables, in vector algebra, symbols assign groups of number variables X=(x,y,z,…) for example triplets of coordinates for points in space.

Solving n-linear equations with n-unknowns using determinants:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= c_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= c_2, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= c_m, \end{aligned}$$

$$\begin{aligned} x_i &= e_i x = e_i \frac{adj\ A}{|A|} c \\ &= \frac{(A_{1i}\ A_{2i}\ \ldots\ A_{ni})\, c}{|A|} \\ &= \frac{(c_1 A_{1i} + c_2 A_{2i} + \ldots + c_n A_{ni})}{|A|} \\ &= \frac{1}{|A|} \begin{vmatrix} a_{11} & \cdots & a_{1,\, i-1} & c_1 & a_{1,\, i+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2,\, i-1} & c_2 & a_{2,\, i+1} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n,\, i-1} & c_n & a_{n,\, i+1} & \cdots & a_{nn} \end{vmatrix}. \end{aligned}$$

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix}.$$

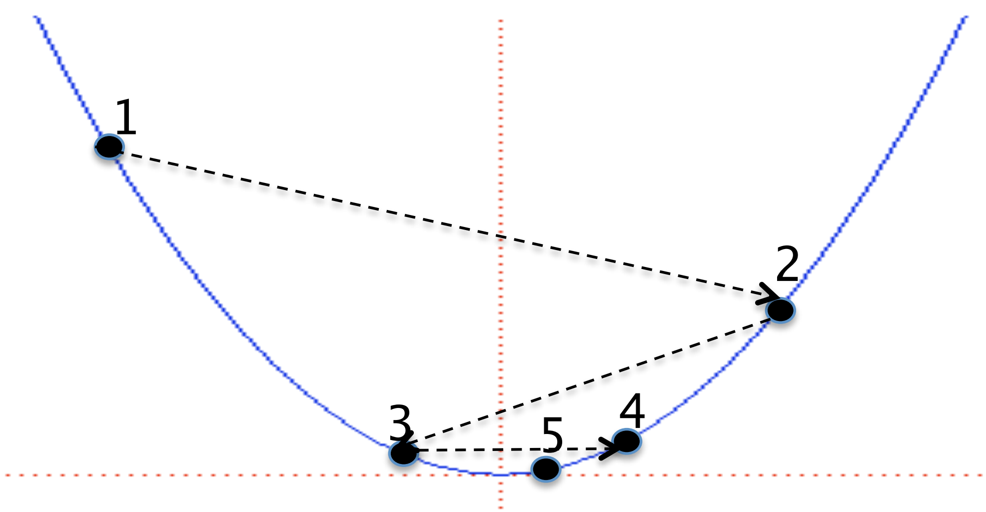
$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} A_{i,\sigma_i}.$$

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{i,j} M_{i,j} = \sum_{i=1}^{n} (-1)^{i+j} a_{i,j} M_{i,j}.$$

Minimization:

Finding the minimum for a function is an often occurring problem in many fields:
Building construction with minimum use of steel and cement
Combining lenses in an optical system with a required resolving power
An investment file with a given combination of risk and bonds
Design of airplanes and cars with minimal air friction
Best matching between pictures (e.g. fingerprints, or face identification).

A simple and effective algorithm is called “**catch a lion in the desert**”:
It applies when we know that the minimum lies within a given range. In this case we test the value of the function at two points, find the point with lower value, and test another point in between closer to the lower value. If we cut the range to half at each step, we approach to minimum to within $1/2^n$ of the initial range. This is much better than finding the value at n equidistant points in the range.



	1,2
1>3	3,2
2>4	3,4
4>5	3,5

Minimization: Simplex

This method can be extended from one-dimensional to n-dimensional range. The range is defined by n-dimensional Dalton, with n+1 vertices X_i

1. We find the values of the function at X_i and sort in increasing order
2. Center of mass of the first n vertices X_0
3. Reflected point to last vertex X_{n+1} $X_r=X_0+\alpha(X_0-X_{n+1})$
4. If $f(X_1) \le f(X_r) \le f(X_n)$ exchange X_{n+1} by X_r and return to 1

   If $f(X_r) < f(X_1)$ calculate extending point $X_e=X_0+\gamma(X_0-X_{n+1})$ {green}

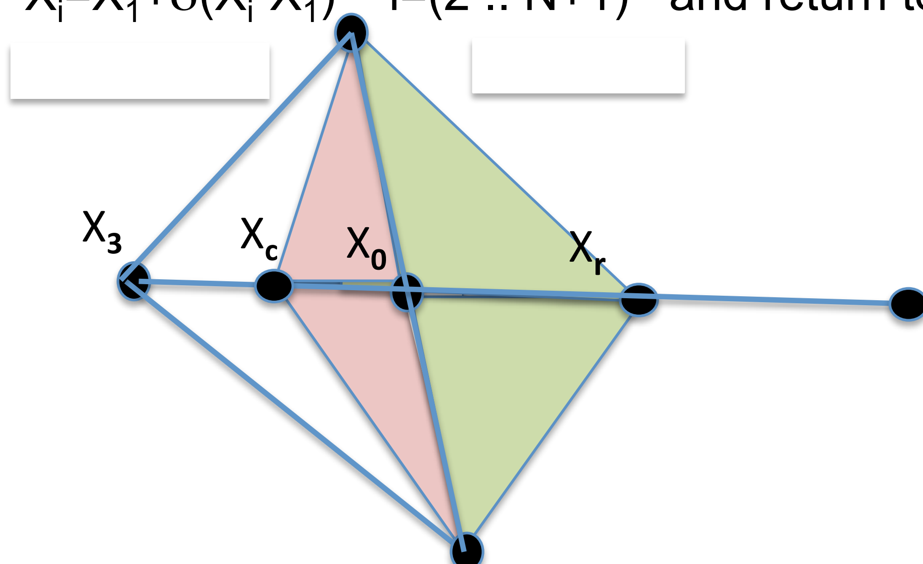
   If $f(X_e) < f(X_r)$ exchange X_{n+1} by X_e and return to 1

   Otherwise exchange X_{n+1} by X_r and return to 1

   Otherwise $f(X_r) \ge f(X_n)$ continue
5. Now $f(X_r) \ge f(X_n)$ calculate shrinking point $X_c=X_0+\rho(X_0-X_{n+1})$ {red}

   If $f(X_c) < f(X_{n+1})$ exchange X_{n+1} by X_c and return to 1

   Otherwise continue
6. Exchange all but first vertex $X_i=X_1+\sigma(X_i-X_1)$ i=(2 .. N+1) and return to 1



Minimization by Newton-Rapson steepest descent

The fastest method for smooth functions close to the minimum.

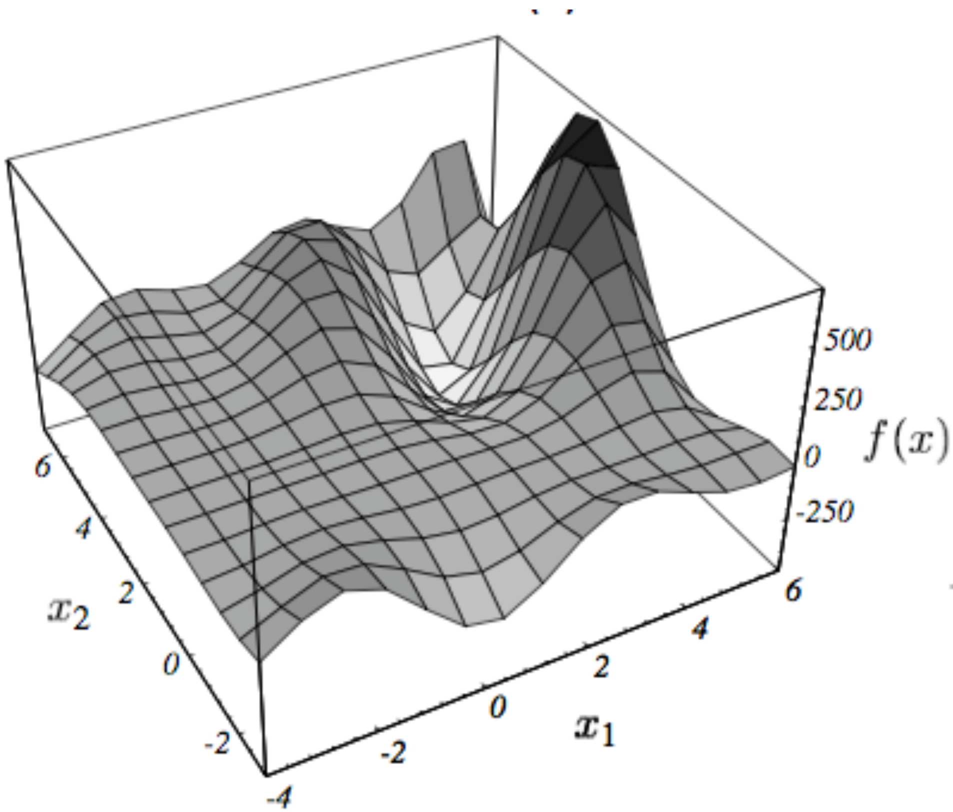
Problems: Calculation of derivatives is sensitive to errors.

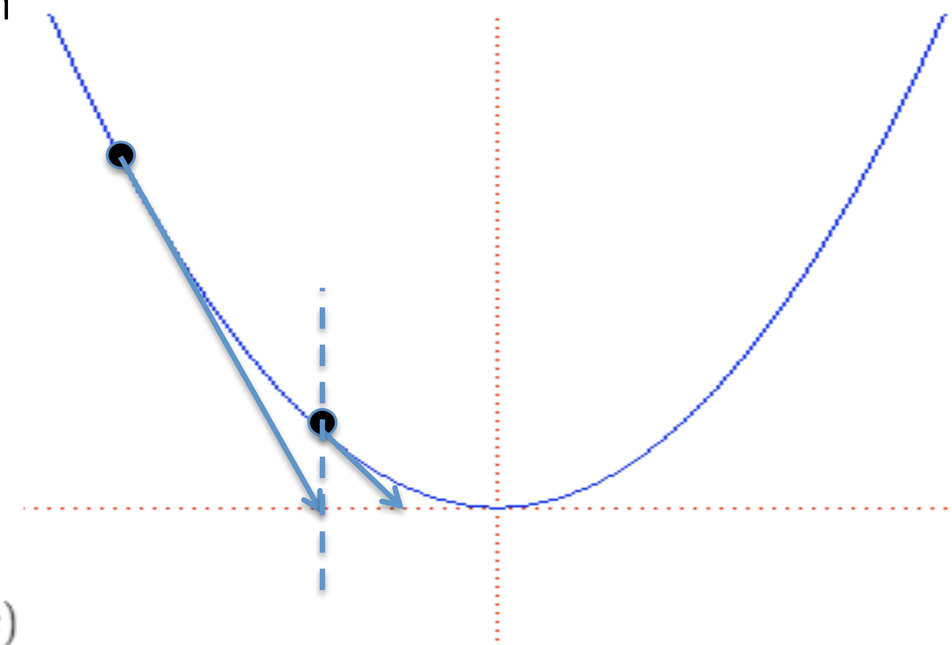
Stuck in local minima

Solved by scanning the function on a course grid to locate global minimum

Throws the extrapolation far away at the minimum.

Solved by conjugated gradien





The following are two mathematical books, Greek and Chinese, that preceded the Arabic Algebra by hundreds of years, but applied all algebraic tools – equations with unknowns solved by elimination, quadratic and higher order equations, etc. We list here their table of contents.

200–284 BC Diophantus – "Arithmetica"

http://www.math.tamu.edu/~dallen/masters/Greek/diophant.pdf

Diophantus is considered the father of algebra, although the name was created at the middle ages by **al Karaji 980-1030 AC**, as well as number theory, that inspired **Pierre de Fermat 1601-1665** and following mathematicians. Diophantic equations are equations that have positive integers as solutions. Diophantus did not consider negative integers acceptable solutions. He did not write expressions as we write algebra today, e.g.. for $(12+6n)/(n^2-3)$ he writes: Six times the number increased by 12, that is divided by the difference that its square is bigger by 3…

The content of the 13 volumes of **“Arithmetica” by Diophantus**

VOLUME I

Simple problems presenting algebraic thinking. Numbers are positive rationales only. Include quadratic equations. Problems sometimes have several solutions.

VOLUME II & III

Presentation of squares as a sum of two rational squares, of non-square numbers as sum of squares (a solution not always exists), and as difference of two squares. Solutions only for numbers that do not have 4n+3 as a factor at even powers. Diophantus presents one solution, but mention cases with infinite number of solutions.

VOLUME IV - VI

Extending problems of second order unknowns (quadratic equations)

VOLUME VII - IX

More complex problems, e.g. number presentation as a sum of two contiguous squares, or arbitrary neighboring squares, or sum of three squares. Numbers of the form 8n+7.

VOLUME X

Right-angle triangles with rational orthogonal edges.

VOLUME XI - XIII

Were lost, but according to the introduction they deal with third-order (cubic) equations

An example of Diophantic problem inscribed on his grave stone:

'Here lies Diophantus,' the wonder behold. Through art algebraic, the stone tells how old: 'God gave him his boyhood one-sixth of his life, One twelfth more as youth while whiskers grew rife; And then yet one-seventh ere marriage begun; In five years there came a bouncing new son. Alas, the dear child of master and sage After attaining half the measure of his father's life chill fate took him. After consoling his fate by the science of numbers for four years, he ended his life.'

How old was Diophantus when he died ?

Lets assume he lived X years, and his son X/2 years
The son was born when the father age was X/6+X/7+X/12+5
The father died 4 years after the son, therefore X/6+X/7+X/12+5 + 4+X/2 = X
9=X-X(14+12+7+42)/84=9/84X
X=84

Diophantus considered negative solutions as “absurd” therefor considered the equation 4=4X+20 with no solution. Today we say X=-4 is the solution…
For this reason he failed to discover that quadratic equations have two solutions. He sorted them into three groups: 1. $aX^2+bX=c$; 2. $aX^2=bX+c$; 3. $aX^2+c=bX$

In a few following examples there are Diophantic equations that require to scan possibilities and find the solution.

Problem 1: Two equations with two unknowns : $y+z=10 \quad yz=9$
Diophantus substituted x from $y-z=2x$ and got second order equation
Adding y-z and y+z we get 2y=10+2x or y=5+x
Subtracting we get 2z=10-2x or z=5-x
Substituting in xy=9 we get $(5+x)(5-x)=25-x^2=9$ therefore $x^2=25-9=16$
or $x=4 \quad y=9 \quad z=1$

Problem 2: Find such x that both 10x+9 and 5x+4 are squares.
solution: x=28

Problem 3: Find such x that x^3-3x^2+3x+1 is a square,
$4x+2$ is a cube
$2x+1$ is a square
solution: x=3/2

Problem 4: Find a fraction between 2 and 5/4 that is a square
solution: 25/16

Problem 5: Find two fractions which squares <6 and sum=13
solution: $66049/10201=(257/101)^2 \quad 66564/10201=(258/101)^2$

Problem 5: Find 3 fractions which squares >3 and sum =10
solution: $(1321/711)=1745041/505521 \quad (1285/711)=1651225/505521$
$(1288/711)=1658944/505521$

Problem 7: Find two numbers which sum=20 and sum of squares=208

we write them as (10+x) & (10-x) thus: $(10+x)^2+(10-x)^2=208$

or $2x^2+200=208$ then $2x^2=8$ and x=2 therefore the numbers are 8,12

There is another solution x=-2 that Diophantus ignores, accidently also gives 8,12

Problem 8: How many positive integers are solutions to: $1/x+1/y=1/n$

For example n=4: ¼=1/5+1/20=1/6+1/12=1/8+1/8

Conjectures in **number theory** that Diaphanous did not prove, and engaged mathematicians for generations:

It is impossible to find numbers of the form 4n-3 or 4n-1 as sum of squares.
It is impossible to find numbers of the form 24n+7 as sum of three squares.
Every number can be written as a sum of four squares.
Every pair of integers, a,b, has c,d so that $a^3 - b^3 = c^3 + d^3$
If p is a prime, (p-1)!+1 divides by p

Fermat tried but failed to prove it. **Lagrange** proved it based on theorems of **Euler**

Is it possible to find integer solutions to:

$$x^n + y^n = z^n$$

Called Fermat last theorem. Was only proved recently, and the proof depends on high math beyond me…

Problem 9: Sum of three odd contiguous integers is 231, what is the large number?
Problem 10: Sum of five consecutive integers is 200, what is the small one?
Problem 11: How many liters I could buy for $100 if one liter costs $7.38 ?
Problem 12: The sum of 100 integers is 3000, what is their average?
Problem 13: From 12 meter long paper strip three pages were cut, each ½ meter shorter. What is the length of the pages?
Problem 14: Four passengers measure the speed of the train: 1^{st}: 27.5km/0.5hours, 2^{nd}: 55 km/hour 3^{rd}:82.5km/1.5hours 4^{th}:110km/2hours. Is the train moving in constant speed?
Problem 15: A shop adds 30% to the cost of bicycles, then sell at 30% discount. Another shop adds 25% then sells at 25% discount. Where is the lower price?
Problem 16: The dimensions of a sand box is 5x8x0.5 meters. How many 100 liters sacks are needed to fill the sand box?

Problem 17: I bought a dress for $80 at 20% discount, what was the price before the sale?

Problem 18: 3kg tomatoes cost $9 at 25% discount, what would be the price next week without any discount?

Problem 19: I have $50 in my pocket. What is the tag price of a dress I can buy at 25% sale?

Problem 20: I save $30 at 20% sale discount. What was the price after discount?

Problem 21: I deposited $1000 in a saving account that pays 4% per year. What would be my balance after 4 years.

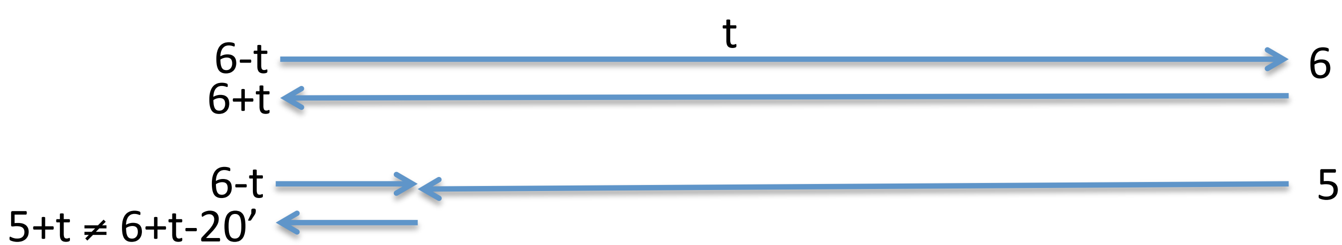
Problem 22: I save $250 every year and deposit in an account paying yearly 4% . How much should I have after 4 years?

Problem 23: My average grade in four exams is 84. What grade must I get in a fifth exam to raise my average to 90?

Distances:

I leave home every day to meet my wife at the station at 6. She finished work earlier and reached the station at 5. Walking towards home we met and reached home 20 minutes earlier than usual. How long is the way from home to the station? (assume my wife and myself walk at the same speed).

Solution: Impossible !!! If my wife reached the station an hour early, she should have reached home too 1 hour earlier, not 20 minutes.



Ages:

In 4 years Sam will be 3 times his age today. How old is he today?

Rachel is older than Sam by 4 years. Two years ago `she was 5 times his age. What is Sam's age today?

Sam is 50 and Rachel 15. When would he be twice Rachel's age?

Sam is 3 times Rachel's age. 20 years ago he was 8 times. What is Sam's age today?

When I reach the age of my father today I'll be 5 times the age of my son now, and my son will be 8 years older than my present age. The sum of my father and my ages is 100. How old am I, and how old is my son?

A man and his grandson celebrate birthday on the same day. During one of the birthday parties the grandfather say: this is the 6^{th} consecutive year that my age is a whole product of my grandson. What are their ages?

Solution: 1-6 and 61-66

The difference between the square of my older brother's age and the square of my age is 749. How old m I ?

Solution: $x^2-y^2=749$ or $(x+y)(x-y)=749$

749=7*107 ; 1*749 is not a reasonable solution, therefore: x=57 y=50

I have 3 kids. The product of their ages is 36. What are their ages?

one cannot tell ! {since many ways to factor 36: 1,2,18; 2,3,6; …)

The sum of their ages equals the number of the house we stand by.

still cannot solve ! (since two factorizations with equal sum: 1+6+6=2+2+9)

Today is the birthday of my eldest son:

Solution: 2,2,9 ; since the other possibility 1,6,6 means twin elder brothers.

My age is twice the product of its two digits. How old am I ?

Solution: $10x+y=2xy \to x=y/(2y-10)$

For positive numbers y>5 and only y=6 yields integers, therefore my age is 36

Coins:

I have 100 coins in my pocket, including dollar, half a dollar and 10 cents coins. Their total value is \$16. What are the coins I have?

Solution: $x+y+z=100 \quad 10x+50y+100z=1600 \;\Rightarrow\; 4y+9z=60$

For positive solutions z=1..6 x=90 y=6 z=4

I deposited a check and received dollars for cents, and cents for dollars. I bought a gum for 5 cents, and was left with twice the value of the original check. What was written on the check?

Solution: a dollars and b cents: $100a+b-5=2(100b+a) \;\Rightarrow\; 100(a-2b)=2a-k+5$

a=2b+1 < 3 only 1 possible. Substitute a-2b therefore: a,k<100

a=63 b=31 or the sum on the check: \$31.63

What are the two integers with sum equal their multiplication ?

Solution: $2+2=2*2$

What are the three numbers with sum equals their multiplication ? Show this is the only solution !

Solution: $1*2*3=1+2+3$ and also $\sqrt{3}+\sqrt{3}+\sqrt{3}=\sqrt{3}*\sqrt{3}*\sqrt{3}$

$xyz=x+y+z$

if x=y=z $x^3=3x$ $x=\sqrt{3}$ else $x\geq y\geq z$ y=z=1 impossible, therefore $y\geq 2$

but y>2 impossible, $xyz\geq yz\geq 3z>x+y+z$ but $3z\leq x+y+z$

$X^2-Dy^2=1$ D=4729494 = 2.3.7.11.29.353

Archimedes cattle problem

Herd of cattle is separated into four groups by color: white, black, brown and spotted. The number of bulls is larger than the number of cows. The number of white bulls equals 1/2+1/3 of the black and brown together. The number of black bulls equals 1/4+1/5 brown and spotted bulls together. Number of spotted bulls equals 1/6+1/7 while and brown bulls together.
The number of white caws equals 1/3+1/4 of all the blacks, number of black caws equals 1/4+1/5 of all spotted, and the spotted caws equal 1/6+1/7 of all whites.
When the while bulls merged with the black they filled the wide fields of Sicily, and when the brown and spotted bulls merged and filled a triangle, standing in lines started with 1 bull, then 2, etc.
Today we say: white + black bulls are a square number. Brown and spotted a triangular number of the form (1+n)*n/2

Solution: 7 equations with 8 unknowns. Solve all unknowns as a function of the 8^{th} unknown. Then multiply the fractions with common denominator, 10,366,482, to make them all integers. The smallest solution for the whole herd, 50,389,082 and the other are multiples of this solution, see next page.

Just imagine how difficult it was for Archimedes to perform all the multiplications with large numbers in Roman number presentation…

.

The equations **the solution**

$$W = \frac{5}{6}B + Y$$

$$B = \frac{9}{20}D + Y$$

$$D = \frac{13}{42}W + Y$$

$$w = \frac{7}{12}(B + b)$$

$$b = \frac{9}{20}(D + d)$$

$$d = \frac{11}{30}(Y + y)$$

$$y = \frac{13}{42}(W + w)$$

B=(267/371)W

D=(297/742)W

Y=(790/1113)W

b=(171580/246821)W

w=(815541/1727747)W

d=(1813071/3455494)W

y=(83710/246821)W

$B = 7,460,514$ *k=4657*1602*k

$W = 10,366,482$ *k=4657*2226*k

$D = 7,358,060$ *k=4657*1580*k

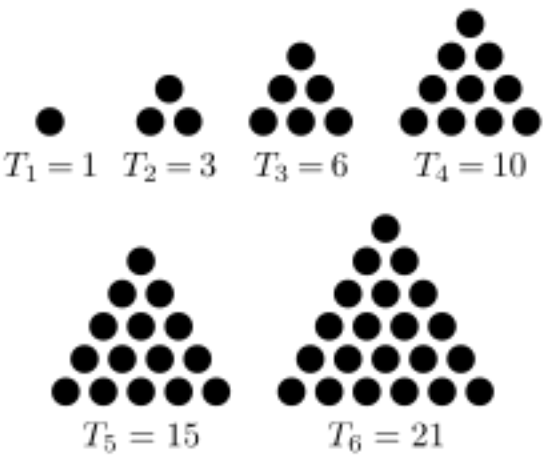
$Y = 4,149,387$ *k=4657*891*k

$b = 4,893,246$ *k

$w = 7,206,360$ *k

$d = 3,515,820$ *k

$y = 5,439,213$ *k



Additional conditions: First: B+W is a square number:

B+W= 7460514k + 10366482k = 2^2*3*11*29*4657k

for B+W to be a square, k = (3)(11)(29)(4657)q^2 q=1,2,…

Second condition: D+Y is a triangular number:

D+Y=t(t+1)/2 solution for t and substitute into D,Y gives: p^2-4*609*7766*4657^2q^2=1

This is Pell's equation: x^2-Dy^2=1 ; D=4729494=2*3*7*11*29*353=609*7766

When x=p y=q*2*4657 and the solution is the above times n for integer j

n = $(w^{4658j} - w^{-4658j})^2$ / (4657*79072) w^2 = u+v √(609*7766)

w=300426607914281713365* √609 + 84129507677858393258* √7766

Chinese problems

http://www-history.mcs.st-and.ac.uk/HistTopics/Chinese_problems.html

Nine chapters of "the art of mathematics" by unknown author,
Composed at roughly Euclid's time, and summarizes centuries of studies

Interesting to compare Greek and Chinese subjects of study. Every-day problems motivated similar problems, as well as inspired non-practical problems. Yet, the Greeks evaluated the logical process of a rigorous proof, but Chinese just quote theorems for practical use without attempting to prove them. They may have known ways to prove theorems, but did not consider important to documents the proofs.

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Chapter 1. Land surveys:
Area of shapes: rectangle, triangle, diamond, circle, the value of π
Addition, subtraction, multiplication and division of integers and rational numbers
The largest common factor
Chapter 2. Rice commerce, lentils etc.:
Ratios and percent to calculate exchange values of goods.
Chapter 3. Division by proportions:
Direct and inverse ratio, computation of debts and credits.
Chapter 4. Area:
The width of a field if length and area are preserved, if length=1+1/2+1/3+1/4+…1/n what is the width. Calculation of square & cube roots. The volume of a sphere (wrong equation…).
Chapter 5. Engineering problems:
Volumes and quantities, channels, pyramid volume (infinitesimal calculation).
Chapter 6. Just division of goods:
Problems of journey, taxation, and proportional divisions.
Chapter 7. Excess and shortage
Solution of linear equation by two guesses and getting to the solution from the errors.
Chapter 8. Matrix calculations
Solution of sets of linear equations by Gauss elimination.
Chapter 9. Right angle triangles, and similar triangles
Uses of Pythagoras law ("Guru circle"), and geometric solution to quadratic equations.

Problem 1. If a rooster cost 5 coins, hen 3 coins, and a 3 chicks cost 1 coin, how many of each we can buy for 100 coins if we buy 100 birds.

Solution: X roosters, y hens, z chicks:

$x+y+z=100$; $5x+3y+z/3=100$ or $15x+9y+z=300$

Subtract $14x+8y=200$; $7x+4y=100$; $y=(100-7x)/4$

Try all multiples of 4:

$x=16$, $y=$ negative

$x=12$, $y=4$, $z=84$

$x=8$, $y=11$, $z=81$

$x=4$, $y=18$, $z=78$

$x=0$, $y=25$, $z=75$

Problem 2. A pool is filled from 5 channels. Chan.1 fills the pool in 1/3 day, chan.2 in 1 day, chan.3 in 2 ½ days, chan.4 in 3 days and chan.5 in 5 days. How long would it take to fill the pool with all channels open at once ?

Solution: will be filled in x days. Chan.1 will fill x/(1/3) of the pool: 1/3 day 1pool
x days 1*x/(1/3) pool

etc. therefore 3x+x+2x/5+x/3+x/5 = 1 x=15/74 days

Problem 3. Man A runs 100 km when man B runs 60km. After man B ran 100km man A start running from the same place after him. Where will he catch man B ?

Solution: will catch him after running x km. Man B ran for (x-100)/60 time units, and man A ran for x/100 time units. At the time they met: (x-100)/60 =x/100 x=250 km

Problem 4. How many people are buying together, and what is the price? If every man pay 8 coins they will have 3 coins in access. If everyone pays 7 coins the will need to add 4 coins.

Solution: x people: 8x-3=7x+4 x=7 people and the total price 49+4=56-3=53

Problem 5. Two piles of coins have equal weights. First one contains 9 gold coins, the second 11 silver coins. If we exchange one coin between the piles, the first pile weights 13 units less than the second. What are the weights of gold and silver coins ?

Solution: gold=x silver=y $9x=11y$ & $8x+y+13=10y+x$ or $7x+13=9y$

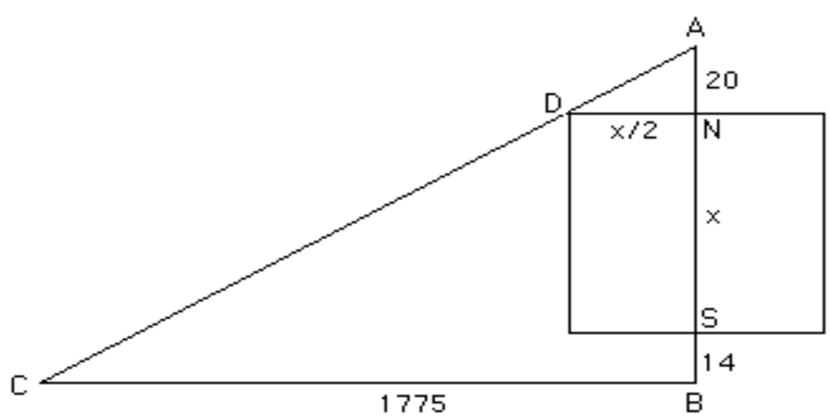
by elimination: $y=117/4$ $x=143/4$

Problem 6. The city walls make a square, with a gate at the center of each side. There is a tree 20 meters out of the north gate. If we go south 14 meters out of the south gate, then 1775 meters east the tree appears behind the walls. What is the length of each wall ?

Solution: x meters. Similar triangles: $20/(x/2)=(20+x+14)/1175$

therefore: $x^2+34x-40*1775=0$ $x=(-34\pm\text{sqrt}(34^2+4*7100))/2=(-34\pm534)/2$

$x=250$ meters, ignoring the negative solution



Problem 7. Shepard A with one sheep asks Shepard B how many sheep he has. B answers: Add twice my herd, plus half plus quarter plus your sheep to have 100. How many sheep B has ?

Solution: $100=2x+x/2+x/4+1=100 \quad x=36$

Problem 8. The weight of a jade cube 1cm in size is 7gr. The weight of a rock cube of identical dimensions is 6 gr. The weight of 3cm cube made of mixture of jade and rock is 176gr. What is the weight of pure jade in this cube ?

Solution: Weight of jade x gr. $3^3=x/7+(176-x)/6 \quad x=98$gr

Problem 9. On my trip I saw 9 hills, on each 9 trees, each with 9 branches, 9 bird nests on each branch, 9 birds in each nest, each has 9 chicks, every chick grew 9 feathers, each feather has 9 different colors along its length. How many feathers of each color combination ?

Solution: $9^7=4,782,969$ feathers $9!=362,880$ combinations ~13.2 feathers of each combination of colors.

Problem 10. When we group marbles in triplets 2 are left. If in fives 3 left, and in sevens 2 left. How many marbles I have ?

Solution: No algebraic solution, but we can scan possible solutions, fastest scan for

groups of sevens plus two:	7+2, 14+2, 21+2, 28+2, 35+2, 42+2, 56+2, 63+2
fives plus 3	5+4, 15+1, 20+3, 30+0 35+2, 40+4, 55+3, 65+0
threes plus 2	9+0, 15+1, 21+2, 30+0, 36+1, 42+2, 58+0, 63+2

Thus number of marbles = 7*3+2=23 but also 7*(3+15n)+2 (adding 7*5*3 n-times)

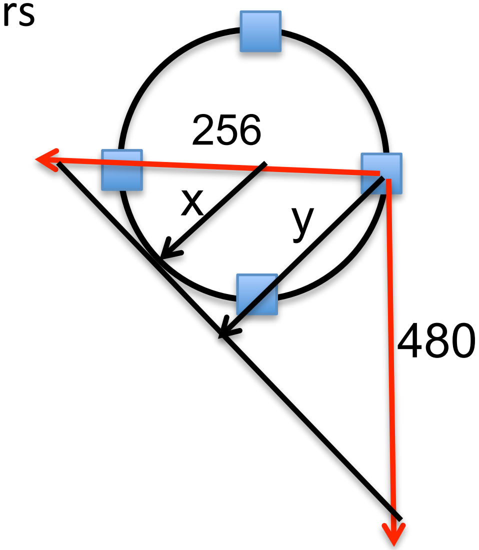
Problem 11. A city has circular wall around it with 4 gates towards north, south, east & west. Two people leave the west gate, first walks 256 meters east and stand there, the second 480 meters south, and just sees the first beyond the wall. What is the wall diameter ?

Solution: Diameter is 2x. The line between the people is tangential to the circular wall.

Apply similar triangles: (256-x)/x =256/y x=256/(256/y+1)

The area of the large triangle 256*480=y*sqrt($256^{2+}480^2$)=544y y=256*480/544

x=256/(256/256/480*544+1)=120 The wall diameter is 240 meters



Problem 12. City wall as in problem 11. A man walks out of the west gate 480 meters. A second man walks from the east gate eastwards 16 meters when he sees the first. What is the wall diameter ?

Solution: Same configuration as Problem 11.

Problem 14. City wall as in problem 11. There is a tree 135 meters south of the south gate. A man going 15 meters north of the north gate, then 208 meters east just sees the tree beyond the city wall. What is the wall diameter ?

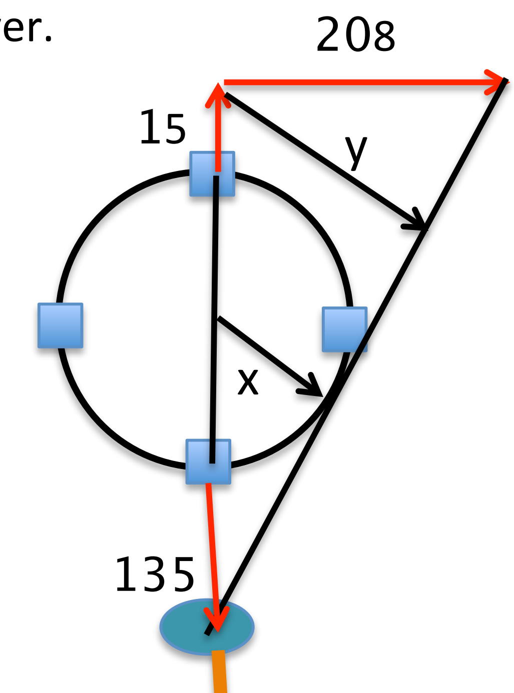
Solution: Wall radius x. Similar triangles: $x/(135+x) = y/(135+2x+15)$

the area of the large triangle: $208*(135+15+2x)=y*\mathrm{sqrt}(208^2+(135+15+2x)^2)$

We get 4^{th} order equation. The Chinese did not know to solve it algebraically, but tried by successive approximations: x=100 is underestimation. X=140: over.

x=120: correct solution.

See and behold, it is the same city…



Problem 15. Same city configuration. A tree 3km north. If a man walks east of the south gate 9km he just sees the tree. What is the wall diameter ?

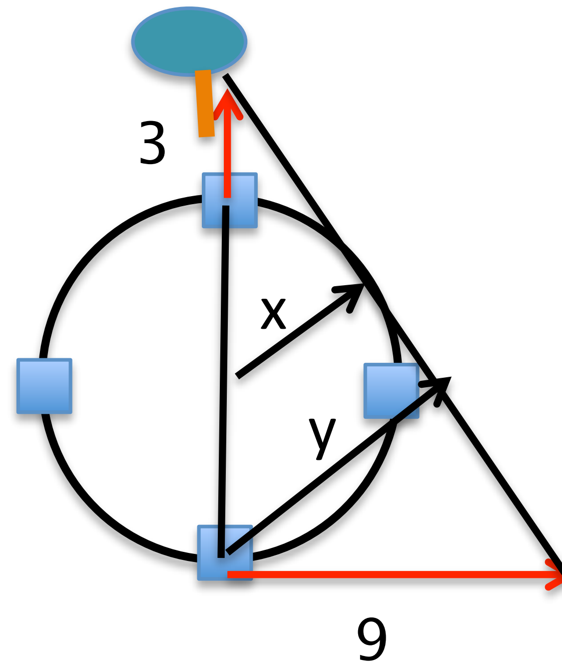
Solution: Similar triangles: x/(3+x)=y/(3+2x)

and triangle area: $9(3+2x)=y*\text{sqrt}((3+2x)^2+9^2)$

again 4^{th} order equation, try solutions:

x=4 underestimate, x=2 over, x=3 correct.

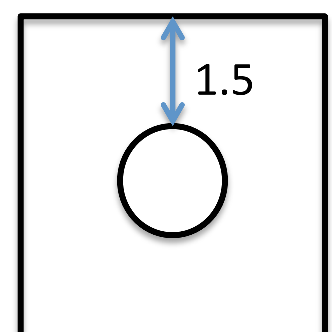
The city wall diameter is 6km.



Problem 15. A square farm has a circular well at its center. The area of the farm is 12 square meters plus 2.5 tenths. The well is 1.5 meters from its edge. What is the farm edge, and what is the well diameter ?

Solution: edge=sqrt(12.25)=3.5 meters

well diameter=3.5-1.5*2=0.5 meters



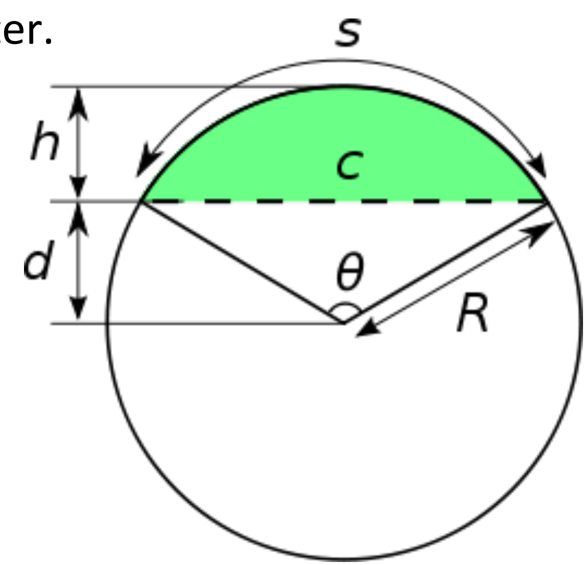
Problem 17. Rice is piled against the wall. Its base perimeter is 6.28 meters, height 2 meters. Another pile is against th4e corner, perimeter is 3.14 meters, height 2 meters. What are their volumes ?

Solution: We have half and quarter cones, full base perimeter=2*3.14r therefore r=1 for half and quarter bases against the wall and in the corner.

The volume of half cones are ½$3.14r^2h/3$ or 4.19 & 2.09 cube meters respectively.

Note: if perimeter is half, the area is four times smaller. But here perimeter is half because of the corner, and so is the base area and the volumes are half.

Problem 18. A small spring runs through a circular field at its center. If the diameter of the field is 4 meters, and the spring width 2 Meters. What is the area out of the water ?



Solution: The green area, A, is twice the area of the section minus the triangle area. Here d=1 R=2 :

Section area= $sR/2=\theta R^2/2=R^2\cos^{-1}(d/R)=4\pi/3$

Area of the triangle=$cd/2=d*\mathrm{sqrt}(R^2-d^2)=\mathrm{sqrt}(3)$

$A=2*[4\pi/3 -\mathrm{sqrt}(3)]=4.91$

Problem 19. For a right angle triangle with edges $a>b>c$ we have $a+b=81$ $a+c=72$. What are the three edges?

Solution: Pythagoras: $a^2=b^2+c^2$

Thus: $a^2=(81-a)^2+(72-a)^2$; $a^2-306a+11745=0$

$a=261$ yields negative values for b,c

therefore, the second solution holds: $a=45$, $b=36$, $c=27$

Problem 20a. The area of right angle triangle is $30cm^2$ and the sum of its perpendicular edges is 17. What are the three edges ?

Solution: bc=60 ; b+c=17; Pythagoras: $a^2=b^2+c^2$ gives $b^2-17b+60=0$

b=17/2±sqrt(289-240)/2=(17±7)/2 ; b=12,5 c=5,12 a=13

Problem 20b. ab=706.2 ; a=c+36.9

Solution: Pythagoras: $a^2=b^2+c^2$ $f(a)=a^2-(706.2/a)^2-(a-36.9)^2=0$ a 4th order equation.

solve by iterative guess: a=10 ; f(10)=-5610.79

a=20 ; f(20)=-1131.7

a=30 ; f(30)=+298.26

can interpolate as if f(a) is linear: a=30-x x=(30-20)*298.26/(298.26+1131.7)=2.086

a=27.914 f(27.914)=58.397

in one or two more steps get good accuracy.

Problem 21. The length of a circular road around the hill is 325 meters. Three people run in this road. The first runs 150 meters per minute, the second 120 and the third 90 meters per minute. If they start at the same point, when would they meet again for the first time ?

Solution: After the first ran x meters, then the second ran x/150*120 meters and the last x/150*90 meters.

for the first and second: x/150=(x-325)/120 x=325*5

meet after first finished 5 rounds and second 4 rounds lasting 10.833 minutes

for the second and third: x/120=(x-325)/90 x=325*4

meet after the second finished 4 rounds and third 3 rounds lasting 10.8333

So all three will meet after 5 rounds for the first, 4 for the second and 3 for the third.

Problem 22. Three people have sacks full of coins.

A says: if I take 2/3 of the coins of B and 1/3 of the coins of C I'll have 100 coins.

B says: if I take 2/3 of A and 1/2 of C I'll also have 100 coins.

C says: if I take 2/3 of A and 2/3 of B I'll have 100 coins too.

How many coins each has ?

Solution: a+2b/3+c/3=100 ; 2a/3+b+c/2=100 ; 2a/3+2b/3+c=100

giving a=60; b=45; c=30

Problem 23. A pheasant cost \$5 a hen \$3 and 3 chicks \$1. I bought 100 birds for \$100. How much did I buy of each ?

Solution: $x+y+z=100$; $5x+3y+z/3=100$ or $15x+9y+z=300$;

substituting z from first eq. $14x+8y=200$ or $7x+4y=100$

We scan for positive integer solutions. X must be of type 4n if y must be an integer.

$x=4$ $y=18$ $z=78$

$x=8$ $y=11$ $z=81$

$x=12$ $y=4$ $z=84$

$x=16$ y-negative

Problem 24. I bought 100 fruits for \$100 . An orange cost \$7, a pear \$3 and 3 kiwis cost \$1. How much did I buy of each ?

Solution: $x+y+z=100$; $7x+3y+z/3=100$ or $21x+9y+z=300$;

substituting z from first eq. $20x+8y=200$ or $5x+2y=50$

Scan y of the form 5n to have integer x:

$y=0$ $x=10$ $z=90$

$y=5$ $x=8$ $z=87$

$y=10$ $x=6$ $z=84$

$y=15$ $x=4$ $z=81$

$y=20$ $x=2$ $z=78$

$y=25$ $x=0$ $z=75$

$y=30$ x-negative

Problem 25. I have pheasants and rabbits in a cage. I count 35 heads and 94 legs. How much do I have of each kind ?

Solution: $x+y=35$; $2x+4y=94$ the answer is : $x=23$ $y=12$

Problem 26. In a right angle triangle $x<y<z$ we have: $2x+4y+4z=x(y^2-z+x)$ & $2yz=z^2+xy$

Find $d=2x+2y$?

Solution: Pythagoras yields third order equation

Problem 27. The rate of recruiting soldiers increase as the third power: 1 soldier on the 1st day, 2^3 on the 2nd, etc. How many soldiers were recruited after 15 days ? After n days ?

Solution: Assume $1^3+2^3+3^3+4^3+\ldots n^3=an^4+bn^3+cn^2+dn+e$

substitute for $n=1,2, \ldots 5$ to get 5 equations with 5 unknowns a,b,..e

Problem 28. r is the radius of the circumscribed circle in a right angle triangle with edges $x<y<z$.

We know that $dxy=24$ & $y+z=9$ what is x ?

Hint: use the relation: $d=2r=x+y-z$

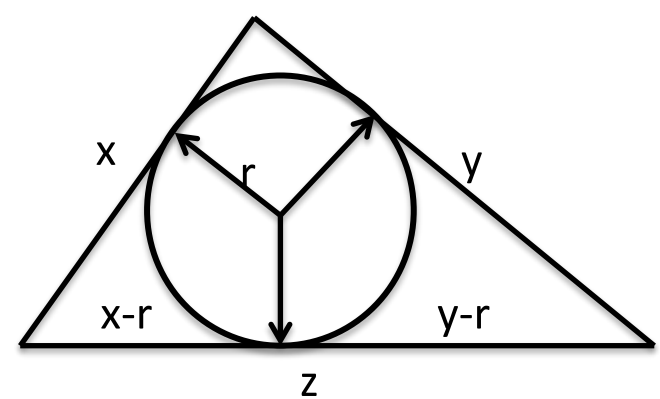
since the triangle area $xy/2=r(x+y+z)/2$ or $xy=r(x+y+z)$

Pythagoras: $x^2+y^2 = z^2=(x-r + y-r)^2=[x^2+y^2+4r^2-4xr-4yr+2xy]$

or $2xy-4xr-4yr+4r^2=0$ $2r(x+y-r)=xy$

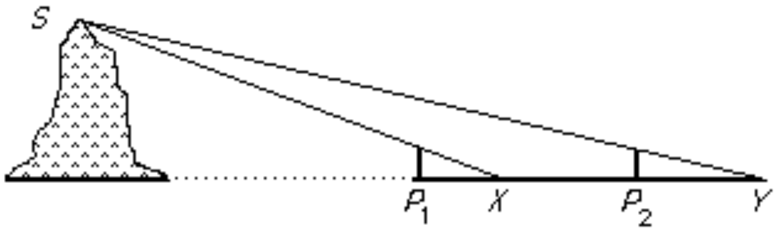
but from the above area relation: $2(x+y-r)=x+y+z$

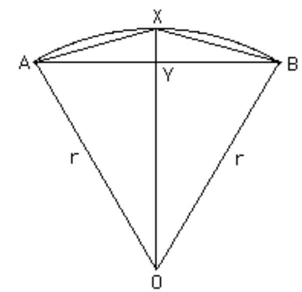
Solution: $x=3$ $y=4$ $z=5$ $r=1$

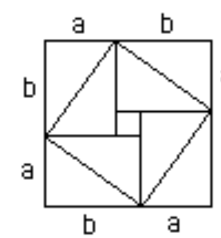


Drawings from the Chinese mathematics book: Pythagoras law, the circumference of a circle, the height of a mountain measured from a distance.

Although at that time the cultures did not communicate, the similarity to Greek books is stunning…



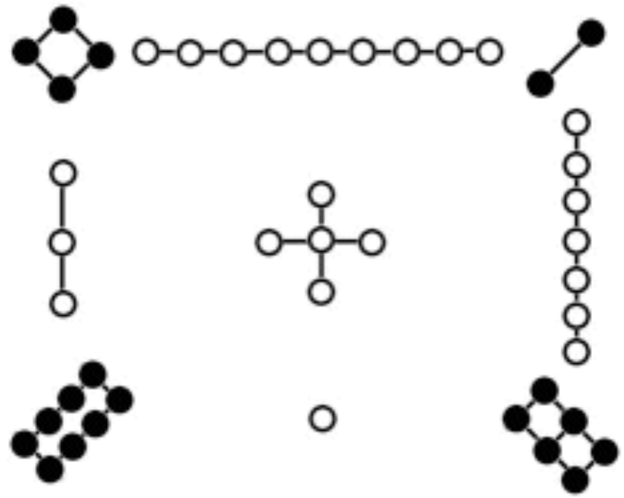




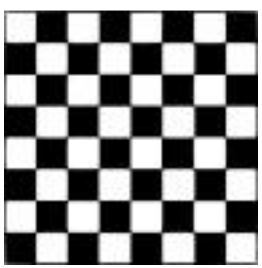
Chinese magic squares:

The sum of rows, columns and diagonals is 15.

Lo Shu presented his magic square as drawn on the left



4	9	2
3	5	7
8	1	6



How many squares we can draw on a chess board?

$$64 + 49 + 63 + 25 + 16 + 9 + 4 + 1 = 204$$

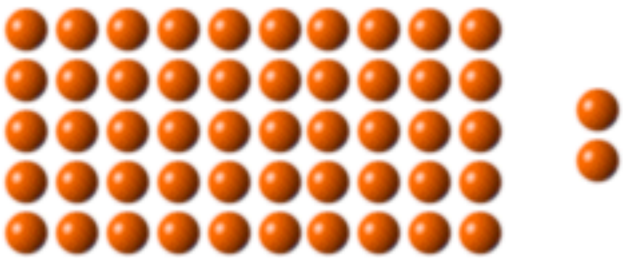
How many rice grains we put on a chess board if we put 1 on the first, and double in next neighboring square:

$$1+2+4+8+\ldots+2^{63}=(2^{64}-1)/(2-1)=1.84467\text{x}10^{19}$$

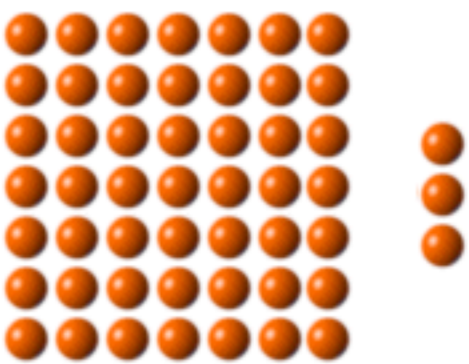
The remnant theorem

If a collection of balls are arranged in rows of 3, there is one ball left over

If arranged in rows of 5, there are two balls left over



If arranged in rows of 7, there are three balls left over



The Chinese Remainder Theorem proves that the smallest number of balls must be 52

$1+3*a = 2+5*b=3+7*c = n$

a=3	-	c=1	n=10
-	b=3	c=2	n=17
-	-	c=3	n=24
a=10	-	c=4	n=31
-	-	c=5	n=38
-	-	c=6	n=45
a=17	b=10	c=7	n=52

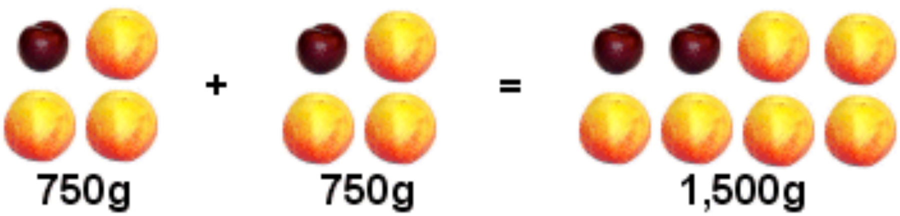
….. C=7 +3*5

a=52	b=31	c=22	n=157

Chinese method of solving equations: by elimination.

Problem:
If one plum and three peaches weigh a total of 750g, and two plums and one peach weigh a total of 500g, how much does a single peach and plum weigh?

First, double the contents of the first scale:



Subtract from this the contents of the second set of scales:

Therefore, a single peach must weigh 200g (1,000 ÷ 5).
Then, take the peach off the second scale:

Therefore, a single plum must weigh 150g (300 ÷ 2)

$X+3y=750$

$2x+y=500$

$2x+6y=1500$ +

$2x+y=500$ -

$5y=1000$

$Y=200$

Algebra school program

Quadratic equations
Inequalities
Functions
Arithmetic and Geometric series. Infinite sums
Equations with $n>1$ unknowns
Induction
Combinatorics
Sets and group theory.
Complex numbers

Solving quadratic equations by complementing terms with x to a square

$ax^2 + bx + c = 0$

$x^2 + (b / a)\, x + c / a = 0$

$x^2 + (b / a)\, x = \mathbf{- c / a}$

$x^2 + (b / a)\, x + (b / 2a)^2 = - c / a + (b / 2a)^2$

$[\, x + (b / 2a)\,]^2 = - c / a + (b / 2a)^2$

$[\, x + (b / 2a)\,]^2 = [\, b^2 - 4a\, c\,] / (\, 4a^2\,)$

$+ (b / 2a) = \pm\, \mathrm{sqrt}\, \{\, [\, b^2 - 4a\, c\,] / (\, 4a^2\,)\, \}$

$x = - b / 2a \pm \mathrm{sqrt}\, \{\, [\, b^2 - 4a\, c\,] / (\, 4a^2\,)\, \}$

$x = - b / 2a \pm \mathrm{sqrt}(b^2 - 4a\, c) / 2\, |\, a\, | = - b / 2a \pm \mathrm{sqrt}(b^2 - 4a\, c) / 2\, |\, a\, |$

$x = [\, -b \pm \mathrm{sqrt}(\, b^2 - 4a\, c\,)\,] / 2\, a$

$b^2 - 4a\, c > 0$ two solutions

$b^2 - 4a\, c = 0$ one solution

$b^2 - 4a\, c > 0$ no solution

Relations:

$x_1 + x_2 = - b / a$ $x_1 x_2 = c / a$ $a(x - x_1)(x - x_2) = ax^2 + b\, x + c$

Graphic description of functions

Consider a parabola: $ax^2 + bx + c = y$

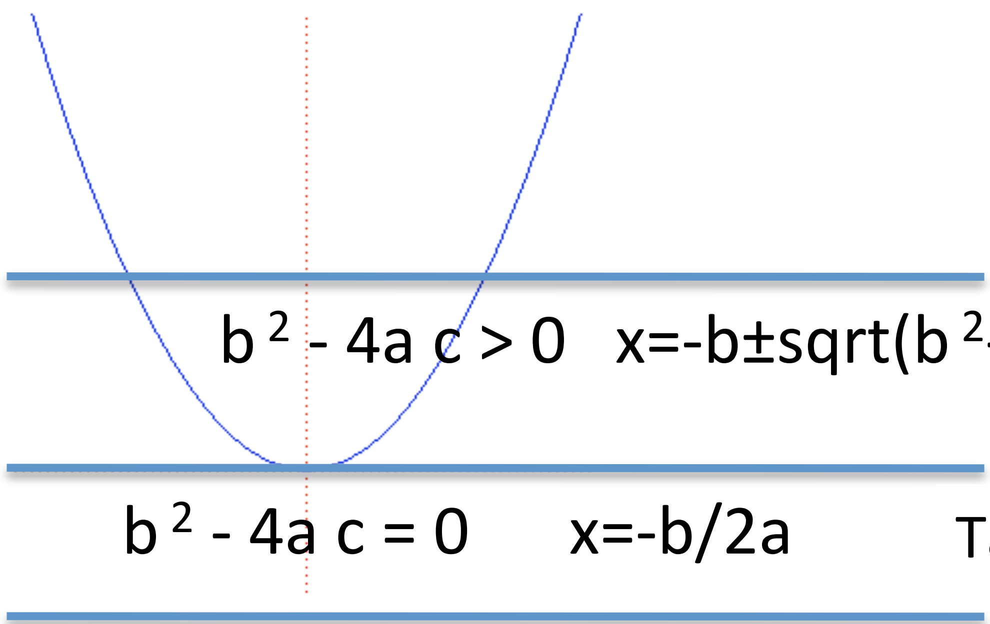
For large or small x y is large (x^2 takes over x)

For x=-b/2a the graph is at its minimum.

Proof: $aX^2=y$ has its minimum at X=0 ; If we substitute X=x+b/2a we get $ax^2+bx+(b/2a)^2=y$

Since it is different than the original equation by a constant, it has the minimum at the same place x.

Sorting the solutions according to the discriminant b^2-4ac



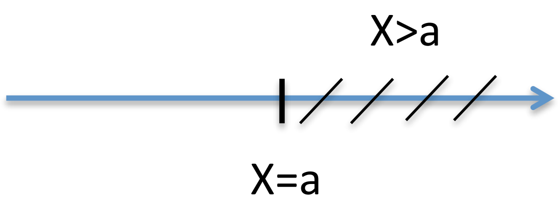
$b^2 - 4ac > 0$ $x=-b\pm sqrt(b^2-4ac)]/2a$ Two solutions

$b^2 - 4ac = 0$ $x=-b/2a$ Tangent – 1 solution

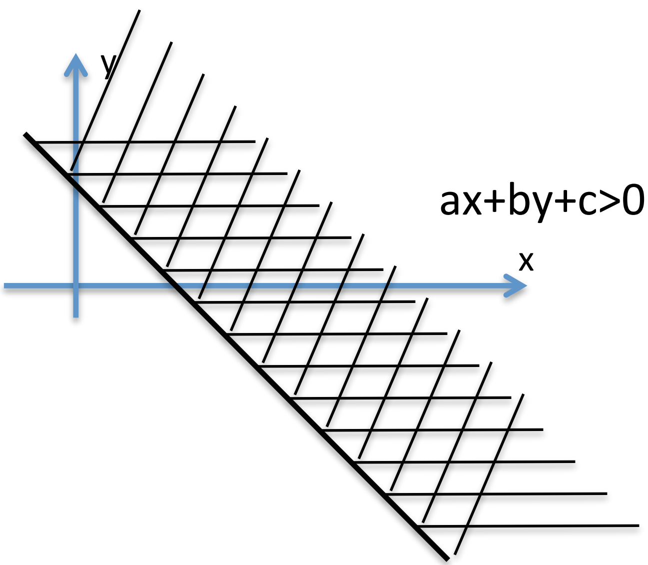
$b^2 - 4ac < 0$ No real solutions

Inequalities: a graphic presentation

One-dimensional: $x>a$

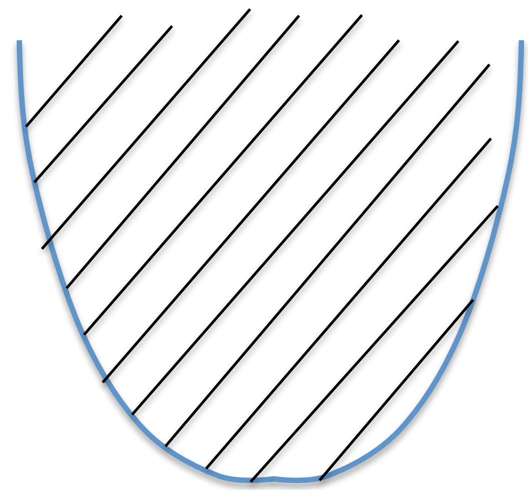


Two-dimensional:



$ax+by+c>0$

$ax^2 + bx + c>0$

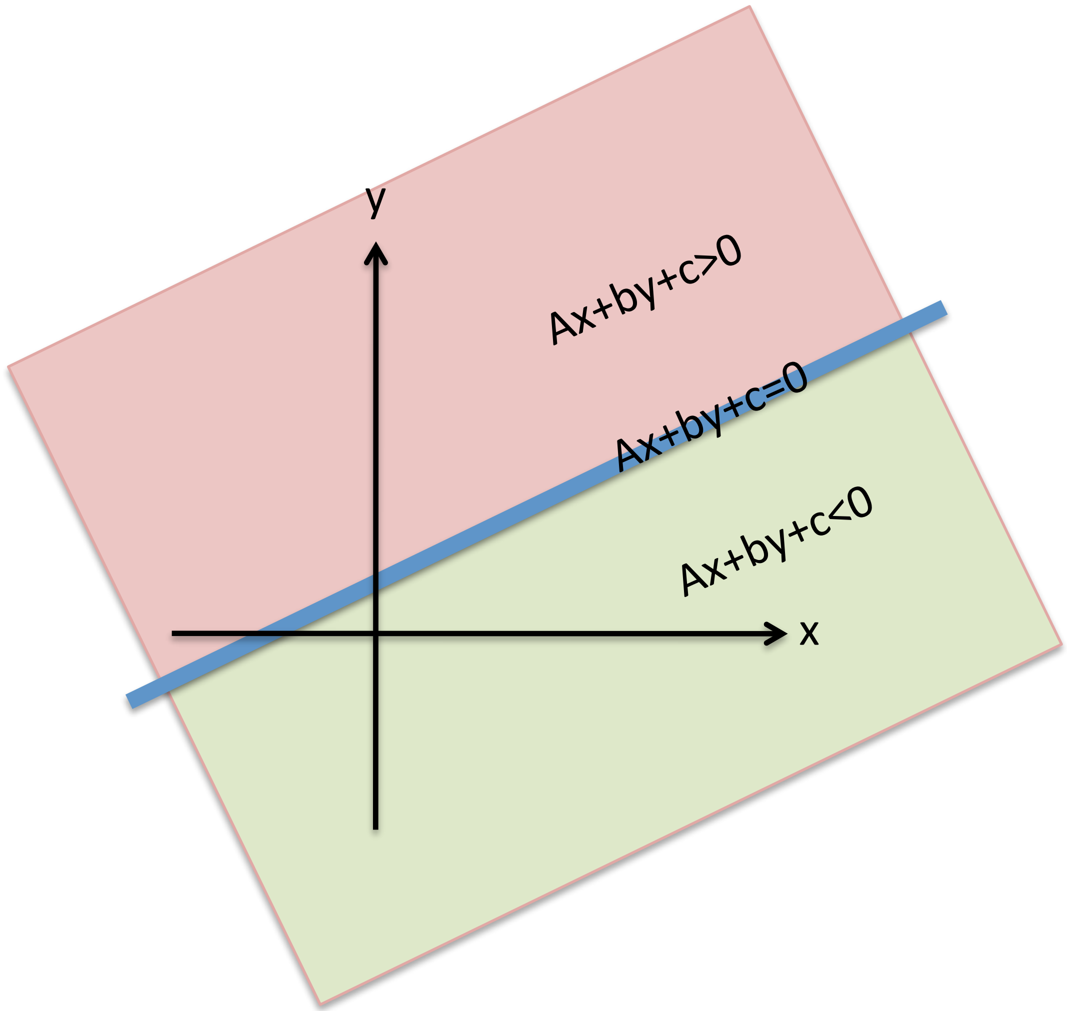


Inequalities: a graphic presentation

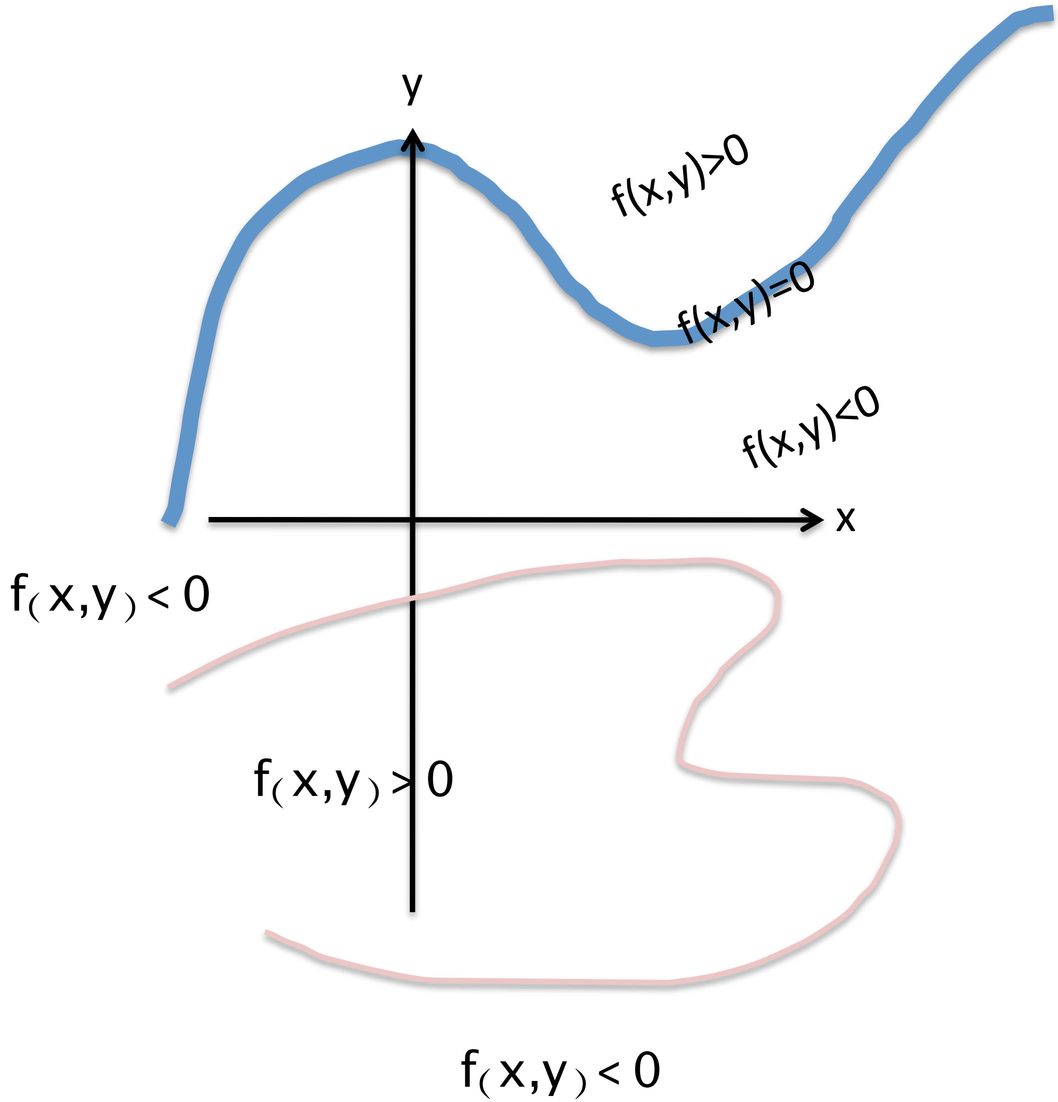
$ax+by+c=0$ is a line in xy plane

$ax+by+c>0$ is half a plane

we can test one point to know which half of the plane we are



In fact, every simple function, f(x,y)=0 divides the plane into two regions: **f(x,y)>0 and f(x,y)<0** true if for every x there is a single y and vice verse, but not if it is multi-value function (bottom graph)



Unlike equalities, where one can multiply or divide both sides by any number, Inequalities reverse their sign > by < if multiplied or divided by a negative number.

if $a>b$ then

$ac>bc$ if $c>0$

$ac<bc$ if $c<0$

This complicates the solution.

If $a=b$ then $b=a$ but: if $a>b$ then $b<a$

Transitivity: if $a=b$ and $b=c$ then $a=c$ also: if $a>b$ and $b>c$ then $a>c$

If $a=b$ then $a+x=b+x$ also: if $a>b$ then $a+x>b+x$

Powers and equalities:

If $a>1$ and $m>n$ then $a^m>a^n$

If $a>0$ & $b>0$ and $a^n=b^n$ then $a=b$

$a^{-n}=1/a^n$

$a^n a^m=a^{n+m}$

Absolute value:

Definition: if $a>0$ $|a|=a$ if $a<0$ $|a|=-a$

Therefore: $|a*b|=|a|*|b|$ and $|a/b|=|a|/|b|$

But $|a\pm b|$ can be any of four values: $-a-b$; $a+b$; $a-b$; $-a+b$

Sums of series, infinite sums and integrals:

Notation:

Sum - $\sum_{i=0}^{i=n} x_i = x_1 + x_2 + x_3 + \ldots + x_n$

Product - $\prod_{i=0}^{i=n} x_i = x_1 * x_2 * x_3 * \ldots * x_n$

A few simple sums −

Arithmetic series:

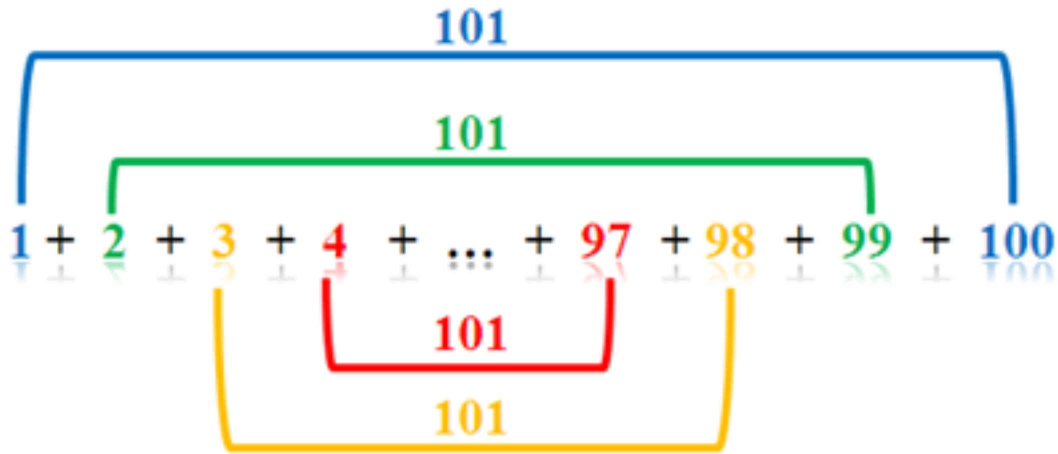
$$\sum_{i=0}^{i=n} i = n(n+1)/2$$

More generally: the sum of an arithmetic series = number of elements * their average

$$\sum_{i=n1}^{i=n2} (a+bi) = (n2-n1+1)*(a+b*n1+a+b*n2)/2$$

$$a_1 + a_2 + \ldots + a_n = n*(a_1 + a_n)/2$$

Gauss at the age of 8 figured out the sum of the numbers 1 till 100 is 50*(1+100)=5050



Sum on powers is always one power higher

$$\sum_{i=0}^{i=n} i^m = an^{m+1} + bn^m + \ldots + xn + y$$

By substituting values for n=0,1,…m+1 we can determine the m+2 unknowns, a,b, … x,y for example for second powers we assume the sum is $a*n^3+b*n^2+x*n$

And solve successively:

$$\sum_{i=0}^{i=0} i^2 = y = 0$$

$$\sum_{i=0}^{i=1} i^2 = a+b+x = 1$$

$$\sum_{i=0}^{i=2} i^2 = 8a+4b+2x = 5$$

$$\sum_{i=0}^{i=3} i^2 = 27a+9b+3x = 14$$

We have three equations with three unknowns, and solve by elimination, first of x:

$a+b+x=1$ x6

8a+4b+2x = 5 x3

27a+9b+3x = 14 x2

have Subtracting 2nd from 3rd & 1st from 2nd

we are left with two equations with two unknowns:

$$a(54-24)+b(18-12)=28-15$$

$$a(24-6)+b(12-6)=15-6$$

$$30a+6b=13$$

$$18a+6b=9$$

Subtract again to eliminate b:

$$12a=4 \quad a=1/3$$

$$6b=(9-18/3)=3 \quad b=1/2$$

$$x=1-1/2-1/3=1/6$$

Now test:

$$\sum_{i=0}^{i=n} i^2 = n^3/3+n^2/2+n/6 = (2n^2+3n+1)n/6 = (2n+1)(n+1)n/6$$

$$n=0 \quad sum=0$$

$$n=1 \quad sum=3*2/6= \quad 1$$

$$n=2 \quad 1+4=sum=5*3*2/6= \quad 5$$

$$n=3 \quad 1+4+9=sum=7*4*3/6=14$$

$$n=4 \quad 1+4+9+16=sum=9*5*4/6=30$$

If one can guess a solution, it can be proved by **induction**:
True for 1;
Assume true for n=1;
Prove true for n+1.
e.g. prove that:

$$\sum_{i=0}^{i=n} i^2 = (2n+1)(n+1)n/6$$

n=1: 0+1=3*2/6=1

N+1:

$$\sum_{i=0}^{i=n+1} i^2 = (2n+2+1)(n+2)(n+1)/6 =v= (2n+1)(n+1)n/6 + (n+1)^2$$

The sign =v= means need to be proved equal.
We can divide both sides by (n+1)/6

$$2n+2+1)(n+2)/6 =v= (2n+1)n + 6(n+1)$$

$$2n2+(3+4)n+6 =v= 2n2+7n+6$$

Ideed true.

Geometric series:

$$\sum_{i=0}^{i=n} aq^i = a + aq + aq^2 + \dots aq^n = a(q^{n+1}-1)/(q-1) = a(1-q^{n+1})/(1-q)$$

Proof: by induction.

Application: Compounded interest in bank deposits, Serial dilutions.

Radioactive decay:

$\lim_{n\to\infty} \{ [1+1/n]^n \} = e$ $\lim_{n\to\infty} \{ [1+p/n]^n \} = e^p$ $\lim_{n\to\infty} \{ [1-p/n]^n \} = e^{-p}$

$\lim_{n\to\infty} \{a(q^{n+1}-1)/(q-1) \} = ae^{-p}$ $q=1-p/n$

Power series:

$1^2+2^2+3^2+\dots+n^2=n(n+1)(2n+2)/6$

Converging infinite sums:

$$\sum_{i=0}^{i=n} 1/i = ?????$$

$$\sum_{i=0}^{i=n} 1/i^2 = ?????$$

We find an interesting feature: although both $1/i$ and $1/i^2$ are diminishingly small for large i, the first sum growth indefinitely.

The proof was given by **Nicole Oresme 1382-1325**:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \cdots$$

$$> 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16} + \cdots .$$

$$1 + \left(\frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{16} + \cdots + \frac{1}{16}\right) + \cdots$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots = \infty.$$

$$\sum_{n=1}^{2^k} \frac{1}{n} \geq 1 + \frac{k}{2}$$

but the second infinite sum converges: the proof is complex, and was given by **Leonhard Euler 1707-1783 and Augustin-Louis Cauchy 1789-1857**

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$$

Combinatorics

The number of ways n people can be ordered in a row is **n! = n*(n-1)*** … **2*1**

n possible way to chose the first in the row

n-1 possible way to chose the second in the row

etc. till 1 last left

The number of possibilities of choosing m ordered row from n people **= n!/(n-m)!**

n possible way to chose the first in the row

n-1 possible way to chose the second in the row

etc. till n-m+1

The number of possibilities of choosing m from n people (no order) **= n!/(n-m)!/m!**

as above, divided by m! order changes.

Important for calculation probabilities.

Newton's binomial equation

$$(a+b)^n = \sum_{m=0}^{m=n} n!/(n-m)!/m!\ a^m b^{(n-m)}$$

Pascal's triangle − recursive formula to build the binomial coefficients: (see in "numbers")

$$n!/(n-m)!/m! = (n-1)!/(n-m-1)!/m! + (n-1)!/(n-m)!/(m-1)!$$

Group theory:

Group is a set of elements with a binary operation.
Example: the group of 3 elements: (-1,0,1) for multiplication.
the operation table:

	-1	0	1
-1	1	0	-1
0	0	0	0
1	-1	0	1

Incomplete groups: positive integers for subtraction, or division: because subtraction of two positive integers may be negative, and division of two integers may be a rational number that is not an integer.

Another incomplete group: irrationals to addition division or multiplication: because the result may be a rational number.

Logarithm:

If $a^x=b$ then $x=\log_a b$ a is the logarithm base.

Properties: $\log(a*b)=\log a+\log b$ $\log(a/b)=\log a-\log b$

Change of base: $\log_c b / \log_c a=\log_a b$

Polynomials:

If P(x) is a polynomial of degree m and P(r)=0 then P divides by (x-r), or:

P(x)=(x-r)*Q(x), and Q(x) is a polynomial of degree m-1